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ON NEW CONFORMABLE FRACTIONAL INTEGRAL INEQUALITIES FOR PRODUCT OF DIFFERENT KINDS OF CONVEXITY

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Abstract. Certain Hermite-Hadamard type inequalities involving various fractional integral operators for products of two functions have, recently, been presented. We aim to establish several Hermite-Hadamard type inequalities for products of two convex and $s-$ convex functions via new conformable fractional integral operators.

Keywords: Convex function, $s-$convex function, Hermite-Hadamard type inequalities, new conformable fractional integral.

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1. Introduction

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

An s-convex function was introduced in Breckners paper [3] and a number of properties and connections with s-convexity in the first sense are discussed in paper [6]. For more study, see ([2, 5]).

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Definition 1.2. *A function* $f : [0, \infty) \to \mathbb{R}$ *is said to be* s*-convex in the second sense if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0, \infty)$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}^+$ with $a \in \mathbb{R}^+_0$ are defined, respectively, by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad (x > a) \tag{1}$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad (x < b) \tag{2}$$

where Γ is the familiar Gamma function (see, e.g., [15, Section 1.1]). It is noted that $J^1_{a+}f(x)$ and $J^1_{b-}f(x)$ become the usual Riemann integrals. In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

The Euler beta function $B(\alpha, \beta)$ is defined by (see, e.g., [15, Section 1.1][10, p18])

$$B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}^-_0\right). \end{cases}$$

Some Hermite-Hadamard type inequalities for products of two different functions are proposed by Chen and Wu in [4] as follows:

Theorem 1.1. *Let* $f, g : [a,b] \to \mathbb{R}$ $a, b \in [0, \infty)$, $a < b$ *be functions such that and* $g, fg \in L[a,b]$. *If* f *is convex and nonnegative and* g *is* s*-convex on* $[a,b]$ *for some fixed* $s \in [0,1]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b)g(b) + J^{\alpha}_{b-}f(a)g(a)\right] \\ \leq\ & \left(\frac{1}{\alpha+s+1} + B(\alpha, s+2)\right) M(a,b) \\ &+ \left(B(\alpha+1, s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)}\right) N(a,b), \end{aligned}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.2. *Let* $f, g : [a,b] \to \mathbb{R}$, $a, b \in [0, \infty)$, $a < b$ *be functions such that* $f, g, fg \in L[a,b]$. *If* f *is* s_1*-convex and* g *is* s_2*-convex function on* $[a,b]$ *for some fixed* $s_1, s_2 \in [0,1]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b)g(b) + J^{\alpha}_{b-}f(a)g(a)\right] \\ \leq\ & \left(\frac{1}{\alpha+s_1+s_2} + B(\alpha, s_1+s_2+1)\right) M(a,b) \\ &+ \left(B(\alpha+s_1, s_2+1) + B(\alpha+s_2, s_1+1)\right) N(a,b), \end{aligned}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.3. *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$ be functions such that $fg\in L[a,b]$. If f is convex and nonnegative on $[a,b]g$ is s_2-convex function on $[a,b]$ for some fixed $s\in[0,1]$, then*

$$\begin{aligned} & 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \le & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)\right] \\ & +\frac{1}{2}M(a,b)\left(B(\alpha+1,s+1)\frac{1}{(\alpha+s)(\alpha+s+1)}\right) \\ & +\frac{1}{2}N(a,b)\left(B(\alpha,s+2)+\frac{1}{\alpha+s+1}\right), \end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Definition 1.3. [1] *Let $\alpha\in(n,n+1]$, $n=0,1,2,...$ and set $\beta=\alpha-n$. Then the left conformable fractional integral of any order $\alpha>0$ is defined by*

$$(I^a_\alpha f)(t)=\frac{1}{n!}\int_a^t (t-x)^n(x-a)^{\beta-1}f(x)dx.$$

Analogously, the right conformable fractional integral of any order $\alpha>0$ is defined by

$$({}^bI_\alpha f)(t)=\frac{1}{n!}\int_t^b (x-t)^n(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha=n+1$ then $\beta=\alpha-n=n+1-n=1$ and hence $(I^a_\alpha f)(t)=(J^a_{n+1}f)(t)$. Some recent result and properties concerning the fractional integral operators can be found ([1, 11, 12, 13]).

