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ON SOME NEW INEQUALITIES FOR $s-$ CONVEX FUNCTIONS

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Abstract. In this paper, we establish a few new generalization of Hermite-Hadamard inequality using $s-$convex functions in the 2nd sense. For this purpose we used some special inequalities like Hölder's.

Keywords: Convex Function, $s-$ Convex Functions, Hölder Inequality, Hermite-Hadamrd Inequality

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1. Introductions

Definition 1.1. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R} = [0, \infty)$ is said to be s-convex on I if the inequality,*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y) \quad (1)$$

holds for all $x, y \in I$ and $t \in [0,1]$ with $t + (1-t) = 1$ and for some fixed $s \in (0,1]$. This class of s- convex functions is usually denoted by K_s^2(see:[17]).

It can be easilly that for $s = 1$, s-convexity reduces to ordinary convexity of funtions defined on $[0, \infty)$.

One of the most famous inequality for the class of convex functions is known as Hermite-Hadamard inequality which is,

$f : I \subseteq \mathbb{R} \to \mathbb{R}$ be convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \quad (2)$$

Within the past thirty years, different variants of this kind of inequalities have been obtained. A few of them can be found in the papers ([5]-[28]).

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Theorem 1.1. *Suppose that $f:[0,\infty)\rightarrow[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$ and let $a,b\in[0,\infty)$,$a<b$, $f\in L^1[0,1]$, then the following inequalities hold*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx\leq\frac{f(a)+f(b)}{s+1}. \quad (3)$$

In [8], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s- convex functions.

Theorem 1.2. *Let f be a s- convex in the second sense on $I=[a,b]$ and let $w:[a,b]\rightarrow\mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned}2^{s-1}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx &\leq \int_a^b f(x)w(x)dx\\ &\leq \frac{f(a)+f(b)}{2}\int_a^b\left[\left(\frac{b-x}{b-a}\right)^s+\left(\frac{x-a}{b-a}\right)^s\right]w(x)dx \quad (4)\end{aligned}$$

see:([18]*).*

Theorem 1.3. *Let $f,\ w:[a,b]\rightarrow\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that w and f are in $L^1([a,b])$. If f is s-convex in the second sense and nonnegative on $[a,b]$ for some fixed $s\in(0,1)$, Then for all $t\in[0,1]$, we have,*

$$\begin{aligned}2f\left(\tfrac{a+b}{2}\right)w\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{b-a}\int_a^b f(x)w(x)dx\\ &+\tfrac{1}{(s+1)(s+2)}M(a,b)+\tfrac{1}{(s+2)}N(a,b)\end{aligned} \quad (5)$$

where

$$\begin{aligned}M(a,b)&=f(a)w(a)+f(b)w(b)\\ N(a,b)&=f(a)w(b)+f(b)w(a)\end{aligned} \quad (6)$$

see:([19]).

2. Hermite- Hadamard Type Inequality for s -Convex Functions

Theorem 2.1. *Let $f,w:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a s-convex in the second sense and nonnegative function on $I=[a,b]$. If w is symmetric about $\frac{a+b}{2}$ then for all $t\in[0,1]$, we have*

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx\leq\frac{s!s!}{(2s+1)!}M(a,b)+\frac{1}{2s+1}N(a,b) \quad (7)$$

where $M(a,b)$ and $N(a,b)$ are given by (6).

Proof. Since w is symmetric about $\frac{a+b}{2}$ and f,w be s-convex functions in the second sense and then $a+b-x=x$ we have

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx=\frac{1}{b-a}\int_a^b f(x)w(a+b-x)dx$$

So $x = ta + (1-t)b$ and $dx = (a-b)dt \Longleftrightarrow dt = \frac{dx}{a-b}$. By integrating limit values $t \to 1$ and $t \to 0$. Therefore, we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)w(a+b-x)dx &= \int_0^1 f(ta+(1-t)b)w(a+b-(ta+(1-t)b))dt \\ &= \int_0^1 f(ta+(1-t)b)w((1-t)a+tb)dt\end{aligned}$$

