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ON THE HYPERBOLIC FIBONACCI MATRIX FUNCTIONS

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Abstract. In this study, we will introduce a new class of hyperbolic matrix functions. By comparing Binet formulas for the Fibonacci and Lucas numbers to the formulas of classical hyperbolic matrix functions, we will define hyperbolic Fibonacci matrix functions and we will deal with some of their properties.

Keywords: Matrix Function, hyperbolic functions, Fibonacci Numbers.

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1. Introduction

Many problems of applied sciences are described by differential systems, so differential systems have many applications in applied sciences such as mathematics and engineering $[1-7]$ and references therein. Computing matrix functions plays a very important role in the solutions to the differential systems. Some important matrix functions are exponential, cosine, sine, hyperbolic sine and hyperbolic cosine of a matrix. Their importances arise in solving differential systems of first and second order or coupled partial differential systems, for example see $[8-11]$. It is well known that $x(t) = e^{At}x_0$ is the solution to differential system

$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0 \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$. Hyperbolic sine and cosine of a matrix play a similar role in coupled partial differential systems. An exact solution of coupled hyperbolic systems

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of the form

$$\left.\begin{aligned} u_{tt}(x,t) &= Au_{xx}(x,t), \quad 0<x<1, \quad t>0, \\ u(0,t)+B_1u_x(0,t) &= 0, \quad t>0, \\ A_2u(1,t)+B_2u_x(1,t) &= 0, \quad t>0, \\ u(x,0) &= f(x), \quad 0\le x\le 1, \\ u_t(x,0) &= g(x), \quad 0\le x\le 1, \end{aligned}\right\} \tag{2}$$

where A, B_1, A_2, B_2 are $n\times n$ complex matrices, and u, f, g are $n-$vector valued functions, was constructed by Jodar et al. [11] in terms of a series which used hyperbolic cosine and sine of a matrix, respectively defined by

$$\cosh(A) = \frac{e^A + e^{-A}}{2} \quad \text{and} \quad \sinh(A) = \frac{e^A - e^{-A}}{2},\ A \in \mathbb{C}^{n\times n}. \tag{3}$$

Many different algorithms for computing the matrix functions mentioned above have been reported in the last decades $[2, 8, 9, 11 - 17]$. For example, Moore [15] has put forward the idea of expanding in either Chebyshev, Legendre or Laguerre orthogonal polynomials for the matrix exponentials. An algorithm for computing the cosine matrix function has been presented by Sastre et al. [17] based on Taylor series and the cosine double angle formula. Defez and Jodar [13] have presented some new methods for computing matrix exponential, sine and cosine based on Hermite matrix polynomial series. Recently, a method for computing hyperbolic matrix functions, $\sinh(A)$ and $\cosh(A)$, has been proposed by Defez et al. [2] based on Hermite matrix polynomial expansions.

In this work we introduce a new class of hyperbolic matrix functions called hyperbolic Fibonacci matrix functions by comparing Binet formulas for the Fibonacci and Lucas numbers to the formulas of classical hyperbolic matrix functions in (3) and we present some of their recursive and hyperbolic properties.

This work is organized as follows. In section 3, we first define matrix power of the golden ratio and give some of its properties. After, we introduce hyperbolic Fibonacci matrix functions. For this purpose, we give some preliminaries about hyperbolic Fibonacci functions in section 2. Our main results about properties of hyperbolic matrix functions are given in section 4.

2. Preliminaries

The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ $(n \ge 1)$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ $(n \ge 1)$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. The characteristic equation of F_n and L_n is $[18, 19]$:

$$t^2 - t - 1 = 0. \tag{4}$$

The roots of Equation (4) are $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ and α and β are the only numbers such that the reciprocal of each is obtained by subtracting 1 from it, that is, $t-1=\frac{1}{t}$, where $t=\alpha$ or β. Thus α is the only positive number that has the properties $\alpha - 1 = \frac{1}{\alpha}$, $\alpha^2 = 1+\alpha$ and $\alpha^{-2} = 1-\alpha^{-1}$. Moreover, the number $\alpha = \frac{1+\sqrt{5}}{2}$ is called golden ratio which has been very attractive for researchers because it occurs ubiquitous such as in nature, art, architecture, and anatomy. The relations between golden ratio and the