In [14], authors have proved the following inequalities for different kinds of convexity via conformable fractional integrals:

Theorem 1.4. *Let $f,g:[a,b]\to\mathbb{R}$, be functions with $0\le a<b$ and $f,g,fg\in L_1[a,b]$. If f is convex and nonnegative and g is s-convex on $[a,b]$ for some fixed $s\in[0,1]$, then one has the following inequality for conformable fractional integrals:*

$$\begin{aligned} & \frac{1}{(b-a)^\alpha}\left[I^a_\alpha f(b)g(b)+{}^bI_\alpha f(a)g(a)\right] \\ \le & \frac{M(a,b)}{n!}\left[B(n+s+2,\alpha-n)+B(n+1,\alpha-n+s+1)\right] \\ & +\frac{N(a,b)}{n!}\left[B(n+2,\alpha-n+s)+B(s+n+1,\alpha-n+1)\right] \end{aligned}$$

with $\alpha\in(n,n+1]$. ($M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$)

Theorem 1.5. *Suppose that $f,g:[a,b]\to\mathbb{R}$ be functions with $0\le a<b$ and $f,g,fg\in L_1[a,b]$. If f is s_1-convex and g is s_2-convex function on $[a,b]$ for some fixed $s_1,s_2\in[0,1]$, then one has the following inequality for conformable fractional integrals:*

$$\begin{aligned} & \frac{1}{(b-a)^\alpha}\left[I^a_\alpha f(b)g(b)+{}^bI_\alpha f(a)g(a)\right] \\ \le & \frac{1}{n!}M(a,b)\left[B(s_1+s_2+n+1,\alpha-n)+B(n+1,s_1+s_2+\alpha-n)\right] \\ & +\frac{1}{n!}N(a,b)\left[B(n+s_1+1,\alpha-n+s_2)+B(n+s_2+1,\alpha-n+s_1)\right]. \end{aligned}$$

where $\alpha\in(n,n+1]$ with $M(a,b)$ and $N(a,b)$ as in Theorem 1.4.

Theorem 1.6. *Let $f,g:[a,b]\to\mathbb{R}$, be functions with $0\leq a<b$ and $f,g,fg\in L_1[a,b]$. If f is convex and g is s-convex on $[a,b]$ for some fixed $s\in[0,1]$, then one has the following inequality for conformable fractional integrals:*

$$\begin{aligned}
& 2^s B(n+1,\alpha-n) f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\\
\leq\; & \frac{\Gamma(n+1)}{2(b-a)^\alpha}\left[I_\alpha^a f(b)g(b)+^b I_\alpha f(a)g(a)\right]\\
& +\frac{1}{2}M(a,b)\left[B(n+2,\alpha-n+s)+B(s+n+1,\alpha-n+1)\right]\\
& +\frac{1}{2}N(a,b)\left[B(n+1,\alpha-n+s+1)+B(n+s+2,\alpha-n)\right]
\end{aligned}$$

where $\alpha\in(n,n+1]$ and $M(a,b)$ and $N(a,b)$ as in Theorem 1.4.

Jarad et. al. [7] has defined a new fractional integral operator. Also, they gave some properties and relations between the some other fractional integral operators, as Riemann-Liouville fractional integral, Hadamard fractional integrals, generalized fractional integral operators..., with this operator.

Let $\beta\in\mathbb{C}$, $Re(\beta)>0$, then the left and right sided fractional conformable integral operators has defined respectively, as follows;

$$ {}_a^\beta\mathfrak{J}^\alpha f(x) = \frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt; \tag{3}$$

$$ {}^\beta\mathfrak{J}_b^\alpha f(x) = \frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt. \tag{4}$$

The fractional integral in (3) coincides with the Riemann-Liouville fractional integral (1) when $a=0$ and $\alpha=1$. It also coincides with the Hadamard fractional integral [9] once $a=0$ and $\alpha\to 0$ with the Katugampola fractional integral [8], when $a=0$. Similarly, Notice that, $(Qf)(t)=f(a+b-t)$ then we have ${}_a^\beta\mathfrak{J}^\alpha f(x)=Q({}^\beta\mathfrak{J}_b^\alpha)f(x)$. Moreover (4) coincides with the Riemann-Liouville fractional integral (2), when $b=0$ and $\alpha=1$. It also coincides with the Hadamard fractional integral [9] once $b=0$ and $\alpha\to 0$ with the Katugampola fractional integral [8], when $b=0$.

