TWMS J. App. Eng. Math. V.8, N.1a, 2018, pp. 208-219

ON THE BEHAVIORS OF A CLASS OF SINGULAR TYPE ROUGH HIGHER ORDER COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper, we study the boundedness of a class of singular type rough higher order commutators defined by

$$T_{\Omega}^{A,m} f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \left(A(x) - A(y)\right)^m f(y) dy$$

and

$$M_{\Omega}^{A,m} f(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|x-y|<r} |\Omega(x-y)| \, |A(x) - A(y)|^m \, |f(y)| \, dy,$$

where $m \in \mathbb{N}$ and $\Omega \in L_s(S^{n-1})\,(s>1)$ is a homogeneous function of degree 0 on $\mathbb{R}^n$ and satisfies the integral zero property over the unit sphere S^{n-1} on generalized weighted Morrey spaces, respectively. As an application, we get the boundedness of these operators on weighted Morrey spaces, respectively. Keywords: Higher order ($=m$-th order) commutator operators, rough kernel, $A_{\frac{p}{s'}}$ weight, generalized weighted Morrey space.

AMS Subject Classification: 42B20, 42B25

1. Introduction and main results

Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0$, $x \in \mathbb{R}^n \setminus \{0\}$ and satisfy the cancellation condition

$$\int\limits_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

We first recall the definitions of the rough Calderón-Zygmund(C-Z) singular integral operator T_{Ω} and a related rough Hardy-Littlewood(H-L) maximal operator M_{Ω}.

Definition 1.1. *Let $f \in L_1^{loc}(\mathbb{R}^n)$. The the rough C-Z singular integral operator T_{Ω} and the rough H-L maximal operator M_{Ω} are defined by*

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§ Manuscript received: January 2, 2017; accepted: March 5, 2018.
TWMS Journal of Applied and Engineering Mathematics, Vol.8, No.1a,

$$T_\Omega f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|x-y|<r} |\Omega(x-y)| \, |f(y)| \, dy.$$

On the other hand, in 1965, Calderón [2] introduced the commutator $[A, S]$ on $\mathbb{R}$ which is defined by

$$\begin{aligned} [A, S] f(x) &= A(x) Sf(x) - S(Af)(x), \\ &= (-1) \, p.v. \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{A(x) - A(y)}{x-y} \frac{f(y)}{x-y} dy, \end{aligned}$$

where $A \in Lip(\mathbb{R})$ and the operator $S := \frac{d}{dx} \circ H$, H denotes the Hilbert transform defined by

$$Hf(x) = p.v. \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

The operator $[A, S]$ is also so called Calderón commutator. Note that the commutator $[A, S]$ can be rewritten as $\left[A, \sqrt{-\Delta}\right]$, where $\Delta = \frac{d^2}{dx^2}$ is the Laplacian operator in $\mathbb{R}$. Thus, the study of the commutator $[A, S]$ plays an important role in some characterizations of function spaces and so on (see [5] for example). Moreover, in [2], Calderón proved that if $A \in Lip(\mathbb{R})$, then the Calderón commutator $[A, S]$ is bounded on $L_p(\mathbb{R})$ for all $1 < p < \infty$. In the same paper [2], Calderón also gave a generalization of the commutator $[A, S]$ is in higher dimensions:

$$S_\Omega^{A,1} f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \frac{A(x) - A(y)}{|x-y|} f(y)dy.$$

Later, Bajsanski and Coifman [1] studied the generalized Calderón commutator as follows:

$$S_\Omega^{A,m} f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \frac{R_m(A; x, y)}{|x-y|^m} f(y)dy,$$

here $R_m(A; x, y)$ is the difference between a function $A(x)$ defined on $\mathbb{R}^n$ and its Taylor polynomial of degree $m-1$ with center y:

$$R_m(A; x, y) = A(x) - \sum_{|\gamma| \le m-1} \frac{1}{\gamma!} D^\gamma A(y) (x-y)^\gamma,$$

