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SOME RESULTS ON LEFT $(\sigma,\tau)-$JORDAN IDEALS AND ONE SIDED GENERALIZED DERIVATIONS

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Abstract. Let R be a prime ring with characteristic not 2 and $\sigma,\tau,\alpha,\beta,\lambda,\mu,\gamma$ automorphisms of R. Let $h: R\longrightarrow R$ be a nonzero **left (resp. right)- generalized** $(\alpha,\beta)-$ derivation associated with $(\alpha,\beta)-$ derivation d_1 (**resp.** d). Let W,V be nonzero left $(\sigma,\tau)-$Jordan ideals of R and I a nonzero ideal of R. In this paper we also study the situations. (1) $ah(R)b\subset C_{\lambda,\mu}(R)$ (2) $bh(I,a)_{\sigma,\tau}=0$ or $h(I,a)_{\sigma,\tau}b=0$, (3) $bh(I)\subset C_{\lambda,\mu}(W)$ or $h(I)b\subset C_{\lambda,\mu}(W)$, (4) $h(I)\subset C_{\lambda,\mu}(J)$, (5) $(h(R),a)_{\alpha,\beta}\subset C_{\alpha,\beta}(R)$, (6) $(h(I)b,a)_{\lambda,\mu}=0$, (7) $b\gamma(W)\subset C_{\lambda,\mu}(V)$ or $\gamma(W)b\subset C_{\lambda,\mu}(V)$.

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1. Introduction

Let R be a ring and σ,τ two mappings of R. For each $r,s\in R$ we set $[r,s]_{\sigma,\tau}=r\sigma(s)-\tau(s)r$ and $(r,s)_{\sigma,\tau}=r\sigma(s)+\tau(s)r$. Let U be an additive subgroup of R. If $(U,R)\subset U$ then U is called a Jordan ideal of R. The definition of $(\sigma,\tau)-$Jordan ideal of R is introduced in [8] as follows: (i) U is called a right $(\sigma,\tau)-$Jordan ideal of R if $(U,R)_{\sigma,\tau}\subset U$, (ii) U is called a left $(\sigma,\tau)-$Jordan ideal if $(R,U)_{\sigma,\tau}\subset U$. (iii) U is called a $(\sigma,\tau)-$Jordan ideal if U is both right and left $(\sigma,\tau)-$Jordan ideal of R. Every Jordan ideal of R is a $(1,1)-$Jordan ideal of R, where $1:R\to R$ is a identity map. The following example is given in [8]. If $R=\{\left(\begin{smallmatrix}x&y\\0&0\end{smallmatrix}\right)\mid x \text{ and } y \text{ are integers}\}$, $U=\{\left(\begin{smallmatrix}x&0\\0&0\end{smallmatrix}\right)\mid x \text{ is integer}\}$, $\sigma\left(\begin{smallmatrix}x&y\\0&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}x&0\\0&0\end{smallmatrix}\right)$ and $\tau\left(\begin{smallmatrix}x&y\\0&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}x&-y\\0&0\end{smallmatrix}\right)$ then U is $(\sigma,\tau)-$right Jordan ideal but not a Jordan ideal of R.

A derivation d is an additive mapping on R which satisfies $d(rs)=d(r)s+rd(s),\forall r,s\in R$. The notion of generalized derivation was introduced by Brešar [3] as follows. An additive mapping $h:R\to R$ will be called a generalized derivation if there exists a derivation d of R such that $h(xy)=h(x)y+xd(y)$ for all $x,y\in R$.

An additive mapping $d:R\to R$ is said to be a $(\sigma,\tau)-$derivation if $d(rs)=d(r)\sigma(s)+\tau(r)d(s)$ for all $r,s\in R$. Every derivation $d:R\to R$ is a $(1,1)-$derivation. Chang [4] gave the following definition. Let R be a ring, σ and τ automorphisms of R and

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$d : R \to R$ a $(\sigma,\tau)-$derivation. An additive mapping $h : R \to R$ is said to be a right generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = h(x)\sigma(y)+\tau(x)d(y)$, for all $x, y \in R$ and h is said to be a left generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = d(x)\sigma(y)+\tau(x)h(y)$ for all $x, y \in R$. h is said to be a generalized $(\sigma,\tau)-$derivation of R associated with d if it is both a left and right generalized $(\sigma,\tau)-$derivation of R associated with d.