In this paper, some new fractional Hermite-Hadamard type inequalities for products two different kinds of convex functions are obtained but now for new conformable fractional integral operators.

2. Main Results

Theorem 2.1. *Let $f,g:[a,b]\to\mathbb{R}$, be functions with $0\leq a<b$ and $f,g,fg\in L_1[a,b]$. If f is convex and nonnegative and g is s-convex on $[a,b]$ for some fixed $s\in[0,1]$, then one has the following inequality for new conformable fractional integrals:*

$$\begin{aligned}
& \alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}_a^\beta\mathfrak{J}^\alpha fg(b)+^\beta\mathfrak{J}_b^\alpha fg(a)\right] \qquad (5)\\
\leq\; & \left[\beta_1(s+2,\alpha)-\beta_1(s+2,\alpha\beta)+\frac{1}{\alpha+s+1}-\frac{1}{\alpha\beta+s+1}\right]M(a,b)\\
& +\left[\beta_1(2,\alpha+s)-\beta_1(2,\alpha\beta+s)+\beta_1(s+1,\alpha+1)-\beta_1(s+1,\alpha\beta+1)\right]N(a,b)
\end{aligned}$$

where $\alpha,\beta>0$ and β_1 is Euler Beta function.
$(M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a))$

Proof. By using the definitions of f and g, we can write

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b) \tag{6}$$

and

$$g(ta+(1-t)b) \leq t^s g(a)+(1-t)^s g(b). \tag{7}$$

By multiplying (6) and (7), we have

$$\begin{aligned} & f(ta+(1-t)b)g(ta+(1-t)b) \\ \leq\ & t^{s+1}f(a)g(a)+(1-t)^{s+1}f(b)g(b) \\ & +t(1-t)^s f(a)g(b)+t^s(1-t)f(b)g(a). \end{aligned} \tag{8}$$

By a similar argument, we get

$$\begin{aligned} & f((1-t)a+tb)g((1-t)a+tb) \\ \leq\ & (1-t)^{s+1}f(a)g(a)+t^{s+1}f(b)g(b) \\ & +t^s(1-t)f(a)g(b)+t(1-t)^s f(b)g(a). \end{aligned} \tag{9}$$

By adding (8) and (9), we obtain

$$\begin{aligned} & f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb) \\ \leq\ & \left(t^{s+1}+(1-t)^{s+1}\right)[f(a)g(a)+f(b)g(b)] \\ & +(t(1-t)^s+t^s(1-t))\,[f(a)g(b)+f(b)g(a)]\,. \end{aligned} \tag{10}$$

If we multiply both sides of (10) by $\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\,[fg(ta+(1-t)b)+fg((1-t)a+tb)]\,dt \\ \leq\ & \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[t^{s+1}+(1-t)^{s+1}\right]M(a,b)dt \\ & +\int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\,[t(1-t)^s+t^s(1-t)]\,N(a,b)dt. \end{aligned}$$

By calculating the above integrals and simplifying, we get

$$\begin{aligned} & \alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}^{\beta}_{a}\mathfrak{J}^{\alpha}fg(b)+{}^{\beta}\,\mathfrak{J}^{\alpha}_{b}fg(a)\right] \\ \leq\ & \left[\beta_1(s+2,\alpha)-\beta_1(s+2,\alpha\beta)+\frac{1}{\alpha+s+1}-\frac{1}{\alpha\beta+s+1}\right]M(a,b) \\ & +[\beta_1(2,\alpha+s)-\beta_1(2,\alpha\beta+s)+\beta_1(s+1,\alpha+1)-\beta_1(s+1,\alpha\beta+1)]\,N(a,b) \end{aligned}$$

which completes the proof. □

Corollary 2.1. *If we choose $s=1$ in the inequality* (5)*, then Theorem 2.1 reduces to the following inequality:*

$$\begin{aligned} & \alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}^{\beta}_{a}\mathfrak{J}^{\alpha}fg(b)+{}^{\beta}\,\mathfrak{J}^{\alpha}_{b}fg(a)\right] \\ \leq\ & \left[\beta_1(3,\alpha)-\beta_1(3,\alpha\beta)+\frac{1}{\alpha+2}-\frac{1}{\alpha\beta+2}\right]M(a,b) \\ & +[\beta_1(2,\alpha+1)-\beta_1(2,\alpha\beta+1)+\beta_1(2,\alpha+1)-\beta_1(2,\alpha\beta+1)]\,N(a,b). \end{aligned}$$