Since f and w are s-convex functions in the second sense, we have

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)w((1-t)a+tb)dt &\leq \int_0^1 \left[t^s f(a) + (1-t)^s f(b)\right]\left[(1-t)^s w(a) + t^s w(b)\right] dt \\ &= \left\{\int_0^1 t^s(1-t)^s f(a)w(a) + t^{2s} f(a)w(b)\right. \\ &\left. + \int_0^1 (1-t)^{2s} f(b)w(a) + t^s(1-t)^s f(b)w(b)dt\right\} \\ &= \left\{\int_0^1 t^s(1-t)^s \left[f(a)w(a) + f(b)w(b)\right] dt\right. \\ &\left. + \int_0^1 t^{2s} f(a)w(b)dt + (1-t)^{2s} f(b)w(a)dt\right\}\end{aligned}$$

By using the fact that $\int_0^1 t^s(1-t)^s dt = \beta(s+1,s+1)$ and therefore,

$$\begin{aligned}&\int_0^1 t^s(1-t)^s \left[f(a)w(a) + f(b)w(b)\right] dt + \int_0^1 t^{2s} f(a)w(b)dt + (1-t)^{2s} f(b)w(a)dt \\ = \;& \beta(s+1,s+1)\left[f(a)w(a) + f(b)w(b)\right] + \left.\frac{t^{2s+1}}{2s+1}\right|_0^1 f(a)w(b) + -\left.\frac{(1-t)^{2s+1}}{2s+1}\right|_0^1 f(b)w(a)\end{aligned}$$

Using Beta function, $\beta(s+1,s+1) = \frac{\Gamma(s+1)\Gamma(s+1)}{\Gamma(2s+2)} = \frac{s!s!}{(2s+1)!}$

$$\begin{aligned}&= \frac{\Gamma(s+1)\Gamma(s+1)}{\Gamma(2s+2)}\left[f(a)w(a) + f(b)w(b)\right] + \frac{1}{2s+1} f(a)w(b) + \frac{1}{2s+1} f(b)w(a) \\ &= \frac{s!s!}{(2s+1)!}\left[f(a)w(a) + f(b)w(b)\right] + \frac{1}{2s+1}\left[f(a)w(b) + f(b)w(a)\right] \\ &= \frac{s!s!}{(2s+1)!} M(a,b) + \frac{1}{2s+1} N(a,b)\end{aligned}$$

which completes the proof. □

Remark 2.1. *If we take $s = 1$ and for all $x \in [a,b]$ in Theorem 1.4, the inequality (7) reduce to inequality*

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx \leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

which is proved by Pachpatte in [20].

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I°(the interior I) If $f' \in L_1[a,b]$ for $a, b \in I$*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = (b-a)\int_0^1 p(t) f'(ta+(1-t)b)\,dt \tag{8}$$

where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t-1, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Proof. Proved by Kirmaci [3]. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I°(I interval) and $f' \in L_1[a,b]$ for $a, b \in I$. If $|f'|$ is the s- convex in the second sense on $[a,b]$, then following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le (b-a)\left\{\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\left[\frac{2^{s+1}-1}{(s+1)(s+2)2^{s+1}}\right]\right\} \tag{9}$$

Proof. From Lemma 2.1 and s-convexity in the second sense of $|f'|$ function , we obtained

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,d(x) - f\left(\frac{a+b}{2}\right)\right| &\le (b-a)\int_0^1 |p(t)|\left|f'(ta+(1-t)b)\right| dt \\ &= (b-a)\left\{\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right| dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 |t-1|\left|f' ta+(1-t)b\right| dt\right\}\end{aligned}$$

$$\begin{aligned}&\le (b-a)\left\{\int_0^{\frac{1}{2}} t\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right] dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 (1-t)\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right] dt\right\} \\ &= (b-a)\left\{\int_0^{\frac{1}{2}} \left\{tt^s\left|f'(a)\right| + t(1-t)^s\left|f'(b)\right|\right\} dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 \left\{(1-t)t^s\left|f'(a)\right| + (1-t)(1-t)^s\left|f'(b)\right|\right\} dt\right\}\end{aligned}$$

