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AN APPLICATION OF FACTORABLE SURFACES IN EUCLIDEAN $4-$SPACE $\mathbb{E}^4$

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Abstract. In the present paper, we consider the factorable surfaces in Euclidean 4-space $\mathbb{E}^4$. We characterize such surfaces in terms of their Gaussian curvature, Gaussian torsion and mean curvature. Further, we classify flat, semiumbilical and minimal factorable surfaces in $\mathbb{E}^4$.

Factorable surface, Euclidean 4-space, monge patch, minimal surface.

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1. Introduction

Let M be a smooth surface given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$ in $\mathbb{E}^4$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M is spaned $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, \ F = \langle X_u, X_v \rangle, \ G = \langle X_v, X_v \rangle,$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^4$.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the spaces of smooth vector fields tangent to M and normal to M, respectively. Given any local vector fields X_1, X_2 tangent to M, consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i,j \leq 2. \tag{1}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^4$, respectively. This map is well-defined, symmetric and bilinear.

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The equation (1) is called Gaussian formula, and the following equation

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \qquad 1 \le i, j \le 2.$$

is satisfied for any arbitrary orthonormal frame field $\{N_1, N_2\}$ of M, where c_{ij}^k are the coefficients of the second fundamental form [5].

The Gaussian curvature and Gaussian torsion of a regular patch $X(u, v)$ are given by

$$K = \frac{1}{W^2} \sum_{k=1}^{2} \left(c_{11}^k c_{22}^k - \left(c_{12}^k \right)^2 \right) \tag{2}$$

and

$$K_N = \frac{1}{W^2} \left(E \left(c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1 \right) - F \left(c_{11}^1 c_{22}^2 - c_{11}^2 c_{22}^1 \right) + G \left(c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1 \right) \right), \tag{3}$$

respectively.

Further, the mean curvature vector of a regular patch $X(u, v)$ is given by

$$\vec{H} = \frac{1}{2W^2} \sum_{k=1}^{2} \left(c_{11}^k G + c_{22}^k E - 2c_{12}^k F \right) N_k. \tag{4}$$

The norm of the mean curvature vector $\left\| \vec{H} \right\|$ is called the mean curvature of M.

A surface M is said to be flat (minimal) if its Gauss curvature (mean curvature vector) vanishes identically [5]. In addition, a point $p \in M$ is semiumbilic if and only if $K_N(p) = 0$ and a surface M immersed in $\mathbb{E}^4$ is said to be semiumbilical provided all its points are semiumbilic [6].

A factorable surfaces (also known homotethical surfaces) in $\mathbb{E}^3$, which can be parametrized, locally, as $X(u, v) = (u, v, f(u)g(v))$, where f and g are smooth functions [8]. Some authors have considered factorable surfaces in Euclidean space and in semi-Euclidean spaces [4, 7, 9]. In [8], Van de Woestyne proved that the only minimal factorable non-degenerate surfaces in $\mathbb{L}^3$ are planes and helicoids.

Many studies can be found about surfaces in 4- dimensional Euclidean space $\mathbb{E}^4$ (see, [1, 2, 3] .

In this work, we consider a factorable surface in Euclidean 4-space. We define the surface which locally can be written as a monge patch

$$X(u, v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v)),$$

for some differentiable functions, $f_i(u), g_i(v),\ i = 1, 2$. We characterize such surfaces in terms of their Gaussian curvature, Gaussian torsion and mean curvature functions and give the conditions for such surfaces to become flat, semiumbilical, and minimal in $\mathbb{E}^4$.