Corollary 2.2. *If we choose $f(x)=1$, we obtain*

$$\begin{aligned}&\alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}_{a}^{\beta}\mathfrak{J}^{\alpha}fg(b)+^{\beta}\mathfrak{J}_{b}^{\alpha}fg(a)\right]\\ \leq\ &\left[\beta_1(3,\alpha)-\beta_1(3,\alpha\beta)+\frac{1}{\alpha+2}-\frac{1}{\alpha\beta+2}\right](g(a)+g(b))\\ &+\left[\beta_1(2,\alpha+1)-\beta_1(2,\alpha\beta+1)+\beta_1(2,\alpha+1)-\beta_1(2,\alpha\beta+1)\right](g(a)+g(b)).\end{aligned}$$

Theorem 2.2. *Suppose that $f,g:[a,b]\to\mathbb{R}$ be functions with $0\leq a<b$ and $f,g,fg\in L_1[a,b]$. If f is s_1-convex and g is s_2-convex function on $[a,b]$ for some fixed $s_1,s_2\in[0,1]$, then one has the following inequality for new conformable fractional integrals:*

$$\begin{aligned}&\alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}_{a}^{\beta}\mathfrak{J}^{\alpha}fg(b)+^{\beta}\mathfrak{J}_{b}^{\alpha}fg(a)\right] \qquad (11)\\ \leq\ &\left[\beta_1(s_1+s_2+1,\alpha)-\beta_1(s_1+s_2+1,\alpha\beta)+\frac{1}{\alpha+s_1+s_2}-\frac{1}{\alpha\beta+s_1+s_2}\right]M(a,b)\\ &+\left[\beta_1(s_1+1,\alpha+s_2)-\beta_1(s_1+1,\alpha\beta+s_2)+\beta_1(s_2+1,\alpha+s_1)-\beta_1(s_2+1,\alpha\beta+s_1)\right]N(a,b)\end{aligned}$$

where $\alpha,\beta>0$ and β_1 is Euler Beta function with $M(a,b)$ and $N(a,b)$ as in Theorem 2.1.

Proof. From the definition of s_1-convexity, we can write

$$f(ta+(1-t)b)\leq t^{s_1}f(a)+(1-t)^{s_1}f(b) \qquad (12)$$

and

$$g(ta+(1-t)b)\leq t^{s_2}g(a)+(1-t)^{s_2}g(b). \qquad (13)$$

By multiplying both side of (12) and (13), we get

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)\\ \leq\ &t^{s_1+s_2}f(a)g(a)+(1-t)^{s_1+s_2}f(b)g(b)\\ &+t^{s_1}(1-t)^{s_2}f(a)g(b)+t^{s_2}(1-t)^{s_1}f(b)g(a). \qquad (14)\end{aligned}$$

By a similar way, it is easy to write,

$$\begin{aligned}&f((1-t)a+tb)g((1-t)a+tb)\\ \leq\ &(1-t)^{s_1+s_2}f(a)g(a)+t^{s_1+s_2}f(b)g(b)\\ &+(1-t)^{s_1}t^{s_2}f(a)g(b)+t^{s_1}(1-t)^{s_2}f(b)g(a). \qquad (15)\end{aligned}$$

By adding (14) and (15), we have

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\ \leq\ &\left(t^{s_1+s_2}+(1-t)^{s_1+s_2}\right)[f(a)g(a)+f(b)g(b)]\\ &+\left(t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right)[f(a)g(b)+f(b)g(a)]. \qquad (16)\end{aligned}$$

If we multiply both sides of (16) by $\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}$, then by integrating with respect to t over $[0,1]$, we deduce

$$\begin{aligned}&\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[fg(ta+(1-t)b)+fg((1-t)a+tb)\right]dt\\ \leq\ &\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[t^{s_1+s_2}+(1-t)^{s_1+s_2}\right]M(a,b)dt\\ &+\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]N(a,b)dt.\end{aligned}$$