If we change the variable with $(1-t)=u$ then right hand side of the last inequality.

$$\begin{aligned}
&= (b-a)\left\{ \left|f'(a)\right| \int_0^{\frac{1}{2}} t^{s+1}dt - \left|f'(b)\right| \int_{\frac{1}{2}}^{1} (1-u)\, u^s du \right. \\
&\quad \left. + \left|f'(a)\right| \int_{\frac{1}{2}}^{1} (1-t)t^s + \left|f'(b)\right| \int_{\frac{1}{2}}^{1} (1-t)^{s+1}dt \right\} \\
&= (b-a)\left\{ \left|f'(a)\right| \left| \left(\frac{t^{s+2}}{s+2} \right)_0^{\frac{1}{2}} - \left|f'(b)\right| \right| \left(\frac{u^{s+1}}{s+1} - \frac{u^{s+2}}{s+2} \right)_1^{\frac{1}{2}} \right. \\
&\quad \left. + \left|f'(a)\right| \left| \left(-\frac{t^{s+2}}{s+2} + \frac{t^{s+1}}{s+1} \right)_{\frac{1}{2}}^{1} + \left|f'(b)\right| \right| \left(-\frac{(1-t)^{s+2}}{s+2} \right)_{\frac{1}{2}}^{1} \right\}
\end{aligned}$$

$$\begin{aligned}
&= (b-a)\left\{ \left|f'(a)\right| \left(\frac{2}{2^{s+2}(s+2)} + \frac{-s-1+s+2}{(s+1)(s+2)} - \frac{1}{2^{s+1}(s+1)} \right) \right. \\
&\quad \left. + \left|f'(b)\right| \left(\frac{2}{2^{s+2}(s+2)} + \frac{s+2-s-1}{(s+1)(s+2)} - \frac{1}{2^{s+1}(s+1)} \right) \right\} \\
&= (b-a)\left\{ \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \right. \\
&\quad \times \left[\frac{2^{-(s+1)}2^{s+1}(s+1)}{(s+1)(s+2)2^{s+1}} + \frac{2^{s+1}}{(s+1)(s+2)2^{s+1}} + \frac{-s-2}{(s+1)(s+2)2^{s+1}} \right] \\
&= (b-a)\left\{ \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \left[\frac{2^{s+1}-1}{(s+1)(s+2)2^{s+1}} \right] \right\}
\end{aligned}$$

So the theroem is proved. □

Remark 2.2. *If we take $s=1$ and for all $x \in [a,b]$ in Theorem 2.2., the inequality (9) reduce to inequality.(see:* [3]*)*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2} \right) \right| \leq \frac{(b-a)}{8} \left\{ \left|f'(a)\right| + \left|f'(b)\right| \right\}$$

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°(the interior I) and $f' \in L_1[a,b]$ for $a, b \in I$.If $|f'|^q$ is s- convex in the second sense on $[a,b]$, $q>1$ then the folloving inequalities hold:*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \qquad (10) \\
&\leq (b-a) \left(\tfrac{2^{-p-1}}{(p+1)} \right)^{1/p} \left(\tfrac{1}{sq+1} \right)^{1/q} \{ [|f'(a)|] + [|f'(b)|] \}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, using Hölder's inequality and s-convex in the second sense of $|f'|$ functions , we obtained