2. An Application of Factorable Surfaces

In [2], the authors studied the surfaces given with the representation of the form

$$X(u, v) = (u, v, z(u, v), w(u, v)), \tag{5}$$

where z and w are some smooth functions. The parametrization (5) is called a Monge patch in $\mathbb{E}^4$. Now we define the factorable surface in $\mathbb{E}^4$ as follows:

Definition 2.1. *Let M be a surface in four dimensional Euclidean surface $\mathbb{E}^4$. If the surface is denoted by $z(u,v) = f_1(u)g_1(v)$ and $w(u,v) = f_2(u)g_2(v)$ in (5) where f_1, f_2, g_1, g_2 are differentiable functions, then the surface is called a factorable surface in $\mathbb{E}^4$. Thus, the factorable surface is defined as a monge patch*

$$X(u,v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v)). \tag{6}$$

In [4], some calculations can be found about tangent vectors, normal vectors, first and second fundamental form coefficients of the surface M. Hence, for classification of semiumbilical, flat and minimal surfaces, we use Gaussian torsion, Gaussian curvature and mean curvature functions.

Theorem 2.1. [4]*Let M be a factorable surface in $\mathbb{E}^4$. Then the Gaussian curvature is given by*

$$K = \frac{\left(f_1'' f_1 g_1'' g_1 - f_1'^2 g_1'^2\right)\widetilde{G} - \left(f_1'' f_2 g_1 g_2'' + f_1 f_2'' g_1'' g_2 - 2 f_1' f_2' g_1' g_2'\right)\widetilde{F} + \left(f_2'' f_2 g_2'' g_2 - f_2'^2 g_2'^2\right)\widetilde{E}}{W^4},$$

where $\widetilde{E} = 1 + \left(f_1' g_1\right)^2 + \left(f_1 g_1'\right)^2$, $\widetilde{F} = f_1' f_2' g_1 g_2 + f_1 f_2 g_1' g_2'$, *and* $\widetilde{G} = 1 + \left(f_2' g_2\right)^2 + \left(f_2 g_2'\right)^2$.

Theorem 2.2. *Let M be a factorable surface in $\mathbb{E}^4$. If M has one of the following parametrizations in $\mathbb{E}^4$, then it is flat:*

$$\begin{aligned}
(i) X(u,v) &= (u, v, c_1 g_1(v), c_2 g_2(v)),\\
(ii) X(u,v) &= (u, v, c_1 f_1(u), c_2 f_2(u)),\\
(iii) X(u,v) &= (u, v, c_1 g_1(v), c_2 f_2(u)),\\
(iv)\ X(u,v) &= (u, v, c_1 f_1(u), c_2 g_2(v)),\\
(v)\ X(u,v) &= (u, v, c, \exp(c_1 u + d_1)\exp(c_2 v + d_2)),\\
(vi)\ X(u,v) &= \left(u, v, c, (c_1 u + d_1)^{\frac{1}{1-l_1}} (c_2 v + d_2)^{\frac{l_1}{l_1 - 1}}\right),\\
(vii)\ X(u,v) &= \left(u, v, \exp(c_1 u + d_1)\exp(c_2 v + d_2), \exp(c_3 u + d_3)\exp(c_3 \frac{c_i}{c_j} v + d_4)\right),\\
(viii)\ X(u,v) &= (u, v, r(u)\cos v, r(u)\sin v),
\end{aligned}$$

the function $r(u)$ satisfies

$$u = \pm \int \sqrt{\frac{c_1 r^2(u) - 1}{r^2(u) + 1}}\, dr(u)$$

where $i, j = 1, 2$, $i \neq j$ and c_k, d_k, $k = 1, ..., 4$ are real constants.

Proof. Let M be a factorable surface given with the parametrization (6) in $\mathbb{E}^4$.

If $f_1'(u) = 0$, $f_2'(u) = 0$ or $g_1'(v) = 0$, $g_2'(v) = 0$ or $f_1'(u) = 0$, $g_2'(v) = 0$ or $g_1'(v) = 0$, $f_2'(u) = 0$, then we obtain the cases (i), (ii), (iii) and (iv), respectively.

If $f_1'(u) = 0$, $g_1'(v) = 0$, then we have

$$f_2'' f_2 g_2'' g_2 - f_2'^2 g_2'^2 = 0. \tag{7}$$

This differential equation has the solutions

$$\begin{aligned}
f_2(u) &= \exp(c_1 u + d_1), \\
g_2(v) &= \exp(c_2 v + d_2),
\end{aligned} \tag{8}$$

and

$$f_2(u) = (c_1u+d_1)^{\frac{1}{1-l_1}}, \qquad (9)$$
$$g_2(v) = (c_2v+d_2)^{\frac{l_1}{l_1-1}},$$

which gives the cases (v) and (vi).