and we have used the notations: γ is a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}_+^n$. Moreover, $|\gamma| = \sum\limits_{i=1}^{n} \gamma_i$, $\gamma! = \prod\limits_{i=1}^{n} \gamma_i!$ and $x^\gamma = \prod\limits_{i=1}^{n} x_i^{\gamma_i}$. Similarly, provided that $D_j = \frac{\partial}{\partial x_j}$

$$D^\gamma A(x) = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} A(x) = \frac{\partial^{\sum\limits_{i=1}^{n} \gamma_i}}{\prod\limits_{i=1}^{n} \partial x_i^{\gamma_i}} A(x) = D_1^{\gamma_1} D_2^{\gamma_2} ... D_n^{\gamma_n} A(x)$$

is the partial derivative of A which is assumed to exist in the classical sense almost everywhere on $\mathbb{R}^n$.

Inspired by the above works, Cohen and Gosselin [3] introduced the following generalized commutator of T_Ω,

$$T_\Omega^A f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y) dy$$

and the corresponding generalized commutator of M_Ω is defined by

$$M_\Omega^A f(x) = \sup_{r>0} \frac{1}{r^{n+m-1}} \int\limits_{|x-y|<r} |\Omega(x-y) R_m(A;x,y) f(y)| dy,$$

where $\Omega \in L_s(S^{n-1})$ $(s>1)$ is a homogeneous function of degree 0 and satisfies (1), $m \in \mathbb{N}$, $R_m(A;x,y)$ is as above.

Thus, if $m=1$, T_Ω^A and M_Ω^A reduce to the commutators of T_Ω and M_Ω, respectively:

$$\begin{aligned} [A, T_\Omega] f(x) &= A(x) T_\Omega f(x) - T_\Omega (Af)(x) \\ &= p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y)) f(y) dy \end{aligned}$$

and

$$\begin{aligned} [A, M_\Omega] f(x) &= A(x) M_\Omega f(x) - M_\Omega (Af)(x) \\ &= \sup_{r>0} \frac{1}{r^n} \int\limits_{|x-y|<r} |\Omega(x-y)| |A(x) - A(y)| |f(y)| dy. \end{aligned}$$

On the other hand, since the commutator has a close relation with partial differential equations and pseudo-differential operator, the theory of higher order (= m-th order) commutator has been received extensive studies in the last 3 decades. In the following we list a few of them about a class of singular type higher order (= m-th order) commutator operators which are related to the study in this article.

Now, let us consider the following higher order (= m-th order) commutator operator of T_Ω:

$$\begin{aligned} T_\Omega^{A,m} f(x) &= T_\Omega\left((A(x) - A(\cdot))^m f(\cdot)\right)(x), \qquad m = 0,1,2,\ldots, \\ &= p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y))^m f(y) dy \end{aligned} \tag{2}$$

and the corresponding higher order (= m-th order) commutator operator of M_Ω:

$$\begin{aligned} M_\Omega^{A,m} f(x) &= M_\Omega\left((A(x) - A(\cdot))^m f(\cdot)\right)(x), \qquad m = 0,1,2,\ldots, \\ &= \sup_{r>0} \frac{1}{r^n} \int\limits_{|x-y|<r} |\Omega(x-y)| |A(x) - A(y)|^m |f(y)| dy. \end{aligned} \tag{3}$$

Moreover, the following pointwise inequality holds:

$$M_\Omega^{A,m} f(x) \le \widetilde{T}_{|\Omega|}^{A,m} (|f|)(x) \qquad x \in \mathbb{R}^n, \tag{4}$$

for all positive measurable function f. Indeed, in order to do this, we need to define an operator by

$$\widetilde{T}_{|\Omega|}^{A,m}\left(|f|\right)(x) = p.v.\int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \left|A\left(x\right)-A\left(y\right)\right|^m |f(y)|\, dy, \tag{5}$$

where $\Omega \in L_1(S^{n-1})\,(s>1)$ is a homogeneous function of degree 0 on $\mathbb{R}^n$. On the other hand, for any $r>0$, we get

$$\begin{aligned}\widetilde{T}_{|\Omega|}^{A,m}\left(|f|\right)(x) &\geq \int\limits_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^n} \left|A\left(x\right)-A\left(y\right)\right|^m |f(y)|\, dy \\ &\geq \frac{1}{r^n}\int\limits_{|x-y|<r} \left|\Omega\left(x-y\right)\right| \left|A\left(x\right)-A\left(y\right)\right|^m |f(y)|\, dy.\end{aligned}$$

