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q-STARLIKE FUNCTIONS OF ORDER ALPHA

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ABSTRACT. For all $q \in (0,1)$ and $0 \leq \alpha < 1$ we define a class of analytic functions, so-called q-starlike functions of order α on the open unit disc $\mathbb{D} = \{z : |z| < 1\}$. We will study this class of functions and explore some inclusion properties with the well-known class Starlike functions of order α.

Keywords: q-starlike functions, distortion theorem, growth theorem, coefficient inequality.

AMS Subject Classification: 30C45

1. INTRODUCTION

In the field of geometric functions theory, the concept of q-calculus (including fractional q-calculus) has been used by several authors. One may refer to the recent papers [6], [7], [8] and [9] on the subject. Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0, |\phi(z)| < 1$ for all $z \in \mathbb{D}$. We denote by $P(q)$ the family of functions of the form $p(z) = 1 + p_1(z) + p_2 z^2 + \cdots$ regular in the open unit disc $\mathbb{D}$ and satisfying

$$\left| p(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \qquad (z \in \mathbb{D}, q \in (0,1)) \tag{1}$$

and let us denote by $\mathcal{A}$ the class of functions $f(z)$ normalized by $f(0) = 0, f'(0) = 1$ that are analytic in the open unit disc $\mathbb{D}$. In other words, the function $f(z)$ in $\mathcal{A}$ have the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Let $f_1(z)$ and $f_2(z)$ be two elements of $\mathcal{A}$, if there exists a function $\phi(z) \in \Omega$ such that $f_1(z) = f_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$. If $f_2(z)$ is univalent, then $f_1(z) \prec f_2(z)$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ which implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, $\mathbb{D}_r = \{z : |z| < r < 1\}$. (Subordination principle [1]).

Let $|q| < 1$ be a fixed real number and we recall here q-fractional calculus for the analytic functions $f(z) \in \mathcal{A}$.

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(i) A subset $\mathbb{B}$ of $\mathbb{C}$ is called q-geometric, if $zq \in \mathbb{B}$ whenever $z \in \mathbb{B}$. If $\mathbb{B}$ is q-geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in \mathbb{B}$.

(ii) Let f be a function (real or complex valued) defined on q-geometric set $\mathbb{B}$, $|q| \neq 1$, the q-difference operator, which was introduced by Jackson [5] and may go back to E. Heine or Euler is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad (z \in \mathbb{B} - \{0\}) \tag{2}$$

The q-difference operator (2) sometimes called Jackson q-difference operator. If $0 \in \mathbb{B}$, the q-derivative at zero is defined for $|q| < 1$, by

$$D_q f(0) = \lim_{n\to\infty} \frac{f(q^n z) - f(0)}{zq^n}, \tag{3}$$

provided the limit exists and does not depend on z. In addition, q-derivative at zero is defined for $|q| > 1$, by

$$D_q f(0) = D_{q^{-1}} f(0).$$

Under the hypothesis of the definition of q-difference operator, we have the following rules [3]

$$D_q z^k = \frac{1-q^k}{1-q} z^{k-1},$$

therefore we have

(1) $D_q f(z) = D_q \left(z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots\right) = 1 + \sum_{n=2}^{\infty} \dfrac{1-q^n}{1-q} a_n z^{n-1}$.

(2) Let $f(z)$ and $g(z)$ be defined on a q-geometric set $\mathbb{B} \subset \mathbb{C}$ such that q-derivative of f and g exist for all $z \in \mathbb{B}$, then
   (a) $D_q\left(af(z) \pm bg(z)\right) = aD_q f(z) \pm bD_q g(z)$ where a and b are real or complex numbers.
   (b) $D_q\left(f(z) \cdot g(z)\right) = g(z)D_q f(z) + f(qz)D_q g(z)$.
   (c) $D_q\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{g(z)D_q f(z) - f(z)D_q g(z)}{g(z)g(qz)}, \quad g(z)g(qz) \neq 0.$
   (d) The q-differential is defined as

$$d_q f(z) = f(z) - f(qz),$$

therefore

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(z) - f(qz)}{(1-q)z} \Rightarrow d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} d_q z.$$

The following theorem is an analogue of the fundamental theorem of calculus.