Further, with the help of Gaussian curvature in Theorem 2.1, we can suppose the cases

$$f_1''f_1g_1''g_1 - f_1'^2g_1'^2 = 0, \quad f_2''f_2g_2''g_2 - f_2'^2g_2'^2 = 0, \qquad (10)$$

$$\widetilde{F} = 0 \qquad (11)$$

and

$$f_1''f_1g_1''g_1 - f_1'^2g_1'^2 = 0, \quad f_2''f_2g_2''g_2 - f_2'^2g_2'^2 = 0,$$

$$f_1''f_2g_1g_2'' + f_1f_2''g_1''g_2 - 2f_1'f_2'g_1'g_2' = 0, \qquad (12)$$

where $\widetilde{E} \neq 0$ and $\widetilde{G} \neq 0$. Hence the equations (10) are congruent to equation (7). Therefore, substituting

$$\begin{array}{ll} f_1(u) = \exp(c_1u+d_1), & f_2(u) = \exp(c_3u+d_3), \\ g_1(v) = \exp(c_2v+d_2), & g_2(v) = \exp(c_4v+d_4), \end{array} \qquad (13)$$

into (11) and (12), we obtain the case(vii).

On the other hand, if we suppose $f_1(u) = f_2(u) = r(u)$ and $g_1(v) = \cos v$, $g_2(v) = \sin v$, then by vanishing Gaussian curvature, we get

$$r''(u)r(u)(1+(r(u))^2) + (r'(u))^2(1+(r'(u))^2) = 0.$$

As a result of this equation, we have a solution. Thus, we get the case (viii). □

Theorem 2.3. [4]*Let M be a factorable surface in $\mathbb{E}^4$. Then the Gaussian torsion is given by*

$$K_N = \frac{E\,(f_1'f_2g_1'g_2'' - f_1f_2'g_1''g_2') - F\,(f_1''f_2g_1g_2'' - f_1f_2''g_1''g_2) + G\,(f_1''f_2'g_1g_2' - f_1'f_2''g_1'g_2)}{W^4}. \qquad (14)$$

where $E = 1+(f_1'g_1)^2+(f_2'g_2)^2$, $F = f_1'f_1g_1'g_1 + f_2'f_2g_2'g_2$, $G = 1+(f_1g_1')^2+(f_2g_2')^2$ are the first fundamental form coefficients of the surface M.

Corollary 2.1. *Let M be a factorable surface with the parametrization (6) in $\mathbb{E}^4$. If the functions $f_1(u), f_2(u), g_1(v)$ and $g_2(v)$ are linear polinomials, then M is a semiumbilical surface.*

Proposition 2.1. *Let M be a factorable surface with the parametrization (6) in $\mathbb{E}^4$. If the functions $f_1(u), f_2(u), g_1(v)$ and $g_2(v)$ satisfy the equations*

$$\begin{array}{l} f_2'(u) = f_1(u), \\ g_2'(v) = g_1(v), \end{array} \qquad (15)$$

then the Gaussian curvature K coincides with the Gaussian torsion K_N.

Proof. Let M be a factorable surface with the parametrization (6) in $\mathbb{E}^4$. Suppose that, the equation (15) is satisfied, then we get $E = \widetilde{E}$, $F = \widetilde{F}$, $G = \widetilde{G}$. Further, by the use of Theorem 2.1 and Theorem 2.3, we obtain $K = K_N$. This completes the proof. □

Example 2.1. *For the surface given with the parametrization*

$$M_1 : X(u,v) = \left(u, v, \frac{-1}{u}\sin v, \ln u \cos v\right), \qquad (u \neq 0) \tag{16}$$

Gaussian curvature K coincides with the Gaussian torsion K_N.