According to Chang's definition, every $(\sigma,\tau)-$derivation $d : R \to R$ is a generalized $(\sigma,\tau)-$derivation associated with d and every derivation $d : R \to R$ is a generalized $(1,1)-$derivation associated with d. A generalized $(1,1)-$derivation is simply called a generalized derivation. The definition of generalized derivation which is given in [3] is a right generalized derivation associated with derivation d according to Chang's definition.

The mapping $h(r) = (a,r)_{\sigma,\tau}$ for all $r \in R$ is a left-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation $d_1(r) = [a,r]_{\sigma,\tau}$ for all $r \in R$ and right-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation $d(r) = -[a,r]_{\sigma,\tau}$ for all $r \in R$.

Throughout the paper, R will be a prime ring with centre Z, characteristic not 2 and $\sigma,\tau,\alpha,\beta,\lambda,\mu,\gamma$ automorphisms of R. We set $C_{\sigma,\tau}(R) = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall r \in R\}$, and shall use the following relations frequently:

$[rs,t]_{\sigma,\tau} = r[s,t]_{\sigma,\tau} + [r,\tau(t)]s = r[s,\sigma(t)] + [r,t]_{\sigma,\tau}s$

$[r,st]_{\sigma,\tau} = \tau(s)[r,t]_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t)$

$(rs,t)_{\sigma,\tau} = r(s,t)_{\sigma,\tau} - [r,\tau(t)]s = r[s,\sigma(t)] + (r,t)_{\sigma,\tau}s$

$(r,st)_{\sigma,\tau} = \tau(s)(r,t)_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t) = -\tau(s)[r,t]_{\sigma,\tau} + (r,s)_{\sigma,\tau}\sigma(t)$

2. Results

Lemma 2.1. [2, Lemma 1] *Let $d : R \longrightarrow R$ be a nonzero $(\sigma,\tau)-$derivation of R and U a nonzero right ideal of R. If $a \in R$ such that $d(U) = 0$ then $d = 0$.*

Lemma 2.2. [5, Theorem 2. 12] *Let W be a left $(\sigma,\tau)-$Jordan ideal of R and $b \in R$. (i) If $[W,b]_{\lambda,\mu} = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$. (ii) If $[b,W]_{\lambda,\mu} = 0$ then $b \in C_{\lambda,\mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.*

Lemma 2.3. [6, Theorem 2.7] *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation d and I, J nonzero ideals of R. If $a \in R$ such that $ah(I) \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $d = 0$.*

Lemma 2.4. [5, Lemma 2.2] *Let I be a nonzero ideal of R and $a, b \in R$. If $b\gamma(I,a)_{\alpha,\beta} = 0$ or $\gamma(I,a)_{\alpha,\beta}b = 0$ then $b = 0$ or $a \in Z$.*

Theorem 2.1. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d. Let I be a nonzero ideal of R and $a, b \in R$.*

(i) If $h\lambda(I)b = 0$ then $b = 0$.
(ii) If $h\lambda(I,a)_{\sigma,\tau} = 0$ then $a \in Z$ or $d\lambda\tau(a) = 0$.
(iii) If $ah(I)b = 0$ then $b = 0$ or $ad\beta^{-1}(a) = 0$.

Proof. (i) If $h\lambda(I)b = 0$ then we have, for all $r \in R$, $x \in I$

$$0 = h\lambda(rx)b = d\lambda(r)\alpha\lambda(x)b + \beta\lambda(r)\mathbf{h}\lambda(\mathbf{x})\mathbf{b} = d\lambda(r)\alpha\lambda(x)b$$

and so $d\lambda(R)\alpha\lambda(I)b = 0$. Since $\lambda(I)$ is a nonzero ideal of R and $d \neq 0$ then the last relation gives that $b = 0$.