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
=\ & \left|(b-a)\int_0^1 p(t)f'(ta+(1-t)b)\,dt\right| \\
=\ & \left|(b-a)\left\{\int_0^{1/2} tf'(ta+(1-t)b)\,dt + \int_{1/2}^1 (t-1)\,f'(ta+(1-t)b)\,dt\right\}\right| \\
\leq\ & |(b-a)|\left\{\int_0^{1/2} \left|tf'(ta+(1-t)b)\right|dt + \int_{1/2}^1 \left|\,(t-1)\,f'(ta+(1-t)b)\right|dt\right\}
\end{aligned}$$

and then using Hölder's inequality,

$$\begin{aligned}
\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq\ & |(b-a)|\ \left\{\int_0^{1/2} \left|tf'(ta+(1-t)b)\right|dt\right. \\
& \left. + \int_{1/2}^1 \left|\,(t-1)\,f'(ta+(1-t)b)\right|dt\right\} \\
\leq\ & (b-a)\left\{\left(\int_0^{1/2} t^p dt\right)^{1/p}\right. \\
& \times \left(\int_0^{1/2} \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
& + \left(\int_{1/2}^1 |t-1|^p\,dt\right)^{1/p} \\
& \left. \times \left(\int_{1/2}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}\right\}
\end{aligned}$$

furthermore,

$$\begin{aligned} I_1 &= \left(\int_0^{1/2} \left| f'\left(ta+(1-t)b\right) \right|^q dt \right)^{1/q} \\ I_2 &= \left(\int_{1/2}^{1} \left| f'\left(ta+(1-t)b\right) \right|^q dt \right)^{1/q} \end{aligned}$$

If we take it as, Using to s-convex in the second sense of $|f'|$ functions and $\sum_{k=1}^{n}(a_k+b_k)^r \leq \sum_{k=1}^{n} {a_k}^r + \sum_{k=1}^{n} {b_k}^r$,

$$\begin{aligned} I_1 &= \left(\int_0^{1/2} \left| f'\left(ta+(1-t)b\right) \right|^q dt \right)^{1/q} \leq \left(\int_0^{1/2} \left[t^s \left| f'(a) \right| + (1-t)^s \left| f'(b) \right| \right]^q dt \right)^{1/q} \\ &\leq \left(\int_0^{1/2} \left[t^s \left| f'(a) \right| \right]^q dt + \int_0^{1/2} \left[(1-t)^s \left| f'(b) \right| \right]^q dt \right)^{1/q} \end{aligned}$$

$$\begin{aligned} &= \left(\left| f'(a) \right|^q \int_0^{1/2} \left[t^{sq} \right] dt + \left| f'(b) \right|^q \int_0^{1/2} \left[(1-t)^{sq} \right] dt \right)^{1/q} \\ &= \left(\left| f'(a) \right|^q \left. \frac{t^{sq+1}}{sq+1} \right|_0^{1/2} + \left| f'(b) \right|^q \left. -\frac{(1-t)^{sq+1}}{sq+1} \right|_0^{1/2} \right)^{1/q} \\ &= \left(\frac{1}{sq+1} \right)^{1/q} \left(\left[2^{-sq-1} \left| f'(a) \right|^q \right] + \left[\left(1-2^{-sq-1} \right) \left| f'(b) \right| \right] \right)^{1/q} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left(\int_{1/2}^{1} \left| f'\left(ta+(1-t)b\right) \right|^q dt \right)^{1/q} \leq \left(\int_{1/2}^{1} \left[t^s \left| f'(a) \right| + (1-t)^s \left| f'(b) \right| \right]^q dt \right)^{1/q} \\ &\leq \left(\int_{1/2}^{1} \left[t^s \left| f'(a) \right| \right]^q dt + \int_{1/2}^{1} \left[(1-t)^s \left| f'(b) \right| \right]^q dt \right)^{1/q} \end{aligned}$$

$$
\begin{aligned}
&= \left(\left|f'(a)\right|^q \int_{1/2}^{1} \left[t^{sq}\right] dt + \left|f'(b)\right|^q \int_{1/2}^{1} \left[(1-t)^{sq}\right] dt \right)^{1/q} \\
&= \left(\left|f'(a)\right|^q \left(\left.\frac{t^{sq+1}}{sq+1}\right|_{1/2}^{1} \right) + \left|f'(b)\right|^q \left(- \left.\frac{(1-t)^{sq+1}}{sq+1}\right|_{1/2}^{1} \right) \right)^{1/q} \\
&= \left(\frac{1}{sq+1}\right)^{1/q} \left(\left[\left(1-2^{-sq-1}\right)\left|f'(a)\right|^q\right] + \left[\left(2^{-sq-1}\right)\left|f'(b)\right|^q\right] \right)^{1/q}
\end{aligned}
$$