Fibonacci and Lucas numbers

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}$$

are well-known. Also, the Binet formulas for F_n and L_n are [18, 19]:

$$F_n=\begin{cases}\frac{\alpha^n+\alpha^{-n}}{\sqrt{5}}, & n \text{ odd},\\ \frac{\alpha^n-\alpha^{-n}}{\sqrt{5}}, & n \text{ even},\end{cases}\tag{5}$$

$$L_n=\begin{cases}\alpha^n-\alpha^{-n}, & n \text{ odd},\\ \alpha^n+\alpha^{-n}, & n \text{ even}.\end{cases}\tag{6}$$

The Fibonacci numbers have many properties, continuous versions and generalizations [18 − 26]. Stakhov and Tkachenko [23] have introduced a new class of hyperbolic functions called hyperbolic Fibonacci functions replacing the discrete variable n in Equation (5) with the continuous variable x that takes its values from the set of the real numbers. Based on an analogy between Binet formula, Equation (5), and the classical hyperbolic functions

$$\sinh(x)=\frac{e^x-e^{-x}}{2} \text{ and } \cosh(x)=\frac{e^x+e^{-x}}{2},$$

Stakhov and Rozin [24] have defined the so-called symmetrical hyperbolic Fibonacci and Lucas functions as follows:

$$sFs(x)=\frac{\alpha^x-\alpha^{-x}}{\sqrt{5}} \text{ and } cFs(x)=\frac{\alpha^x+\alpha^{-x}}{\sqrt{5}},\tag{7}$$

$$sLs(x)=\alpha^x-\alpha^{-x} \text{ and } cLs(x)=\alpha^x+\alpha^{-x},\tag{8}$$

where $sFs(x)$, $cFs(x)$, $sLs(x)$ and $cLs(x)$ denote symmetrical hyperbolic Fibonacci sine, cosine, symmetrical hyperbolic Lucas sine and cosine functions, respectively. The graphs of the symmetrical hyperbolic Fibonacci functions have a symmetric form and are similar to the graphs of the classical hyperbolic functions. Also, the symmetrical hyperbolic Fibonacci functions $sFs(x)$ and $cFs(x)$ are increasing on $(0,+\infty)$. The graphs of the symmetrical hyperbolic Fibonacci functions are given in [24]. The symmetrical hyperbolic Fibonacci functions have properties that are similar to the classical hyperbolic functions. Some of them are [24]:

$$cFs(x)=cFs(-x),\ sFs(x)=-sFs(-x) \text{ and } [cFs(x)]^2-[sFs(x)]^2=\frac{4}{5},$$

$$sFs(x+2)=cFs(x+1)+sFs(x) \text{ and } cFs(x+2)=sFs(x+1)+cFs(x),$$

$$\frac{2}{\sqrt{5}}cFs(x+y)=cFs(x)cFs(y)+sFs(x)sFs(y).$$

Also, the derivatives of hyperbolic Fibonacci functions are [24]:

$$[cFs(x)]^{(n)}=\begin{cases}(\ln\alpha)^n\, sFs(x), & \text{for } n \text{ odd},\\ (\ln\alpha)^n\, cFs(x), & \text{for } n \text{ even},\end{cases}$$

$$[sFs(x)]^{(n)}=\begin{cases}(\ln\alpha)^n\, cFs(x), & \text{for } n \text{ odd},\\ (\ln\alpha)^n\, sFs(x), & \text{for } n \text{ even}.\end{cases}$$

Recently, Bahşi [20] has introduced the analog of the Wilker inequality and the parameterized Wilker inequality for the hyperbolic Fibonacci functions. For more information and the generalizations about hyperbolic Fibonacci functions see [20 − 26] the references cited therein.

3. Hyperbolic Fibonacci Matrix Functions

Let us consider the diferential system

$$\frac{dx}{dt} = (\ln \alpha)\, Ax, \qquad x(0) = x_0 \tag{9}$$

where $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$ and α is the golden ratio. It is clear that the solution to the Equation (9) is $x(t) = \alpha^{At}x_0$. The Taylor series expansions of the functions e^t and α^t are

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

and

$$\alpha^t = 1 + \ln(\alpha)t + \frac{[\ln(\alpha)]^2}{2!}t^2 + \frac{[\ln(\alpha)]^3}{3!}t^3 + \cdots$$

where $t \in \mathbb{R}$ and $\alpha^t = e^{t\ln(\alpha)}$. The series expansions above give us the matrix functions