(ii) If $h\lambda(I,a)_{\sigma,\tau}=0$ then we get, for all $x\in I$

$$\begin{aligned}0&=h\lambda(\tau(a)x,a)_{\sigma,\tau}=h\lambda\{\tau(a)(x,a)_{\sigma,\tau}-[\tau(a),\tau(a)]x\}\\&=h\{\lambda\tau(a)\lambda(x,a)_{\sigma,\tau}\}=d\lambda\tau(a)\alpha\lambda(x,a)_{\sigma,\tau}+\beta\lambda\tau(a)h\lambda(\mathbf{x},\mathbf{a})_{\sigma,\tau}\\&=d\lambda\tau(a)\alpha\lambda(x,a)_{\sigma,\tau}.\end{aligned}$$

That is $d\lambda\tau(a)\alpha\lambda(I,a)_{\sigma,\tau}=0$. Using 2.4 we obtain that $a\in Z$ or $d\lambda\tau(a)=0$ by the last relation.

(iii) If $ah(I)b=0$ then we have, for all $x\in I$

$$0=ah(\beta^{-1}(a)x)b=ad\beta^{-1}(a)\alpha(x)b+\mathbf{aah}(\mathbf{x})\mathbf{b}=ad\beta^{-1}(a)\alpha(x)b.$$

That is, $ad\beta^{-1}(a)\alpha(I)b=0$. Since $\alpha(I)$ is a nonzero ideal of R then we obtain that $b=0$ or $ad\beta^{-1}(a)=0$ in prime rings. □

Corollary 2.1. *Let I be a nonzero ideal of R and $a,b,c\in R$. If $a(I,c)_{\sigma,\tau}b=0$ then $b=0$ or $a[a,\tau(c)]=0$ (and $a=0$ or $[b,\sigma(c)]b=0$).*

Proof. The mapping defined by $h(r)=(r,c)_{\sigma,\tau},\forall r\in R$ is a left-generalized derivation associated with derivation $d_1(r)=-[r,\tau(c)],\forall r\in R$ and right-generalized derivation associated with derivation $d(r)=[r,\sigma(c)],\forall r\in R$. If $h=0$ then $d=0=d_1$ and so $c\in Z$ is obtained. Let $h\neq 0$.

If $a(I,c)_{\sigma,\tau}b=0$ then we have $ah(I)b=0$. Since h is a left-generalized derivation associated with d_1 then we have $b=0$ or $ad_1(a)=0$ by 2.1(iii). That is $b=0$ or $a[a,\tau(c)]=0$. If $c\in Z$ then $a[a,\tau(c)]=0$. Finally we obtain that $b=0$ or $a[a,\tau(c)]=0$ for any cases.

On the other hand, since h is a right-generalized derivation associated with d then $ah(R)b=0$ gives that $a=0$ or $d(b)b=0$ by [6, Lemma 2.19 (i)]. That is $a=0$ or $[b,\sigma(c)]b=0$. If $c\in Z$ then $[b,\sigma(c)]b=0$. Finally we obtain that $a=0$ or $[b,\sigma(c)]b=0$ for any cases. □

Theorem 2.2. *Let $h:R\longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d. Let I be a nonzero ideal of R and $a,b\in R$.*

(i) If $h(I,a)_{\sigma,\tau}b=0$ then $d\tau(a)=$ or $[b,\sigma(a)]b=0$.
(ii) If $bh\lambda(I)=0$ then $bd\beta^{-1}(b)=0$.

Proof. (i) If $h(I,a)_{\sigma,\tau}b=0$ then we get, for all $x\in I$

$$\begin{aligned}0&=h(\tau(a)x,a)_{\sigma,\tau}b=h\{\tau(a)(x,a)_{\sigma,\tau}-[\tau(a),\tau(a)]x\}b\\&=h\{\tau(a)(x,a)_{\sigma,\tau}\}b=d\tau(a)\alpha(x,a)_{\sigma,\tau}b+\beta\tau(a)h(x,a)_{\sigma,\tau}b\\&=d\tau(a)\alpha(x,a)_{\sigma,\tau}b\end{aligned}$$

which gives that

$$\alpha^{-1}d\tau(a)(I,a)_{\sigma,\tau}\alpha^{-1}(b)=0. \tag{1}$$

Then 1 gives that $d\tau(a)=0$ or $[b,\alpha\sigma(a)]b=0$ by 2.1.