By calculating the above integrals and simplifying, we get

$$\begin{aligned}
&\alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}_a^\beta\mathfrak{J}^\alpha fg(b)+^\beta\mathfrak{J}_b^\alpha fg(a)\right]\\
\leq\ &\left[\beta_1(s_1+s_2+1,\alpha)-\beta_1(s_1+s_2+1,\alpha\beta)+\frac{1}{\alpha+s_1+s_2}-\frac{1}{\alpha\beta+s_1+s_2}\right]M(a,b)\\
&+\left[\beta_1(s_1+1,\alpha+s_2)-\beta_1(s_1+1,\alpha\beta+s_2)+\beta_1(s_2+1,\alpha+s_1)-\beta_1(s_2+1,\alpha\beta+s_1)\right]N(a,b),
\end{aligned}$$

where we use the fact that $(1-(1-t)^\alpha)^{\beta-1}\leq 1-(1-t)^{\alpha\beta-\alpha}$. This completes the proof. □

Remark 2.1. *If we choose $s_1=s_2=1$ in the inequality* (11)*, then Theorem 2.2 reduces to the Corollary 2.1.*

Theorem 2.3. *Let $f,g:[a,b]\to\mathbb{R}$, be functions with $0\leq a<b$ and $f,g,fg\in L_1[a,b]$. If f is convex and g is s-convex on $[a,b]$ for some fixed $s\in[0,1]$, then one has the following inequality for new conformable fractional integrals:*

$$\begin{aligned}
&\frac{2^{s+1}}{\beta\alpha^\beta}fg\left(\frac{a+b}{2}\right)\\
\leq\ &\frac{2^{s+1}}{(b-a)^{\alpha\beta}}\Gamma(\beta)\left[{}_a^\beta\mathfrak{J}^\alpha fg(b)+^\beta\mathfrak{J}_b^\alpha fg(a)\right]\\
\leq\ &\left[\beta_1(2,s+1)-\beta_1(2,\alpha\beta-\alpha+s+1)+\beta_1(s+1,2)-\beta_1(s+1,\alpha\beta-\alpha+2)\right]M(a,b)\\
&+\left[\beta_1(s+2,s+2)-\beta_1(s+2,\alpha\beta-\alpha+s+2)\right]N(a,b)
\end{aligned}$$

where $\alpha,\beta>0$ and β_1 is Euler Beta function with $M(a,b)$ and $N(a,b)$ as in Theorem 2.1.

Proof. By using the definitions, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
\leq\ &f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)\\
\leq\ &\frac{1}{2^{s+1}}\left[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)+g((1-t)a+tb)\right]\\
&+\frac{1}{2^{s+1}}\left[\left(t(1-t)^s+(1-t)t^s\right)M(a,b)+\left((1-t)^{s+1}t^{s+1}\right)N(a,b)\right]. \qquad (17)
\end{aligned}$$

By multiplying both sides of (17) by $\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}fg\left(\frac{a+b}{2}\right)dt\\
\leq\ &\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[fg\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)\right]dt\\
\leq\ &\frac{1}{2^{s+1}}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[fg(ta+(1-t)b)+fg((1-t)a+tb)\right]\\
&+\frac{1}{2^{s+1}}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[t(1-t)^s+t^s(1-t)\right]M(a,b)dt\\
&+\frac{1}{2^{s+1}}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}\left[t^{s+1}+(1-t)^{s+1}\right]N(a,b)dt.
\end{aligned}$$

By computing the above integrals, we get

$$\begin{aligned}
&\frac{2^{s+1}}{\beta\alpha^{\beta}} fg\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{2^{s+1}}{(b-a)^{\alpha\beta}}\Gamma(\beta)\left[{}_{a}^{\beta}\mathfrak{J}^{\alpha} fg(b) + {}^{\beta}\,\mathfrak{J}_{b}^{\alpha} fg(a)\right] \\
\leq\ & \left[\beta_1(2,s+1) - \beta_1(2,\alpha\beta-\alpha+s+1) + \beta_1(s+1,2) - \beta_1(s+1,\alpha\beta-\alpha+2)\right] M(a,b) \\
& + \left[\beta_1(s+2,s+2) - \beta_1(s+2,\alpha\beta-\alpha+s+2)\right] N(a,b)
\end{aligned}$$

where we use the fact that $(1-(1-t)^{\alpha})^{\beta-1} \leq 1-(1-t)^{\alpha\beta-\alpha}$, we get the desired result. □

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