Theorem 2.4. [4]*Let M be a factorable surface in $\mathbb{E}^4$. Then the mean curvature vector is given by*

$$\begin{aligned} \vec{H} &= \frac{f_1''g_1G + f_1g_1''E - 2f_1'g_1'F}{2\sqrt{\widetilde{E}}W^2}\vec{N_1} \\ &+ \frac{\widetilde{E}\left(f_2''g_2G + f_2g_2''E - 2f_2'g_2'\right) - \widetilde{F}\left(f_1''g_1G + f_1g_1''E - 2f_1'g_1'\right)}{2\sqrt{\widetilde{E}}W^3}\vec{N_2}. \end{aligned}$$

Theorem 2.5. *Let M be a factorable surface in $\mathbb{E}^4$. Then M is a minimal surface if and only if*

$$f_i''g_iG + f_ig_i''E - 2f_i'g_i'F = 0, \qquad i = 1, 2. \tag{17}$$

Proof. Let M be a factorable surface in $4-$dimensional Euclidean space $\mathbb{E}^4$. If the surface is minimal then by the use of the previous theorem the mean curvature vector $\vec{H}$ vanishes. Since the mean curvature vector can be written as $\vec{H} = H_1\ \vec{N_1} + H_2\ \vec{N_2}$, then we have $H_1 = H_2 = 0$. Thus, we get the equation (17). The converse statement is trivial. □

Theorem 2.6. *Let M be a factorable surface in $\mathbb{E}^4$. If M has one of the following parametrizations in $\mathbb{E}^4$, then it is minimal:*

$$\begin{aligned}
\text{(i) } X(u,v) &= \left(u, v, (c_1u + c_2)\,d_1, (c_3u + c_4)\,d_2\right), \\
\text{(ii) } X(u,v) &= \left(u, v, c_1\left(d_1v + d_2\right), c_2\left(d_3v + d_4\right)\right), \\
\text{(iii) } X(u,v) &= \left(u, v, (c_1u + c_2)\,d_1, c_3\left(d_3v + d_4\right)\right), \\
\text{(iv) } X(u,v) &= \left(u, v, c, (u + d_1)\tan(c_2v + d_2)\right), \\
\text{(v) } X(u,v) &= \left(u, v, c, \tan(c_1u + d_1)\,(v + d_2)\right), \\
\text{(vi) } X(u,v) &= \left(u, v, r(u)\cos v, r(u)\sin v\right) : \\
r(u) &= \frac{1}{2c_1}\left(c_1^2 e^{\pm\frac{2(u+c_2)}{c_1}} + c_1^2 - 1\right)e^{\pm\frac{(u+c_2)}{c_1}}, \\
\text{(vii) } X(u,v) &= \left(u, v, (u + d_1)\tan(c_2v + d_2), (u + d_1)\tan(c_2v + d_2)\right), \\
\text{(viii) } X(u,v) &= \left(u, v, \tan(c_1u + d_1)\,(v + d_2), \tan(c_1u + d_1)\,(v + d_2)\right), \\
\text{(ix) } X(u,v) &= \left(u, v, c, f_2(u)g_2(v)\right), \\
\text{(x) } X(u,v) &= \left(u, v, f_1(u)g_1(v), f_1(u)g_1(v)\right),
\end{aligned}$$

the functions $f_i(u)$ and $g_i(v)$ satisfy the equations $(i = 1, 2)$

$$\begin{aligned}
u &= \int \frac{df_i(u)}{\sqrt{2k\ln f_i(u) + c_1}}, v = \int \frac{dg_i(v)}{\sqrt{c_2 g_i^4(v) - \frac{m}{2}}}, \\
u &= \int \frac{df_i(u)}{\sqrt{c_2 f_i^4(u) - \frac{k}{2}}},\ v = \int \frac{dg_i(v)}{\sqrt{2m\ln g_i(v) + c_2}}, \\
u &= \int \frac{df_i(u)}{\sqrt{c_1 f_i^{2(1+c)}(u) - c_2}},\ v = \int \frac{dg_i(v)}{\sqrt{c_3 g_i^{2(1-c)}(v) - c_4}},
\end{aligned}$$

where $k, m, c, c_1, c_2, c_3, c_4$ are real constants.