(ii) If $bh\lambda(I)=0$ then we have, for all $x\in I$

$$\begin{aligned}0&=bh\lambda(\lambda^{-1}\beta^{-1}(b)x)=bh(\beta^{-1}(b)\lambda(x))\\&=bd\beta^{-1}(b)\alpha\lambda(x)+b\mathbf{bh}\lambda(\mathbf{x})=bd\beta^{-1}(b)\alpha\lambda(x)\end{aligned}$$

and so $bd\beta^{-1}(b)\alpha\lambda(I)=0$. Since $\alpha\lambda(I)$ is a nonzero ideal of R then we obtain that $bd\beta^{-1}(b)=0$. □

Remark 2.1. [5, Corollary 2.11] *Let $d:R\longrightarrow R$ be a nonzero $(\alpha,\beta)-$ derivation and W a nonzero left $(\sigma,\tau)-$Jordan ideal of R. If $d\gamma(W)=0$ then $\sigma(v)-\tau(v)\in Z$ for all $v\in W$.*

Theorem 2.3. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d. Let I be a nonzero ideal of R and $a, b \in R$.*

(i) If $h\lambda(I,a)_{\sigma,\tau} = 0$ then $a \in Z$ or $d\lambda\sigma(a) = 0$.
(ii) If $bh\lambda(I) = 0$ then $b = 0$.
(iii) If $bh(I,a)_{\sigma,\tau} = 0$ then $b[b,\tau(a)] = 0$ or $d\sigma(a) = 0$.

Proof. (i) If $h\lambda(I,a)_{\sigma,\tau} = 0$ then we get, for all $x \in I$

$$\begin{aligned} 0 &= h\lambda(x\sigma(a),a)_{\sigma,\tau} = h\lambda\{x[\sigma(\mathbf{a}),\sigma(\mathbf{a})] + (x,a)_{\sigma,\tau}\sigma(a)\} \\ &= h\{\lambda(x,a)_{\sigma,\tau}\lambda\sigma(a)\} = h\lambda(x,a)_{\sigma,\tau}\alpha\lambda\sigma(a) + \beta\lambda(x,a)_{\sigma,\tau}d\lambda\sigma(a) \\ &= \beta\lambda(x,a)_{\sigma,\tau}d\lambda\sigma(a) \end{aligned}$$

That is $\beta\lambda(I,a)_{\sigma,\tau}d\lambda\sigma(a) = 0$. Using 2.4 we obtain that $a \in Z$ or $d\lambda\sigma(a) = 0$ by the last relation.

(ii) If $bh\lambda(I) = 0$ then we have, for all $r \in R$, $x \in I$

$$0 = bh\lambda(xr) = bh\lambda(x)\alpha\lambda(r) + b\beta\lambda(x)d\lambda(r) = b\beta\lambda(x)d\lambda(r)$$

and so $b\beta\lambda(I)d\lambda(R) = 0$. Since $\lambda(I)$ is a nonzero ideal of R and $d \neq 0$ then the last relation gives that $b = 0$.