Thus, taking the supremum for $r>0$ in the inequality above, we obtain (4), which completes the proof. Moreover, for $m=1$ above, $T_{\Omega}^{A,m}$ and $M_{\Omega}^{A,m}$ obviously reduce to the above commutators $[A,T_{\Omega}]$ and $[A,M_{\Omega}]$, respectively. Also, $T_{\Omega,\alpha}^{A,k}$ and $M_{\Omega,\alpha}^{A,k}$ are trivial generalizations of the above commutators $[A,T_{\Omega}]$ and $[A,M_{\Omega}]$, respectively.

Here and henceforth, $F\approx G$ means $F\gtrsim G\gtrsim F$; while $F\gtrsim G$ means $F\geq CG$ for a constant $C>0$; and p' always denotes the conjugate index of any $p>1$, that is, $\frac{1}{p'}:=1-\frac{1}{p}$ and also C stands for a positive constant that can change its value in each statement without explicit mention. Throughout the paper we assume that $x\in\mathbb{R}^n$ and $r>0$ and also let $B(x,r)$ denotes x-centred Euclidean ball with radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)|=v_n r^n$, where $v_n=|B(0,1)|$.

Now, we recall the definition of weighted Lebesgue spaces as follows.

Definition 1.2. (***Weighted Lebesgue space***) *Let $1\leq p\leq\infty$ and given a weight function $w\left(x\right)\in A_p\left(\mathbb{R}^n\right)$, we shall define weighted Lebesgue spaces as*

$$L_p(w)\equiv L_p(\mathbb{R}^n,w)=\left\{f: \|f\|_{L_{p,w}}=\left(\int\limits_{\mathbb{R}^n}|f(x)|^p w(x)dx\right)^{\frac{1}{p}}<\infty\right\}, \qquad 1\leq p<\infty.$$

$$L_{\infty,w}\equiv L_{\infty}(\mathbb{R}^n,w)=\left\{f: \|f\|_{L_{\infty,w}}=\operatorname*{esssup}_{x\in\mathbb{R}^n}|f(x)|w(x)<\infty\right\}.$$

Here and later, A_p denotes the Muckenhoupt classes. That is, $w\left(x\right)\in A_p\left(\mathbb{R}^n\right)$ for some $1<p<\infty$ if

$$\left(\frac{1}{|B|}\int\limits_B w(y)dy\right)\left(\frac{1}{|B|}\int\limits_B w(y)^{-\frac{1}{p-1}}dy\right)^{p-1}\leq C$$

for all balls B (see [6] for more details). By Hölder's inequality,

$$|B|\lesssim w\left(B\right)^{\frac{1}{p}}\left\|w^{-\frac{1}{p}}\right\|_{L_{p'}(B)} \tag{6}$$

is valid. Moreover, by (1.3) in [6] and (6),

$$\|w\|_{L_1(B)}^{1/p}\|w^{-1/p}\|_{L_{p'}(B)}\approx|B|$$

is also valid.

Set $\frac{p}{s'} > 1$. Since $w \in A_{\frac{p}{s'}}$, by (1.3) in [6] we get

$$\left\| w^{-\frac{1}{p}} \right\|_{L_{s'\left(\frac{p}{s'}\right)'}(B(x_0,t))} \lesssim t^{\frac{n}{s'}} \|w\|_{L_1(B(x_0,t))}^{-1/p}. \tag{7}$$

Also, for $s' < p < \infty$, it is clear that $w \in A_{\frac{p}{s'}}$ implies $w \in A_p$.