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$\alpha^A = I + \ln(\alpha)A + \frac{[\ln(\alpha)]^2}{2!}A^2 + \frac{[\ln(\alpha)]^3}{3!}A^3 + \cdots \tag{10}$$

where $A \in \mathbb{C}^{n\times n}$ and I is $n-$dimensional identity matrix. By using series expansion (10), we can see easily that the matrix power of the golden ratio, α^A, has similar properties to that of matrix exponential, e^A. Some of them are:

1. Zero matrix power of the golden ratio is an identity matrix. That is,

$$\alpha^0 = I.$$

2. Identitiy matrix power of the golden ratio is α times identity matrix. That is,

$$\alpha^I = \alpha I \quad \text{and} \quad \alpha^{mI} = \alpha^m I \quad \text{for } m \in \mathbb{Z}.$$

3. Tranposed matrix power of the golden ratio equals the transpose of the matrix power of the golden ratio. That is,

$$\alpha^{\left(A^T\right)} = \left(\alpha^A\right)^T.$$

4. The inverse of the matrix power of the golden ratio exists and is given by

$$\left(\alpha^A\right)^{-1} = \alpha^{-A}.$$

5. The derivative of the matrix power of the golden ratio is given by

$$\frac{d\alpha^{At}}{dt} = \ln(\alpha)A\alpha^{At}.$$

6. The power of the matrix power of the golden ratio satisfies

$$\left(\alpha^A\right)^m = \alpha^{mA},$$

where $m \in \mathbb{Z}$.

Next we give three lemmas related to matrix powers of golden ratio together with their proofs, because we use them in our main results.

Lemma 3.1. *For the commutable $n \times n$ matrices A and B,*

$$\alpha^{A+B} = \alpha^A\alpha^B.$$

Proof. Series expansions of matrix powers of golden ratio yield

$$\begin{aligned}
\alpha^{A+B} &= I+\ln\alpha\,(A+B)+\frac{[\ln\alpha]^2}{2!}(A+B)^2+\frac{[\ln\alpha]^3}{3!}(A+B)^3+\cdots \\
&= I+\ln\alpha\,(A+B)+\frac{[\ln\alpha]^2}{2!}\left(A^2+AB+BA+B^2\right)+\cdots, \\
\alpha^{A}\alpha^{B} &= \left(I+[\ln\alpha]\,A+\frac{[\ln\alpha]^2}{2!}A^2+\cdots\right)\left(I+[\ln\alpha]\,B+\frac{[\ln\alpha]^2}{2!}B^2+\cdots\right) \\
&= I+\ln\alpha\,(A+B)+\frac{[\ln\alpha]^2}{2!}\left(A^2+2AB+B^2\right)+\cdots
\end{aligned}$$

$$\alpha^{A+B}-\alpha^{A}\alpha^{B}=\frac{[\ln\alpha]^2}{2!}(BA-AB)+\cdots$$

If $AB=BA$, then $\alpha^{A+B}-\alpha^{A}\alpha^{B}=0$. Thus, desired result is obtained. ∎

Lemma 3.2. *The following identities that are analogous to identities for the golden ratio* $\alpha^2=1+\alpha$, $\alpha^{-2}=1-\alpha^{-1}$ *and* $\alpha-\alpha^{-1}=1$ *are valid for matrix power of the golden ratio:*

$$\alpha^{2I}=I+\alpha^{I},$$
$$\alpha^{-2I}=I-\alpha^{-I}$$

and

$$\alpha^{I}-\alpha^{-I}=I.$$

Proof.

$$I+\alpha^{I}=I+\alpha I=(1+\alpha)\,I=\alpha^2 I=\alpha^{2I},$$
$$I-\alpha^{-I}=I-\alpha^{-1}I=\left(1-\alpha^{-1}\right)I=\alpha^{-2}I=\alpha^{-2I}$$

and

$$\alpha^{I}-\alpha^{-I}=\alpha I-\alpha^{-1}I=\left(\alpha-\alpha^{-1}\right)I=I.$$