(iii) If $bh(I,a)_{\sigma,\tau} = 0$ then we get, for all $x \in I$

$$\begin{aligned} 0 &= bh(x\sigma(a),a)_{\sigma,\tau} = bh\{(x,a)_{\sigma,\tau}\sigma(a)\} \\ &= bh(x,a)_{\sigma,\tau}\alpha\sigma(a) + b\beta(x,a)_{\sigma,\tau}d\sigma(a) \\ &= b\beta(x,a)_{\sigma,\tau}d\sigma(a). \end{aligned}$$

That is

$$\beta^{-1}(b)(I,a)_{\sigma,\tau}\beta^{-1}d\sigma(a) = 0. \tag{2}$$

Using 2.1 and 2 we obtain $b[b,\beta\tau(a)] = 0$ or $d\sigma(a) = 0$. □

Corollary 2.2. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d and W be a nonzero left $(\sigma,\tau)-$Jordan ideal of R. If $h\lambda(W) = 0$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.*

Proof. If $h\lambda(W) = 0$ then we have $h\lambda(R,v)_{\sigma,\tau} = 0, \forall v \in W$. This means that, for any $v \in W$

$$v \in Z \text{ or } d\lambda\tau(v) = 0$$

by 2.3(i). This means that W is the union of its additive subgroups $K = \{v \in W \mid v \in Z\}$ and $L = \{v \in W \mid d\lambda\tau(v) = 0\}$. Since a group can not be the union of two of its proper subgroups, we have $W = K$ or $W = L$. We obtain that

$$W \subset Z \text{ or } d\lambda\tau(W) = 0.$$

If $d\lambda\tau(W) = 0$ then we obtain $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by 2.1. On the other hand $W \subset Z$ gives that $\sigma(v) - \tau(v) \in Z$ for all $v \in W$. □

Lemma 2.5. [5, Lemma 2.2] *Let I be a nonzero ideals of R and $a, b \in R$. If $b, ba \in C_{\lambda,\mu}(I)$ or $(b, ab \in C_{\lambda,\mu}(I))$ then $b = 0$ or $a \in Z$.*

Lemma 2.6. *Let W be a nonzero left $(\sigma,\tau)-$Jordan ideal of R and $a, b \in R$. If $b, ba \in C_{\lambda,\mu}(W)$ or $b, ab \in C_{\lambda,\mu}(W)$ then $b = 0$ or $a \in Z$ or $\sigma(v) - \tau(v) \in Z$ for all $v \in W$.*

Proof. $b, ba \in C_{\lambda,\mu}(W)$ then we have $[b, W]_{\lambda,\mu} = 0$ and $[ba, W]_{\lambda,\mu} = 0$. Using this relations and 2.2(ii) we get, for all $v \in W$

$$\{\sigma(v) - \tau(v) \in Z \text{ or } b \in C_{\lambda,\mu}(R)\} \text{ and } \{\sigma(v) - \tau(v) \in Z \text{ or } ba \in C_{\lambda,\mu}(R)\}.$$

This means that

$$\sigma(v) - \tau(v) \in Z \text{ or } \{b \in C_{\lambda,\mu}(R) \text{ and } ba \in C_{\lambda,\mu}(R)\}$$

If $\{b \in C_{\lambda,\mu}(R)$ and $ba \in C_{\lambda,\mu}(R)\}$ then we have $b = 0$ or $a \in Z$ by 2.5 . Finally we obtain that $b = 0$ or $a \in Z$ or $\sigma(v) - \tau(v) \in Z$ for all $v \in W$.

If $b, ab \in C_{\lambda,\mu}(W)$ then, considering as above we get the required result. □

Theorem 2.4. *Let W be a nonzero left $(\sigma, \tau)-$Jordan ideal of R and $a, b \in R$. Let I be a nonzero ideal of R.*

(i) If $(I, a)_{\alpha,\beta} \subset C_{\lambda,\mu}(W)$ then $a \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(ii) If $b\gamma(I, a)_{\alpha,\beta} \subset C_{\lambda,\mu}(W)$ or $\gamma(I, a)_{\alpha,\beta}b \subset C_{\lambda,\mu}(W)$ then $b = 0$ or $a \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. (i) If $(I, a)_{\alpha,\beta} \subset C_{\lambda,\mu}(W)$ then we have, for all $x \in I$

$$C_{\lambda,\mu}(W) \ni (\beta(a)x, a)_{\alpha,\beta} = \beta(a)(x, a)_{\alpha,\beta} - [\beta(a), \beta(a)]x = \beta(a)(x, a)_{\alpha,\beta}$$

and so $\beta(a)(I, a)_{\alpha,\beta} \subset C_{\lambda,\mu}(W)$. Using 2.6 we obtain

$$(I, a)_{\alpha,\beta} = 0 \text{ or } a \in Z \text{ or } \sigma(v) - \tau(v) \in Z \text{ for all } v \in W.$$