Theorem 1.1 (Fundamental theorem of q-calculus, [3])**.** *If $F(z)$ is an antiderivative of $f(z)$ and $F(z)$ is continuous at $z = 0$, we have*

$$\int_a^b f(\zeta) d_q \zeta = F(b) - F(a)$$

where $0 \leq a < b \leq \infty$.

In this paper, we investigate a new class of analytic functions defined in the open unit disk, which are associated with q-calculus operators. In particular, we will give certain inclusion properties for the newly defined class of q-starlike functions of order α, namely,

$$S_q^*(\alpha) = \left\{ f(z) \in \mathcal{A} : z\frac{D_q f(z)}{f(z)} = \alpha + (1-\alpha)p(z), p(z) \in P(q), 0 \leq \alpha < 1 \right\}.$$

2. Main Results

For $p(z) \in P(q)$ it can be easily seen that

$$z\frac{D_q f(z)}{f(z)} = \alpha + (1-\alpha)p(z) \Rightarrow \frac{z\frac{D_q f(z)}{f(z)} - \alpha}{1-\alpha} \in P(q).$$

In a recent work of Polatoğlu et al. [4], authors proved that

Theorem 2.1. *$F(z) \in P(q)$ if and only if $F(z) \prec \frac{1+z}{1-qz}$.*

Therefore we have the following lemma.

Lemma 2.1. *$f(z) \in S_q^*(\alpha)$ if and only if*

$$\frac{z\frac{D_q f(z)}{f(z)} - \alpha}{1-\alpha} \prec \frac{1+z}{1-qz}.$$

Proof. The proof of the Lemma 2.1 is an immediate consequence of the above Theorem. □

Lemma 2.2. *Let the function $f(z) \in \mathcal{A}$, then $f(z) \in S_q^*(\alpha)$ if and only if*

$$z\frac{D_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

where $A = \frac{b^2-a^2+a}{b}, B = \frac{1-a}{b}, a = \frac{1-\alpha q}{1-q}$ *and* $b = \frac{1-\alpha}{1-q}$.

Proof. If $f(z) \in S_q^*(\alpha)$, then we have by (1)

$$\left|\left(z\frac{D_q f(z)}{f(z)} - \alpha\right) - \frac{1-\alpha}{1-q}\right| \leq \frac{1-\alpha}{1-q},$$

and

$$\left|z\frac{D_q f(z)}{f(z)} - \frac{1-\alpha q}{1-q}\right| \leq \frac{1-\alpha}{1-q}.$$

For brevity we say $\frac{1-\alpha q}{1-q} = a$ and $\frac{1-\alpha}{1-q} = b$, thus we have

$$\left|z\frac{D_q f(z)}{f(z)} - a\right| \leq b,$$

and

$$\left|\frac{1}{b}\cdot z\frac{D_q f(z)}{f(z)} - \frac{a}{b}\right| \leq 1.$$

Now we set

$$\psi(z) = \frac{1}{b}\cdot z\frac{D_q f(z)}{f(z)} - \frac{a}{b}.$$

then ψ is an analytic function and has a modulo at most one by (2). Furthermore, for $f(z) \in S_q^*(\alpha)$ we have

$$\psi(z) = \frac{1}{b}\left(z\frac{1+\sum_{n=2}^{\infty}\frac{1-q^n}{1-q}a_n z^{n-1}}{z+\sum_{n=2}^{\infty}a_n z^n}\right) - \frac{a}{b},$$

and

$$\psi(0) = \frac{1}{b} - \frac{a}{b} = \frac{1-a}{b}.$$

Therefore the function

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \overline{\psi(0)}\psi(z)} = \frac{z\frac{D_q f(z)}{f(z)} - 1}{b - \frac{1-a}{b}\left(z\frac{D_q f(z)}{f(z)} - a\right)},$$