∎

Lemma 3.3. *The following identity that is analogous to identitiy for the golden ratio* $\frac{1}{5}\left(\alpha^2+\alpha^{-2}+2\right)=1$ *is valid for matrix power of the golden ratio:*

$$\frac{1}{5}\left(\alpha^{2I}+\alpha^{-2I}+2I\right)=I.$$

Proof. Lemma 2 yields

$$\begin{aligned}
\frac{1}{5}\left(\alpha^{2I}+\alpha^{-2I}+2I\right) &= \frac{1}{5}\left(I+\alpha^{I}+I-\alpha^{-I}+2I\right) \\
&= \frac{4}{5}I+\frac{1}{5}\left(\alpha^{I}-\alpha^{-I}\right) \\
&= \frac{4}{5}I+\frac{1}{5}I \\
&= I
\end{aligned}$$

∎

Now we give the definition of the hyperbolic Fibonacci matrix functions based on an analogy between Binet formulas (5), (6) and the hyperbolic matrix functions (3).

Definition 3.1. *Let α be the golden ratio. The symmetrical hyperbolic Fibonacci sine and cosine matrix functions are defined by, respectively*

$$sFs(A) = \frac{\alpha^A - \alpha^{-A}}{\sqrt{5}} \quad \text{and} \quad cFs(A) = \frac{\alpha^A + \alpha^{-A}}{\sqrt{5}}, \tag{11}$$

where A is $n \times n$ matrix. Similarly, the symmetrical hyperbolic Lucas sine and cosine matrix functions are defined by, respectively

$$sLs(A) = \alpha^A - \alpha^{-A} \quad \text{and} \quad cLs(A) = \alpha^A + \alpha^{-A}. \tag{12}$$

The classical hyperbolic matrix functions and symmetrical hyperbolic Fibonacci matrix functions have series expansions as follows:

$$\cosh(A) = I + \frac{1}{2!}A^2 + \frac{1}{4!}A^4 + \cdots$$

$$\sinh(A) = A + \frac{1}{3!}A^3 + \frac{1}{5!}A^5 + \cdots$$

$$cFs(A) = \frac{2}{\sqrt{5}}I + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^2}{2!}A^2 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^4}{4!}A^4 + \cdots \tag{13}$$

$$sFs(A) = \frac{2}{\sqrt{5}}(\ln\alpha)A + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^3}{3!}A^3 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^5}{5!}A^5 + \cdots \quad . \tag{14}$$

From the properties of α^A and the definitions of the hyperbolic Fibonacci matrix functions, we have some basic properties of $cFs(A)$ and $sFs(A)$:

1. For the zero matrix, $cFs(0) = \frac{2}{\sqrt{5}}I$ and $sFs(0) = 0$.
2. For the identity matrix, $cFs(I) = I$ and $sFs(I) = \frac{1}{\sqrt{5}}I$
3. $cFs(-A) = cFs(A)$ and $sFs(-A) = -sFs(A)$.
4. For the transpose matrix, $[cFs(A)]^T = cFs(A^T)$ and $[sFs(A)]^T = sFs(A^T)$.
5. $cFs(A) = \frac{2}{\sqrt{5}}\cosh(A\ln\alpha)$ and $sFs(A) = \frac{2}{\sqrt{5}}\sinh(A\ln\alpha)$.

Throughout this paper $sFs(A)$, $cFs(A)$, $sLs(A)$ and $cLs(A)$ denote the symmetrical hyperbolic Fibonacci and Lucas matrix functions given in (11), (12) and α denotes the golden ratio, $\alpha = \frac{1+\sqrt{5}}{2}$. Also, in statements of our theorems, we will mention some identities related to hyperbolic Fibonacci functions given in [24] .

4. Recursive and Hyperbolic Properties of the Symmetrical Hyperbolic Fibonacci Matrix Functions

Theorem 4.1. *(Recursive relation). The following correlations that are analogous to the recurrent equations for the hyperbolic Fibonacci Functions $sFs(x+2) = cFs(x+1)+sFs(x)$ and $cFs(x+2) = sFs(x+1) + cFs(x)$ are valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$sFs(A + 2I) = cFs(A + I) + sFs(A),$$

$$cFs(A + 2I) = sFs(A + I) + cFs(A).$$

Also, these correlations correspond to the recurrence relation $F_{n+2} = F_{n+1} + F_n$ for the Fibonacci numbers.