Proof. Let M be a factorable surface with the parametrization (6) in $\mathbb{E}^4$. By the use of (17) with first fundamental coefficients, we get,

$$f_1''g_1\left(1+f_1'^2g_1^2+f_2'^2g_2^2\right)+f_1g_1''\left(1+f_1^2g_1'^2+f_2^2g_2'^2\right)-2f_1'g_1'\left(f_1'f_1g_1'g_1+f_2'f_2g_2'g_2\right)=0, \quad (18)$$

$$f_2''g_2\left(1+f_1'^2g_1^2+f_2'^2g_2^2\right)+f_2g_2''\left(1+f_1^2g_1'^2+f_2^2g_2'^2\right)-2f_2'g_2'\left(f_1'f_1g_1'g_1+f_2'f_2g_2'g_2\right)=0. \quad (19)$$

If $g_1'(u)=0$, $g_2'(u)=0$ or $f_1'(u)=0$, $f_2'(u)=0$ we obtain the cases (i) and (ii), respectively.

If $f_2'(u)=0$, $g_1'(v)=0$, we obtain the case (iii).

If $f_1'(u)=0$, $g_1'(v)=0$, the equality (18) holds and from (19), we get

$$\frac{f_2''(u)}{f_2(u)}+\frac{g_2''(v)}{g_2(v)}+\left(f_2''(u)f_2(u)-f_2'^2(u)\right)g_2'^2(v)+\left(g_2''(v)g_2(v)-g_2'^2(v)\right)f_2'^2(u)=0. \quad (20)$$

If $f_2''(u)=0$ or $g_2''(v)=0$ in (20), we obtain the cases (iv) and (v).

If $f_2''(u)g_2''(v)\neq 0$ in (20), differentiating (20) with respect to u and v, we have

$$\frac{\left(f_2''(u)f_2(u)-f_2'^2(u)\right)'}{\left(f_2'^2(u)\right)'}=-\frac{\left(g_2''(v)g_2(v)-g_2'^2(v)\right)'}{\left(g_2'^2(v)\right)'}=c. \quad (21)$$

If $c=1$, $c=-1$ and $c\neq\pm1$, then, we obtain the case (ix).

Also, if $f_1(u)=f_2(u)=r(u)$ and $g_1(u)=\cos v$, $g_2(v)=\sin v$, then we have

$$r''(u)\left(1+(r(u))^2\right)-r(u)\left(1+(r'(u))^2\right)=0.$$

As a result of this equation, we have a solution. Thus, we get the case (vi).

If $f_1(u)=f_2(u)$, $g_1(v)=g_2(v)$ in (18), the equation (18) coincides with (19). Then we find

$$\frac{f_1''(u)}{f_1(u)}+\frac{g_1''(v)}{g_1(v)}+\left(f_1''(u)f_1(u)-f_1'^2(u)\right)2g_1'^2(v)+\left(g_1''(v)g_1(v)-g_1'^2(v)\right)2f_1'^2(u)=0. \quad (22)$$

If $f_1''(u)=0$ or $g_1''(v)=0$ in (22), we obtain the cases (vii) and (viii), respectively. Also, if $f_2''(u)g_2''(v)\neq 0$, again we obtain the case (x). □

Example 2.2. *By choosing the constants $c_1=c_2=1$ in case (vi) of the previous theorem, the surface given with the parametrization*

$$M_2 : X(u,v)=(u,v,e^{3u+3}\cos v, e^{3u+3}\sin v) \quad (23)$$

is congruent to a factorable minimal surface. We can plot the projection of the surfaces with maple command: plot3d($[s,t,z+w], s=a..b, t=c..d$)

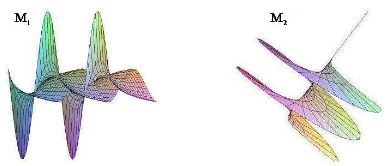

FIGURE 1. Factorable surface M_1 satisfying $K=K_N$ and Factorable minimal surface M_2

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