satisfies the conditions of Schwarz lemma, then we have

$$z\frac{D_q f(z)}{f(z)} = \frac{1 + \frac{b^2-a^2+a}{b}\phi(z)}{1 + \frac{1-a}{b}\phi(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)},$$

and making use of subordination principle, one can easily see that

$$z\frac{D_q f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},$$

where $A = \frac{b^2-a^2+a}{b}$ and $B = \frac{1-a}{b}$. Conversely, if

$$z\frac{D_q f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},$$

then

$$z\frac{D_q f(z)}{f(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)},$$

where , $|\phi(z)| < 1, \phi(0) = 0$. Thus we have

$$z\frac{D_q f(z)}{f(z)} - a = b\left[\frac{\frac{1-a}{b} + \phi(z)}{1 + \frac{1-a}{b}\phi(z)}\right].$$

Since the linear transformation $\dfrac{\frac{1-a}{b} + \phi(z)}{1 + \frac{1-a}{b}\phi(z)}$ maps the unit disc onto itself, it follows from that

$$\left|z\frac{D_q f(z)}{f(z)} - a\right| = b\left|\frac{\frac{1-a}{b} + \phi(z)}{1 + \frac{1-a}{b}\phi(z)}\right| < |b|.$$

□

Theorem 2.2. *Let $f(z)$ be an element of $S_q^*(\alpha)$ then*

$$F_2(\alpha, q, r) \leq |f(z)| \leq F_1(\alpha, q, r), \quad (a \neq 1)$$

$$G_2(\alpha, q, r) \leq |f(z)| \leq G_1(\alpha, q, r), \quad (a = 1)$$

where

$$F_1(\alpha, q, r) = \left[r\left(1 + \frac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}},$$

$$F_2(\alpha, q, r) = \left[r\left(1 - \frac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}},$$

$$G_1(\alpha, q, r) = \left[e^{\frac{b^2-a^2+a}{b}r}\right] r^{\frac{1-q}{\log q^{-1}}},$$

$$G_2(\alpha, q, r) = \left[e^{-\frac{b^2-a^2+a}{b}r}\right] r^{\frac{1-q}{\log q^{-1}}}.$$

Furthermore, we have

$$M_2(\alpha,q,r)\le |D_qf(z)|\le M_1(\alpha,q,r),\quad (a\neq 1)$$
$$N_2(\alpha,q,r)\le |D_qf(z)|\le N_1(\alpha,q,r),\quad (a=1)$$

where

$$M_1(\alpha,q,r)=\frac{1}{r}\left[r\left(1+\frac{1-a}{b}\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}}\frac{1+Ar}{1+Br},$$

$$M_2(\alpha,q,r)=\frac{1}{r}\left[r\left(1-\frac{1-a}{b}\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}}\frac{1-Ar}{1-Br},$$

and

$$N_1(\alpha,q,r)=\frac{1}{r}e^{\frac{b^2-a^2+a}{b}r}r^{\frac{1-q}{\log q^{-1}}}(1+Ar),$$
$$N_2(\alpha,q,r)=\frac{1}{r}e^{-\frac{b^2-a^2+a}{b}r}r^{\frac{1-q}{\log q^{-1}}}(1-Ar).$$

Proof. Using the above Lemma (2.2), we can write if

$$z\frac{D_qf(z)}{f(z)}\prec\frac{1+Az}{1+Bz}$$

then

$$\left|z\frac{D_qf(z)}{f(z)}-\frac{1-ABr^2}{1-B^2r^2}\right|\le\frac{(A-B)\,r}{1-B^2r^2}. \tag{4}$$

Because the linear transformation $\dfrac{1+Az}{1+Bz}$ maps $|z|=r$ onto the disc with centre $c(r)=\left(\frac{1-ABr^2}{1-B^2r^2},0\right)$ and the radius $\rho(r)=\frac{(A-B)r}{1-B^2r^2}$ (this was proved by W. Janowski [2]), therefore using the subordination principle we can write

$$\left|z\frac{D_qf(z)}{f(z)}-\frac{1-ABr^2}{1-B^2r^2}\right|\le\frac{(A-B)\,r}{1-B^2r^2},\ \ (a\neq 1)$$
$$\left|z\frac{D_qf(z)}{f(z)}-1\right|\le Ar,\ \ (a=1) \tag{5}$$