Suppose that $w \in A_p(\mathbb{R}^n)$, by the definition of $A_p(\mathbb{R}^n)$, we know that

$$w^{1-p'} \in A_{p'}(\mathbb{R}^n). \tag{8}$$

If $w \in A_{\frac{p}{s'}}$, by (8) we know

$$w^{1-\left(\frac{p}{s'}\right)'} \in A_{\left(\frac{p}{s'}\right)'}.$$

Since $w^{1-\left(\frac{p}{s'}\right)'} \in A_{\left(\frac{p}{s'}\right)'}$, by (1.3) in [6] we know

$$\left(w^{1-\left(\frac{p}{s'}\right)'} (B(x_0,t)) \right)^{\frac{1}{\left(\frac{p}{s'}\right)' s'}} \lesssim t^{\frac{n}{s'}} \|w\|_{L_1(B(x_0,t))}^{-1/p}. \tag{9}$$

It is known that $A_p \subset A_s$ if $1 \le p < s < \infty$, and that $w \in A_p$ for some $1 < p < s$ if $w \in A_s$ with $s > 1$, and also $[w]_{A_p} \le [w]_{A_s}$.

Now, let us list some definitions and known results:

Definition 1.3. (**BMO *function***) *Denote the bounded mean oscillation function space by*

$$BMO(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \|f\|_* := \sup_{B \subset \mathbb{R}^n} \mathcal{M}_{f,B} < \infty \right\},$$

here and in the sequel

$$\mathcal{M}_{f,B} := \frac{1}{|B|} \int_B |f(x) - f_B| dx, \qquad f_B = \frac{1}{|B|} \int_B f(y) dy.$$

Definition 1.4. (***Weighted* BMO *function***) *Denote the weighted bounded mean oscillation function space by*

$$BMO(\mathbb{R}^n, w) = \left\{ f \in L_{1,w}^{loc}(\mathbb{R}^n) \text{ and } w \in A_\infty(\mathbb{R}^n) : \|f\|_{*,w} := \sup_{B \subset \mathbb{R}^n} \mathcal{M}_{f,B,w} < \infty \right\},$$

here and in the sequel

$$\mathcal{M}_{f,B,w} := \frac{1}{w(B)} \int_B |f(x) - f_{B,w}| w(x) \, dx, \qquad f_{B,w} = \frac{1}{w(B)} \int_B f(y) w(y) \, dy.$$

Lemma 1.1. [4]

(*i*) *Suppose that* $1 \le p < \infty$, $w(x) \in A_\infty(\mathbb{R}^n)$, $f \in BMO(\mathbb{R}^n)$, $m > 0$, $x \in \mathbb{R}^n$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w(B(x,r_1))} \int_{B(x,r_1)} |f(y) - f_{B(x,r_2),w}|^{mp} w(y) dy \right)^{\frac{1}{p}} \lesssim \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^m \|f\|_*^m. \tag{10}$$

(ii) Suppose that $1 < p < \infty$, $w(x) \in A_p(\mathbb{R}^n)$, $f \in BMO(\mathbb{R}^n)$, $m > 0$, $x \in \mathbb{R}^n$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w^{1-p'}(B(x,r_1))} \int\limits_{B(x,r_1)} |f(y) - f_{B(x,r_2),w}|^{mp'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \lesssim \left(1 + \left|\ln \frac{r_1}{r_2}\right|\right)^m \|f\|_*^m.$$

Before giving the main results of this paper, we introduce another space which plays important roles in PDE. Except the weighted Lebesgue space $L_p(w)$, the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of $L_p(w)$ is another important function space. The weighted Morrey space $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w)$, $1 \leq p < \infty$, $0 < \kappa < 1$, is the collection of all classes of locally integrable functions f whose weighted Morrey space norm

$$\|f\|_{L_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \|w\|_{L_1(B(x,r))}^{-\frac{\kappa}{p}} \|f\|_{L_p(w,B(x,r))}$$

is finite. Note that for $\kappa = 0$, we have $L_{p,\kappa}(w) = L_p(w)$. This space was introduced in 2009 by Komori and Shirai in [7] in order to study the boundedness of classical operators in harmonic analysis. Then, Guliyev [4] has given a concept of generalized weighted Morrey spaces $M_{p,\varphi}(w)$ which could be viewed as extension of $L_{p,\kappa}(w)$. This generalization can be summarized as follows:

For $1 \leq p < \infty$, positive measurable function $\varphi(x,r)$ on $\mathbb{R}^n \times (0,\infty)$ and nonnegative measurable function w on $\mathbb{R}^n$, $f \in M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L_{p,w}^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|w\|_{L_1(B(x,r))}^{-\frac{1}{p}} \|f\|_{L_p(w,B(x,r))}$$

is finite. Note that for $\varphi(x,r) \equiv \|w\|_{L_1(B(x,r))}^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$ and $\varphi(x,r) \equiv \|w\|_{L_1(B(x,r))}^{-\frac{1}{p}}$, we have $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ and $M_{p,\varphi}(w) = L_p(w)$, respectively.

The aim of the present paper is to study the boundedness of the operators $T_\Omega^{A,m}$ and $M_\Omega^{A,m}$ generated by T_Ω and M_Ω with BMO functions on generalized weighted Morrey spaces, respectively. That is, in this paper we will consider this problem. As an application, we obtain the boundedness of these operators on weighted Morrey spaces, respectively.

Now, let us state our main results as follows.

Theorem 1.1. *Suppose that* $1 < p < \infty$, $s' < p$, $\Omega \in L_s(S^{n-1})$ $(s > 1)$ *is a homogeneous function of degree* 0 *and satisfies (1) such that* $m \in \mathbb{N}$, $w \in A_{\frac{p}{s'}}$, $A \in BMO(\mathbb{R}^n)$ *and* $T_\Omega^{A,m}$ *is defined as (2). Then,*

$$\begin{aligned} \left\| T_\Omega^{A,m} f \right\|_{L_p(w,B(x_0,r))} &\lesssim \|A\|_*^m \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \\ &\times \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt. \qquad (11) \end{aligned}$$

Theorem 1.2. *Suppose that* $1 < p < \infty$, $s' < p$, $\Omega \in L_s(S^{n-1})$ $(s > 1)$ *is a homogeneous function of degree* 0 *and satisfies (1) such that* $m \in \mathbb{N}$, $w \in A_{\frac{p}{s'}}$, $A \in BMO(\mathbb{R}^n)$, $T_\Omega^{A,m}$,

$M_{\Omega}^{A,m}$ *are defined as (2), (3) and the pair* (φ_1, φ_2) *satisfies the condition*

$$\int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|_{L_1(B(x,\tau))}^{\frac{1}{p}}}{\|w\|_{L_1(B(x,t))}^{\frac{1}{p}}} \frac{dt}{t} \lesssim \varphi_2(x,r). \tag{12}$$

Then,

$$\left\| T_{\Omega}^{A,m} f \right\|_{M_{p,\varphi_2}(w,\mathbb{R}^n)} \lesssim \|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)}, \tag{13}$$

$$\left\| M_{\Omega}^{A,m} f \right\|_{M_{p,\varphi_2}(w,\mathbb{R}^n)} \lesssim \|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)}. \tag{14}$$

Corollary 1.1. *Suppose that* $1 < p < \infty$, $s' < p$, $\Omega \in L_s(S^{n-1})\,(s > 1)$ *is a homogeneous function of degree* 0 *and satisfies (1) such that* $m \in \mathbb{N}$, $w \in A_{\frac{p}{s'}}$, $A \in BMO\,(\mathbb{R}^n)$, $0 < \kappa < 1$, $T_{\Omega}^{A,m}$, $M_{\Omega}^{A,m}$ *are defined as (2), (3) and the pair* (φ_1, φ_2) *satisfies the condition (12). Then,*

$$\left\| T_{\Omega}^{A,m} f \right\|_{L_{p,\kappa}(w,\mathbb{R}^n)} \lesssim \|f\|_{L_{p,\kappa}(w,\mathbb{R}^n)},$$

$$\left\| M_{\Omega}^{A,m} f \right\|_{L_{p,\kappa}(w,\mathbb{R}^n)} \lesssim \|f\|_{L_{p,\kappa}(w,\mathbb{R}^n)}.$$

2. Proofs of the main results

We begin with the proof of Theorem 1.1 which plays a great role in the proof of Theorem 1.2.