Proof. By Lemmas 1 and 2, we have

$$\begin{aligned} cFs(A+I)+sFs(A) &= \frac{\alpha^{A+I}+\alpha^{-A-I}}{\sqrt{5}}+\frac{\alpha^{A}-\alpha^{-A}}{\sqrt{5}} \\ &= \frac{\alpha^{A}(\alpha^{I}+I)-\alpha^{-A}(I-\alpha^{-I})}{\sqrt{5}} \\ &= \frac{\alpha^{A}\alpha^{2I}-\alpha^{-A}\alpha^{-2I}}{\sqrt{5}} \\ &= \frac{\alpha^{A+2I}-\alpha^{-A-2I}}{\sqrt{5}} \\ &= sFs(A+2I) \end{aligned}$$

and

$$\begin{aligned} sFs(A+I)+cFs(A) &= \frac{\alpha^{A+I}-\alpha^{-A-I}}{\sqrt{5}}+\frac{\alpha^{A}+\alpha^{-A}}{\sqrt{5}} \\ &= \frac{\alpha^{A}(\alpha^{I}+I)+\alpha^{-A}(I-\alpha^{-I})}{\sqrt{5}} \\ &= \frac{\alpha^{A}\alpha^{2I}+\alpha^{-A}\alpha^{-2I}}{\sqrt{5}} \\ &= \frac{\alpha^{A+2I}+\alpha^{-A-2I}}{\sqrt{5}} \\ &= cFs(A+2I). \end{aligned}$$

∎

Theorem 4.2. *The following Cassini type identities are valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$[sFs(A)]^2-cFs(A+I)cFs(A-I)=-I, \tag{15}$$

$$[cFs(A)]^2-sFs(A+I)sFs(A-I)=I. \tag{16}$$

Proof. By Lemmas 1 and 3, the left hand side of the first identity (LHS) is

$$\begin{aligned} (LHS) &= \frac{(\alpha^{A}-\alpha^{-A})^2-(\alpha^{A+I}+\alpha^{-A-I})(\alpha^{A-I}+\alpha^{-A+I})}{\left(\sqrt{5}\right)^2} \\ &= \frac{\alpha^{2A}-2I+\alpha^{-2A}-\left(\alpha^{2A}+\alpha^{2I}+\alpha^{-2I}+\alpha^{-2A}\right)}{5} \\ &= \frac{-(2I+\alpha^{2I}+\alpha^{-2I})}{5} \\ &= -I \end{aligned}$$

and the left hand side of the second identity (LHS) is

$$\begin{aligned}(LHS) &= \frac{(\alpha^{A}+\alpha^{-A})^2-(\alpha^{A+I}-\alpha^{-A-I})(\alpha^{A-I}-\alpha^{-A+I})}{\left(\sqrt{5}\right)^2}\\ &= \frac{\alpha^{2A}+2I+\alpha^{-2A}-\left(\alpha^{2A}-\alpha^{2I}-\alpha^{-2I}+\alpha^{-2A}\right)}{5}\\ &= \frac{2I+\alpha^{2I}+\alpha^{-2I}}{5}\\ &= I.\end{aligned}$$

∎

The identities (15) and (16) correspond to the Cassini identity, $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$, for the Fibonacci numbers and similar to the equations for the hyperbolic Fibonacci functions $[sFs(x)]^2 - cFs(x+1)cFs(x-1) = -1$ and $[cFs(x)]^2 - sFs(x+1)sFs(x-1) = 1$.

The proofs of the next two theorems are similar to the proofs of the previous theorems. So, we give them without proof.

Theorem 4.3. *The following correlations that are similar to the equations for the hyperbolic Fibonacci Functions $cFs(x+1)+cFs(x-1)=cLs(x)$ and $sFs(x+1)+sFs(x-1)=sLs(x)$ are valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$cFs(A+I)+cFs(A-I)=cLs(A),$$

$$sFs(A+I)+sFs(A-I)=sLs(A).$$

Also, these correlations correspond to the identity $F_{n+1}+F_{n-1}=L_n$ for the Fibonacci and Lucas numbers.

Theorem 4.4. *The following correlations that are similar to the equations for the hyperbolic Fibonacci Functions $cFs(x)+sLs(x)=2sFs(x+1)$ and $sFs(x)+cLs(x)=2cFs(x+1)$ are valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$cFs(A)+sLs(A)=2sFs(A+I),$$

$$sFs(A)+cLs(A)=2cFs(A+I).$$

Also, these correlations correspond to the identity $F_n+L_n=2F_{n+1}$ for the Fibonacci and Lucas numbers.