and

$$
\begin{aligned}
[I_1 + I_2] &= \left(\frac{1}{sq+1}\right)^{1/q} \left(\left[2^{-sq-1}\left|f'(a)\right|^q\right] + \left[\left(1-2^{-sq-1}\right)\left|f'(b)\right|\right] \right)^{1/q} \\
&\quad + \left(\frac{1}{sq+1}\right)^{1/q} \left(\left[\left(1-2^{-sq-1}\right)\left|f'(a)\right|^q\right] + \left[\left(2^{-sq-1}\right)\left|f'(b)\right|^q\right] \right)^{1/q} \\
&= \left(\frac{1}{sq+1}\right)^{1/q} \Big\{ \left(\left[2^{-sq-1}\left|f'(a)\right|^q\right] + \left[\left(1-2^{-sq-1}\right)\left|f'(b)\right|\right] \right)^{1/q} \\
&\quad + \left(\left[\left(1-2^{-sq-1}\right)\left|f'(a)\right|^q\right] + \left[\left(2^{-sq-1}\right)\left|f'(b)\right|^q\right] \right)^{1/q} \Big\} \\
&\leq \left(\frac{1}{sq+1}\right)^{1/q} \Big\{ \left(\left[2^{-sq-1}\left|f'(a)\right|^q\right]^{1/q} + \left[\left(1-2^{-sq-1}\right)\left|f'(b)\right|\right]^{1/q} \right) \\
&\quad + \left(\left[\left(1-2^{-sq-1}\right)\left|f'(a)\right|^q\right]^{1/q} + \left[\left(2^{-sq-1}\right)\left|f'(b)\right|^q\right]^{1/q} \right) \Big\}
\end{aligned}
$$

and then

$$
\begin{aligned}
\left(\int_0^{1/2} t^p dt \right)^{1/p} &= \left(\frac{2^{-p-1}}{(p+1)} \right)^{1/p}, \\
\left(\int_{1/2}^{1} |t-1|^p \, dt \right)^{1/p} &= \left(\int_{1/2}^{1} (1-t)^p \, dt \right)^{1/p} = \left(\frac{2^{-p-1}}{(p+1)} \right)^{1/p}
\end{aligned}
$$

as it can be calculated as

$$
\begin{aligned}
&(b-a) \left\{ \left(\int_0^{1/2} t^p dt \right)^{1/p} I_1 + \left(\int_{1/2}^{1} |t-1|^p \, dt \right)^{1/p} I_2 \right\} \\
&= (b-a) \left\{ \left(\frac{2^{-p-1}}{(p+1)} \right)^{1/p} I_1 + \left(\frac{2^{-p-1}}{(p+1)} \right)^{1/p} I_2 \right\} \\
&= (b-a) \left\{ \left(\frac{2^{-p-1}}{(p+1)} \right)^{1/p} [I_1 + I_2] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (b-a)\left(\frac{2^{-p-1}}{(p+1)}\right)^{1/p}\left(\frac{1}{sq+1}\right)^{1/q} \\
&\left\{\left(\left[2^{-sq-1}\left|f'(a)\right|^{q}\right]^{1/q}+\left[\left(1-2^{-sq-1}\right)\left|f'(b)\right|\right]^{1/q}\right)\right. \\
&\left.+\left(\left[\left(1-2^{-sq-1}\right)\left|f'(a)\right|^{q}\right]^{1/q}+\left[\left(2^{-sq-1}\right)\left|f'(b)\right|^{q}\right]^{1/q}\right)\right\} \\
&= (b-a)\left(\frac{2^{-p-1}}{(p+1)}\right)^{1/p}\left(\frac{1}{sq+1}\right)^{1/q}\left\{\left[\left|f'(a)\right|\right]+\left[\left|f'(b)\right|\right]\right\}
\end{aligned}
$$