If $(I, a)_{\alpha,\beta} = 0$ then $0 = (rx, a)_{\alpha,\beta} = r(x, a)_{\alpha,\beta} - [r, \beta(a)]x = -[r, \beta(a)]x$ for all $r \in R, x \in I$. That is $[R, \beta(a)]I = 0$.This gives that $a \in Z$ in prime rings.

(ii) If $b\gamma(I, a)_{\alpha,\beta} \subset C_{\lambda,\mu}(W)$ then we get, for all $x \in I$

$$C_{\lambda,\mu}(W) \ni b\gamma(x\alpha(a), a)_{\alpha,\beta} = b\gamma(x)\gamma[\alpha(a),\alpha(a)] + b\gamma(x, a)_{\alpha,\beta}\gamma\alpha(a) = b\gamma(x, a)_{\alpha,\beta}\gamma\alpha(a)$$

and so

$$b\gamma(I, a)_{\alpha,\beta}\gamma\alpha(a) \subset C_{\lambda,\mu}(W). \tag{3}$$

If we use hypothesis and 2.6 in 3 then we get

$$b\gamma(I, a)_{\alpha,\beta} = 0 \text{ or } a \in Z \text{ or } \sigma(v) - \tau(v) \in Z \text{ for all } v \in W.$$

If $b\gamma(I, a)_{\alpha,\beta} = 0$ then we obtain that $b = 0$ or $a \in Z$ by 2.4.

If $\gamma(I, a)_{\alpha,\beta}b \subset C_{\lambda,\mu}(W)$ then we have, for all $x \in I$

$$C_{\lambda,\mu}(W) \ni \gamma(\beta(a)x, a)_{\alpha,\beta}b = \gamma\beta(a)\gamma(x, a)_{\alpha,\beta}b - \gamma[\beta(a), \beta(a)]\gamma(x)b = \gamma\beta(a)\gamma(x, a)_{\alpha,\beta}b$$

That is

$$\gamma\beta(a)\gamma(I, a)_{\alpha,\beta}b \subset C_{\lambda,\mu}(W). \tag{4}$$

If we use 2.6 and hypothesis then 4 gives that

$$\gamma(I, a)_{\alpha,\beta}b = 0 \text{ or } a \in Z \text{ or } \sigma(v) - \tau(v) \in Z \text{ for all } v \in W.$$

If $\gamma(I, a)_{\alpha,\beta}b = 0$ then we obtain that $b = 0$ or $a \in Z$ by 2.4. Finally we obtain that $b = 0$ or $a \in Z$ or $\sigma(v) - \tau(v) \in Z$ for all $v \in W$. □

Corollary 2.3. *Let W, V be nonzero left $(\sigma, \tau)-$Jordan ideals of R and $b \in R$.*

(i) If $V \subset C_{\lambda,\mu}(W)$ then $V \subset Z$ or $\sigma(w) - \tau(w) \in Z, \forall w \in W$.

(ii) If $b\gamma(V) \subset C_{\lambda,\mu}(W)$ or $\gamma(V)b \subset C_{\lambda,\mu}(W)$ then $b = 0$ or $V \subset Z$ or $\sigma(w) - \tau(w) \in Z, \forall w \in W$.

Proof. (i) If $V \subset C_{\lambda,\mu}(W)$ then $(R,V)_{\sigma,\tau} \subset C_{\lambda,\mu}(W)$ and so $V \subset Z$ or $\sigma(w)-\tau(w) \in Z, \forall w \in W$ by 2.4(i).