Proof of Theorem 1.1.

Proof. Without loss of generality, it is sufficient for us to show that the conclusion is true for $k = 2$ since there is no essential difference for the general case.

For any $x_0 \in \mathbb{R}^n$, we write as $f = f_1 + f_2$, where $f_1\,(y) = f\,(y)\,\chi_{B(x_0,2r)}\,(y)$, $f_2\,(y) = f\,(y)\,\chi_{(B(x_0,2r))^C}\,(y)$, $r > 0$ and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B\,(x_0, 2r)$. Then

$$\left\| T_{\Omega}^{A,2} f \right\|_{L_p(w,B(x_0,r))} \leq \left\| T_{\Omega}^{A,2} f_1 \right\|_{L_p(w,B(x_0,r))} + \left\| T_{\Omega}^{A,2} f_2 \right\|_{L_p(w,B(x_0,r))}.$$

Let us estimate $\left\| T_{\Omega}^{A,2} f_1 \right\|_{L_p(w,B(x_0,r))}$ and $\left\| T_{\Omega}^{A,2} f_2 \right\|_{L_p(w,B(x_0,r))}$, respectively.

From Corollary 7 in [8] and also by taking $\alpha = 0$ there, it is similar to the estimate of (2.8) in [6], we have

$$\begin{aligned}\left\|T_{\Omega}^{A,2}f_1\right\|_{L_p(w,B(x_0,r))} &\le \left\|T_{\Omega}^{A,2}f_1\right\|_{L_p(w,\mathbb{R}^n)}\\ &\lesssim \|A\|_*^2 \|f_1\|_{L_p(w,\mathbb{R}^n)}\\ &= \|A\|_*^2 \|f\|_{L_p(w,B(x_0,2r))}\\ &\lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.\\ &\lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}}\\ &\times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t}dt.\end{aligned}$$

Now, let us estimate the second part $\left(= \left\|T_{\Omega}^{A,2}f_2\right\|_{L_p(w,B(x_0,r))}\right)$. Firstly,

$$(A(x)-A(y))^2 = \left(A(x)-A_{B(x,r),w}\right)^2 - 2\left(A(x)-A_{B(x,r),w}\right)\left(A(y)-A_{B(x,r),w}\right) + \left(A(y)-A_{B(x,r),w}\right)^2$$

is valid. Next, for any given $x \in B(x_0, r)$ we have

$$\begin{aligned}\left|T_{\Omega}^{A,2}f_2(x)\right| &\lesssim \left|\left(A(x)-A_{B(x,r),w}\right)^2\right| |T_\Omega f_2(x)| +\\ &2\left|\left(A(x)-A_{B(x,r),w}\right)\right| \left|T_\Omega\left(\left(A(y)-A_{B(x,r),w}\right)f_2\right)(x)\right| +\\ &\left|T_\Omega\left(\left(A(y)-A_{B(x,r),w}\right)^2 f_2\right)(x)\right|\\ &:= F_1+F_2+F_3.\end{aligned}$$

(i) For the estimate used in F_1, we first have to prove the below inequality:

$$|T_\Omega f_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t}dt. \tag{15}$$

By (2.11) in [6], we get (15). Thus, we have

$$F_1 \lesssim \left|\left(A(x)-A_{B(x,r),w}\right)^2\right| \int_{2r}^{\infty} \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t}dt.$$

Later, taking $L_p(w, B(x_0, r))$-norm above and by (i) of the Lemma 1.1, we obtain

$$\begin{aligned}\|F_1\|_{L_p(w,B(x_0,r))} &\lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}}\\ &\times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t}dt.\end{aligned}$$