Using q-differential properties and partial q-derivatives, we can write

$$\operatorname{Re} z\frac{D_qf(z)}{f(z)}=r\frac{\partial_q}{\partial r}\log\left|f(re^{i\theta})\right|. \tag{6}$$

Considering (5) and (6)together we obtain

$$\frac{1-Ar}{r(1-Br)}\le\frac{\partial_q}{\partial r}\log\left|f(re^{i\theta})\right|\le\frac{1+Ar}{1+Br},(a\neq 1) \tag{7}$$

$$\frac{1}{r}-A\le\frac{\partial_q}{\partial r}\log\left|f(re^{i\theta})\right|\le\frac{1}{r}+A,(a=1) \tag{8}$$

and taking q-integrals both sides of (7) and (8), we get

$$\left[r\left(1-\frac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}}\le|f(z)|\le\left[r\left(1+\frac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}},a\neq 1 \tag{9}$$

$$e^{-\frac{(b^2-a^2+a)r}{b}}r^{\frac{1-q}{\log q^{-1}}}\le|f(z)|\le e^{\frac{b^2-a^2+a}{b}r}r^{\frac{1-q}{\log q^{-1}}},a=1. \tag{10}$$

On the other hand, we have from (4) that

$$\frac{1-Ar}{1-Br} \le \left| z\frac{D_q f(z)}{f(z)} \right| \le \frac{1+Ar}{1+Br}, \quad (a \neq 1)$$
$$1-Ar \le \left| z\frac{D_q f(z)}{f(z)} \right| \le 1+Ar, \quad (a = 1)$$

and

$$\frac{1}{r}\left|f(z)\right|\frac{1-Ar}{1-Br} \le \left|D_q f(z)\right| \le \frac{1}{r}\left|f(z)\right|\frac{1+Ar}{1+Br}, (a \neq 1)$$
$$\frac{1}{r}\left|f(z)\right|(1-Ar) \le \left|D_q f(z)\right| \le \frac{1}{r}\left|f(z)\right|(1+Ar), (a = 1)$$

thus we obtain from (9) and (10)

$$\frac{1}{r}\left[r\left(1-\tfrac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}}\frac{1-Ar}{1-Br} \le \left|D_q f(z)\right| \le \frac{1}{r}\left[r\left(1+\tfrac{1-a}{b}r\right)^{\frac{b^2-(a-1)^2}{1-a}}\right]^{\frac{1-q}{\log q^{-1}}}\frac{1+Ar}{1+Br}, (a \neq 1)$$
$$\frac{1}{r}e^{-\frac{b^2-a^2+a}{b}r}\frac{1-q}{r\log q^{-1}}(1-Ar) \le \left|D_q f(z)\right| \le \frac{1}{r}e^{\frac{b^2-a^2+a}{b}r}r^{\frac{1-q}{\log q^{-1}}}(1+Ar), (a = 1).$$

□

All these inequalities in the Theorem 2.2 are sharp because extremal function is the solution of

$$z\frac{D_q f(z)}{f(z)} = \alpha + (1-\alpha)p(z) = \alpha + (1-\alpha)\frac{1+z}{1-qz}$$

q-differential equation.

3. Conclusion

We briefly consider some consequences of the results derived in the paper. In this paper, we investigate a new class of analytic functions defined in the open unit disk, which are associated with q-calculus operators. In particular, we gave certain inclusion properties for the newly defined class of q-starlike functions of order α. If we let $q \to 1^-$ and making use of the techniques from q-calculus, we observe that the function class $S_q^*(\alpha)$ and the inequalities of Theorem 2.2 provide the q-extensions of the known class and the related inequalities due to Janowski [2] (see also Goodman [1]).

We also conclude by remarking that the q-calculus operators defined in Section 1 can be used to investigate properties like, coefficient estimates, distortion theorems, etc. of several analytic (or meromorphic) function classes.

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