(ii) If $b\gamma(V) \subset C_{\lambda,\mu}(W)$ or $\gamma(V)b \subset C_{\lambda,\mu}(W)$ then we have $b\gamma(R,V)_{\sigma,\tau} \subset C_{\lambda,\mu}(W)$ or $\gamma(R,V)_{\sigma,\tau}b \subset C_{\lambda,\mu}(W)$. This gives that $b=0$ or $V \subset Z$ or $\sigma(w)-\tau(w) \in Z, \forall w \in W$ by 2.4(ii). □

Lemma 2.7. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$ derivtaion associated with a nonzero $(\alpha,\beta)-$derivation d and I, J nonzero ideals of R. If $h(I) \subset C_{\lambda,\mu}(J)$ then R is commutative.*

Proof. If $h(I) \subset C_{\lambda,\mu}(J)$ then we have, for all $x \in I$, $t \in J$, $r \in R$

$$\begin{aligned}0 &= [h(xr),t]_{\lambda,\mu} = [h(x)\alpha(r)+\beta(x)d(r),t]_{\lambda,\mu}\\ &= h(x)[\alpha(r),\lambda(t)] + [\mathbf{h(x)},\mathbf{t}]_{\lambda,\mu}\alpha(r) + \beta(x)[d(r),t]_{\lambda,\mu} + [\beta(x),\mu(t)]d(r)\\ &= h(x)[\alpha(r),\lambda(t)] + \beta(x)[d(r),t]_{\lambda,\mu} + [\beta(x),\mu(t)]d(r).\end{aligned}$$

That is

$$h(x)[\alpha(r),\lambda(t)] + \beta(x)[d(r),t]_{\lambda,\mu} + [\beta(x),\mu(t)]d(r) = 0 \text{ for all } x \in I, t \in J, r \in R. \quad (5)$$

Replacing r by $\alpha^{-1}\lambda(t)$ in 5 we get

$$\beta(x)[k(t),t]_{\lambda,\mu} + [\beta(x),\mu(t)]k(t) = 0 \text{ for all } x \in I, t \in J \quad (6)$$

where $k(t) = d\alpha^{-1}\lambda(t)$. Replacing x by rx in 6 we obtain, for all $x \in I$, $t \in J$, $r \in R$

$$\begin{aligned}0 &= \beta(rx)[k(t),t]_{\lambda,\mu} + [\beta(rx),\mu(t)]k(t)\\ &= \beta(r)\beta(x)[k(t),t]_{\lambda,\mu} + \beta(r)[\beta(x),\mu(t)]k(t) + [\beta(r),\mu(t)]\beta(x)k(t) = [\beta(r),\mu(t)]\beta(x)k(t)\end{aligned}$$

which gives $[R,\mu(t)]\beta(I)d\alpha^{-1}\lambda(t) = 0$. Since $\beta(I)$ is a nonzero ideal then we have, for any $t \in J$

$$t \in Z \text{ or } d\alpha^{-1}\lambda(t) = 0.$$

Considering as in the proof of 2.2 we get $J \subset Z$ or $d\alpha^{-1}\lambda(J) = 0$. Since d is nonzero then $d\alpha^{-1}\lambda(J) \neq 0$ by 2.1 and so $J \subset Z$ is obtained. This means that R is commutative by [9, Lemma 3]. □

Theorem 2.5. *Let W be a left $(\sigma,\tau)-$Jordan ideal of R and I a nonzero ideal of R. Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with nonzero $(\alpha,\beta)-$ derivation $d : R \longrightarrow R$ and $b \in R$.*

(i) If $h(I) \subset C_{\lambda,\mu}(W)$ then $\sigma(v)-\tau(v) \in Z, \forall v \in W$.
(ii) If $bh(I) \subset C_{\lambda,\mu}(W)$ then $b \in Z$ or $\sigma(v)-\tau(v) \in Z, \forall v \in W$.

Proof. (i) If $h(I) \subset C_{\lambda,\mu}(W)$ then we have $[h(I),W]_{\lambda,\mu} = 0$. This means that, $h(I) \subset C_{\lambda,\mu}(R)$ or $\sigma(v)-\tau(v) \in Z, \forall v \in W$ by 2.2 (ii).