This proof is completed. □

Remark 2.3. *If we take $s = 1$ and for all $x \in [a, b]$ in Theorem 5, the inequality (10) reduce to inequality.(see:* [3]*)*

Lemma 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°(the interior I).If $f' \in L_1[a, b]$ for $a, b \in I$, then the following equality holds:*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_{a}^{b} f(x)\,dx \\
&= \frac{b-a}{2}\int_{0}^{1}(2t-1)\left[f'(tb+(1-t)a)\right]dt. \qquad (11)
\end{aligned}
$$

Proof. Proved by Dragomir and Agarwal in [4]. □

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I°(the interior I) and $|f'| \in L_1\ [a, b]$ for $a, b \in I$, then $|f'|$ is the s-convex in the second sense on $[a, b]$, thenthe following inequality holds;*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b} f(x)\,dx\right| \\
&\leq \frac{b-a}{2}\left(\frac{2^{-s}+s}{s^{2}+3s+2}\right)\left[s\left|f'(b)\right|+s\left|f'(a)\right|\right] \qquad (12)
\end{aligned}
$$

Proof. From Lemma 2.2 and by using s-convexity function of $|f'|$,we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{2}\int_0^1 |2t-1|\left|f'(tb+(1-t)a)\right|dt$$

$$\leq \frac{b-a}{2}\int_0^1 |2t-1|\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right]dt$$

$$= \frac{b-a}{2}\left[\int_0^{\frac{1}{2}} -(2t-1)\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right]dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1 (2t-1)\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'(a)\right|\right]dt\right]$$

$$= \frac{b-a}{2}\left[-\left(\left|f'(b)\right|\frac{2^{-(s+1)}}{s^2+3s+2}-\left|f'(a)\right|\frac{2^{-(s+1)}+s}{s^2+3s+2}\right)\right.$$

$$\left.+\left(\left|f'(b)\right|\frac{2^{-(s+1)}+s}{s^2+3s+2}+\left|f'(a)\right|\frac{2^{-(s+1)}}{s^2+3s+2}\right)\right]$$

$$= \frac{b-a}{2}\left[\left|f'(b)\right|\left(\frac{2^{-(s+1)}}{s^2+3s+2}+\frac{2^{-(s+1)}+s}{s^2+3s+2}\right)\right.$$

$$\left.+\left|f'(a)\right|\left(\frac{2^{-(s+1)}+s}{s^2+3s+2}+\frac{2^{-(s+1)}}{s^2+3s+2}\right)\right]$$

$$= \frac{b-a}{2}\left(\frac{2^{-s}+s}{s^2+3s+2}\right)\left[s\left|f'(b)\right|+s\left|f'(a)\right|\right]$$

which completes the proof. □

Remark 2.4. *If we take $s=1$ and for all $x\in[a,b]$, then inequality (12) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in ([4])*

Theorem 2.5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°(the interior I) and $|f'|^q$ is the s-convex in the second sense on $[a,b]$. $q>1$, the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{13}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.2 by using Hölder's inetgral inequality and s-convex in the second sense of $|f'|$ functions , we heve

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\int_0^1 |2t-1|^p\,dt\right)^{\frac{1}{p}} \times\left(\int_0^1 \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

obtained. And then since $|f'|$ is s-convex in the second sense function,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left[t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\ &= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left.\frac{t^{s+1}}{s+1}\right|_0^1\left|f'(b)\right|^q-\left.\frac{(1-t)^{s+1}}{s+1}\right|_0^1\left|f'(a)\right|^q\right]^{\frac{1}{q}} \\ &= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(b)|^q+|f'(a)|^q}{s+1}\right]^{\frac{1}{q}}\end{aligned}$$

which completes the proof. □

Remark 2.5. *If we take $s=1$ and for all $x\in[a,b]$, then inequality (13) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in* [4].

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