Theorem 4.5. *(Pythagorean theorem). The following correlation that is similar to the equation for the hyperbolic Fibonacci functions $[cFs(x)]^2-[sFs(x)]^2=\frac{4}{5}$ is valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$[cFs(A)]^2-[sFs(A)]^2=\frac{4}{5}I.$$

The equation $[cFs(A)]^2-[sFs(A)]^2=\frac{4}{5}I$ correspond to the identity $[\cosh(x)]^2-[\sinh(x)]^2=1$ for the classical hyperbolic functions.

Proof. From Lemma 1, we have

$$\begin{aligned}[cFs(A)]^2 - [sFs(A)]^2 &= \left(\frac{\alpha^A + \alpha^{-A}}{\sqrt{5}}\right)^2 - \left(\frac{\alpha^A - \alpha^{-A}}{\sqrt{5}}\right)^2 \\ &= \frac{\alpha^{2A} + 2I + \alpha^{-2A} - \alpha^{2A} + 2I - \alpha^{-2A}}{5} \\ &= \frac{4}{5}I.\end{aligned}$$

∎

Theorem 4.6. *The following correlation that is similar to the equation for the hyperbolic Fibonacci functions* $\frac{2}{\sqrt{5}}cFs(x+y) = cFs(x)cFs(y) + sFs(x)sFs(y)$ *is valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$\frac{2}{\sqrt{5}}cFs(A+B) = cFs(A)cFs(B) + sFs(A)sFs(B), \tag{17}$$

where A *and* B *commute.*

Proof. By Lemma 1, the right hand side of the Equation (17) (RHS) is

$$\begin{aligned}(RHS) &= \frac{\alpha^A + \alpha^{-A}}{\sqrt{5}}\frac{\alpha^B + \alpha^{-B}}{\sqrt{5}} + \frac{\alpha^A - \alpha^{-A}}{\sqrt{5}}\frac{\alpha^B - \alpha^{-B}}{\sqrt{5}} \\ &= \frac{\alpha^{A+B} + \alpha^{A-B} + \alpha^{-A+B} + \alpha^{-A-B} + \alpha^{A+B} - \alpha^{A-B} - \alpha^{-A+B} + \alpha^{-A-B}}{5} \\ &= \frac{2\alpha^{A+B} + 2\alpha^{-A-B}}{5} \\ &= \frac{2}{\sqrt{5}}cFs(A+B).\end{aligned}$$

∎

Theorem 4.7. *The following correlation that is similar to the equation for the hyperbolic Fibonacci functions* $\frac{2}{\sqrt{5}}cFs(x-y) = cFs(x)cFs(y) - sFs(x)sFs(y)$ *is valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$\frac{2}{\sqrt{5}}cFs(A-B) = cFs(A)cFs(B) - sFs(A)sFs(B), \tag{18}$$

where A *and* B *commute.*

Proof. By Lemma 1, the right hand side of the Equation (18) (RHS) is

$$\begin{aligned}(RHS) &= \frac{\alpha^A + \alpha^{-A}}{\sqrt{5}}\frac{\alpha^B + \alpha^{-B}}{\sqrt{5}} - \frac{\alpha^A - \alpha^{-A}}{\sqrt{5}}\frac{\alpha^B - \alpha^{-B}}{\sqrt{5}} \\ &= \frac{\alpha^{A+B} + \alpha^{A-B} + \alpha^{-A+B} + \alpha^{-A-B} - \alpha^{A+B} + \alpha^{A-B} + \alpha^{-A+B} - \alpha^{-A-B}}{5} \\ &= \frac{2\alpha^{A-B} + 2\alpha^{-A+B}}{5} \\ &= \frac{2}{\sqrt{5}}cFs(A-B).\end{aligned}$$

∎

Theorem 4.8. *The following correlations that are similar to the equation for the hyperbolic Fibonacci functions* $\frac{2}{\sqrt{5}}sFs(x \pm y) = sFs(x)cFs(y) \pm cFs(x)sFs(y)$ *are valid for the symmetrical hyperbolic Fibonacci matrix functions:*

$$\frac{2}{\sqrt{5}}sFs(A \pm B) = sFs(A)cFs(B) \pm cFs(A)sFs(B), \tag{19}$$

where A *and* B *commute.*

Proof. The proof is similar to the proofs of Theorems 6 and 7. ∎

Next theorem gives us nth derivatives of hyperbolic Fibonacci matrix functions.