If $h(I) \subset C_{\lambda,\mu}(R)$ then we get R is commutative by 2.7 and so $\sigma(v)-\tau(v) \in Z, \forall v \in W$.

(ii) If $bh(I) \subset C_{\lambda,\mu}(W)$ then $[bh(I),W]_{\lambda,\mu} = 0$. Using 2.2 (ii) we have $bh(I) \subset C_{\lambda,\mu}(R)$ or $\sigma(v)-\tau(v) \in Z, \forall v \in W$.

If $bh(I) \subset C_{\lambda,\mu}(R)$ then $b \in Z$ by 2.3. Finally we obtain that $b \in Z$ or $\sigma(v)-\tau(v) \in Z, \forall v \in W$. □

Theorem 2.6. *Let W be a left $(\sigma,\tau)-$Jordan ideal of R and $I \neq 0$ an ideal of R and I a nonzero ideal of R.. Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with nonzero $(\alpha,\beta)-$ derivation d and $a, b \in R$.*

(i) If $h(I) \subset C_{\lambda,\mu}(W)$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
(ii) If $h(I)b \subset C_{\lambda,\mu}(W)$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
(iii) If $(h(R), a)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$ then $a^2 \in Z$ or $d(a^2) = 0$.

Proof. (i) If $h(I) \subset C_{\lambda,\mu}(W)$ then $[h(I), W]_{\lambda,\mu} = 0$. This means that, $h(I) \subset C_{\lambda,\mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by 2.2 (ii).

If $h(I) \subset C_{\lambda,\mu}(R)$ then we get R is commutative by [7, Theorem 2.12] and so $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(ii) If $h(I)b \subset C_{\lambda,\mu}(W)$ then $[h(I)b, W]_{\lambda,\mu} = 0$. This gives $h(I)b \subset C_{\lambda,\mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by 2.2 (ii).

If $h(I)b \subset C_{\lambda,\mu}(R)$ then $b \in Z$ by 2.3. Finally we obtain that $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(iii) Using the hypothesis $(h(R), a)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$ we get, for all $r \in R$

$$\begin{aligned} 0 &= [h(r)\alpha(a) + \beta(a)h(r), a]_{\alpha,\beta} \\ &= h(r)\alpha(a)\alpha(a) + \beta(a)h(r)\alpha(a) - \beta(a)h(r)\alpha(a) - \beta(a)\beta(a)h(r) = [h(r), a^2]_{\alpha,\beta}. \end{aligned}$$

That is $[h(R), a^2]_{\alpha,\beta} = 0$. This means that $a^2 \in Z$ or $d(a^2) = 0$ by [7, Lemma 8]. □

Corollary 2.4. *Let W be nonzero left $(\sigma, \tau)-$Jordan ideal of R and $b \in R$. If $(W, b)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$ then $b^2 \in Z$ or $\sigma(w) - \tau(w) \in Z$ for all $w \in W$.*

Proof. For any $w \in W$ let us define the mapping $h(r) = (r, w)_{\sigma,\tau}, \forall r \in R$. Then h is a left-generalized derivation associated with derivation $d(r) = -[r, \tau(w)], \forall r \in R$.

If $(W, b)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$ then we have $((R, w)_{\sigma,\tau}, b)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$ and so $(h(R), b)_{\alpha,\beta} \subset C_{\alpha,\beta}(R)$.

If $h \neq 0$ then we have $b^2 \in Z$ or $d\beta(b^2) = 0$ by 2.6 (iii) and so

$$b^2 \in Z \text{ or } [\beta(b^2), \tau(w)] = 0.$$

If $h = 0$ then $d = 0$ is obtained. This gives that $w \in Z$ and so $[\beta(b^2), \tau(w)] = 0$. If we consider this argument for all $w \in W$ then we get

$$b^2 \in Z \text{ or } [\tau^{-1}\alpha(b^2), W] = 0.$$

If $[\tau^{-1}\alpha(b^2), W] = 0$ then $b^2 \in Z$ or $\sigma(w) - \tau(w) \in Z$ for all $w \in W$ by 2.2 (ii). □

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