Theorem 4.9. *The* n*th derivatives of hyperbolic Fibonacci matrix functions are:*

$$[cFs(At)]^{(n)} = \begin{cases} (A\ln\alpha)^n\, sFs(At), & \text{for } n \text{ odd,} \\ (A\ln\alpha)^n\, cFs(At), & \text{for } n \text{ even,} \end{cases}$$

$$[sFs(At)]^{(n)} = \begin{cases} (A\ln\alpha)^n\, cFs(At), & \text{for } n \text{ odd,} \\ (A\ln\alpha)^n\, sFs(At), & \text{for } n \text{ even,} \end{cases}$$

where $t \in \mathbb{R}$.

Proof. From the series expansions of the functions $cFs(A)$ and $sFs(A)$, we have

$$\begin{aligned} [cFs(At)]' &= \left[\frac{2}{\sqrt{5}}I + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^2}{2!}A^2t^2 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^4}{4!}A^4t^4 + \cdots\right]' \\ &= A(\ln\alpha)\left[\frac{2}{\sqrt{5}}(\ln\alpha)At + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^3}{3!}A^3t^3 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^5}{5!}A^5t^5 + \cdots\right] \\ &= (\ln\alpha)AsFs(At) \end{aligned}$$

$$\begin{aligned} [cFs(At)]'' &= [(\ln\alpha)AsFs(At)]' \\ &= \left((\ln\alpha)A\left[\frac{2}{\sqrt{5}}(\ln\alpha)At + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^3}{3!}A^3t^3 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^5}{5!}A^5t^5 + \cdots\right]\right)' \\ &= A^2(\ln\alpha)^2\left[\frac{2}{\sqrt{5}}I + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^2}{2!}A^2t^2 + \frac{2}{\sqrt{5}}\frac{(\ln\alpha)^4}{4!}A^4t^4 + \cdots\right] \\ &= A^2(\ln\alpha)^2 cFs(At) \\ &\ldots\ldots \end{aligned}$$

$$[cFs(A)]^{(n)} = \begin{cases} (A\ln\alpha)^n\, sFs(At), & \text{for } n \text{ odd,} \\ (A\ln\alpha)^n\, cFs(At), & \text{for } n \text{ even.} \end{cases}$$

Similarly, one can see that

$$[sFs(At)]^{(n)} = \begin{cases} (A\ln\alpha)^n\, cFs(At), & \text{for } n \text{ odd,} \\ (A\ln\alpha)^n\, sFs(At), & \text{for } n \text{ even.} \end{cases}$$

∎

Theorem 4.10. *The hyperbolic Fibonacci matrix functions have Moivre type equation:*

$$[cFs(A) \pm sFs(A)]^n = \left[\frac{2}{\sqrt{5}}\right]^{n-1} [cFs(nA) \pm sFs(nA)].$$

Proof.

$$\begin{aligned} \left[\frac{2}{\sqrt{5}}\right]^{n-1} [cFs(nA) \pm sFs(nA)] &= \left[\frac{2}{\sqrt{5}}\right]^{n-1} \left[\frac{\alpha^{nA} + \alpha^{-nA}}{\sqrt{5}} \pm \frac{\alpha^{nA} - \alpha^{-nA}}{\sqrt{5}}\right] \\ &= \left[\frac{2}{\sqrt{5}}\right]^{n} \alpha^{\pm nA} = \left[\frac{2}{\sqrt{5}} \alpha^{\pm A}\right]^{n} \\ &= \left[\frac{\alpha^{A} + \alpha^{-A}}{\sqrt{5}} \pm \frac{\alpha^{A} - \alpha^{-A}}{\sqrt{5}}\right]^{n} \\ &= [cFs(A) \pm sFs(A)]^n. \end{aligned}$$

∎

5. Conclusion

In this study, we introduce a new class of hyperbolic matrix functions called hyperbolic Fibonacci matrix functions and investigate their recursive and hyperbolic properties. The similar properties can be obtained for the hyperbolic Lucas matrix functions. Also, we think that some algorithms for the computing hyperbolic Fibonacci matrix functions based on Hermite matrix polynomial expansions can be derived.

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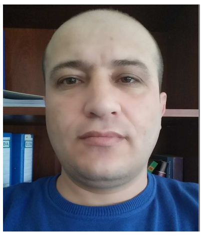

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