(ii) Second, for F_3, it is obvious that $x \in B(x_0, r)$, $y \in (B(x_0, 2r))^C$ implies $|x_0 - y| \approx |x - y|$. Thus, by Fubini's theorem, Hölder's inequality and (2.2) in [5], we get

$$F_3 = \left| T_\Omega \left(\left(A(y) - A_{B(x,r),w} \right)^2 f_2 \right)(x) \right|$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{2r<|x_0-y|\le t} \left| A(y) - A_{B(x_0,r),w} \right|^2 |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} \left| A(y) - A_{B(x_0,r),w} \right|^2 |f(y)| \, |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \int\limits_{2r}^{\infty} \left(\int\limits_{B(x_0,t)} \left| A(y) - A_{B(x_0,r),w} \right|^{2s'} |f(y)|^{s'} dy \right)^{\frac{1}{s'}} \left(\int\limits_{B(x_0,t)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \frac{dt}{t^{n+1}}$$

$$\lesssim \int\limits_{2r}^{\infty} \left(\int\limits_{B(x_0,t)} \left| A(y) - A_{B(x_0,r),w} \right|^{2s'} |f(y)|^{s'} dy \right)^{\frac{1}{s'}} |B(x_0, 2t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}}. \qquad (16)$$

On the other hand, set $\nu = \frac{p}{s'}$. From $w \in A_\nu$, we know $w^{1-\nu'} \in A_{\nu'}$. Since $s' < p$, it follows from Hölder's inequality that

$$\left(\int\limits_{B(x_0,t)} \left| \left(A(y) - A_{B(x_0,r),w} \right) \right|^{2s'} |f(y)|^{s'} dy \right)^{\frac{1}{s'}} \lesssim \|f\|_{L_p(w,B(x_0,t))} \left\| \left(A(\cdot) - A_{B(x_0,r),w} \right)^2 \right\|_{L_{s'\nu'}\left(w^{1-\nu'},B(x_0,t)\right)}. \qquad (17)$$

Later, by (ii) of the Lemma 1.1 and using (9), we get

$$\left\| \left(A(\cdot) - A_{B(x_0,r),w} \right)^2 \right\|_{L_{s'\nu'}\left(w^{1-\nu'},B(x_0,t)\right)} = \left(\int\limits_{B(x_0,t)} \left| A(y) - A_{B(x_0,r),w} \right|^{2s'\nu'} w^{1-\nu'}(y) \, dy \right)^{\frac{1}{s'\nu'}}$$

$$\lesssim \|A\|_*^2 \left(1 + \ln \frac{t}{r} \right)^2 \left(w^{1-\nu'}(B(x_0,t)) \right)^{\frac{1}{\nu' s'}}$$

$$\lesssim \|A\|_*^2 \left(1 + \ln \frac{t}{r} \right)^2 t^{\frac{n}{s'}} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}}. \qquad (18)$$

At last, substituting (17) and (18) into (16), we get

$$F_3 = \left| T_\Omega \left(\left(A(y) - A_{B(x,r),w} \right)^2 f_2 \right)(x) \right|$$

$$\lesssim \|A\|_*^2 \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

Thus, taking $L_p(w, B(x_0, r))$-norm above

$$\|F_3\|_{L_p(w,B(x_0,r))} \lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

Similarly, F_2 has the same estimate above, that is, it is analogous to the estimates of $F_3 = \left|T_\Omega \left(\left(A(y) - A_{B(x,r),w}\right)^2 f_2\right)(x)\right|$ above, we have

$$\left|T_\Omega \left(\left(A(y) - A_{B(x,r),w}\right) f_2\right)(x)\right| \lesssim \|A\|_* \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt. \tag{19}$$

Thus, by (ii) of the Lemma 1.1 and (19), we get

$$F_2 = 2\left|\left(A(x) - A_{B(x,r),w^q}\right)\right| \left|T_{\Omega,\alpha} \left(\left(A(y) - A_{B(x,r),w^q}\right) f_2\right)(x)\right| \lesssim \|A\|_*^2 \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

Here we omit the details, thus the inequality

$$\|F_2\|_{L_p(w,B(x_0,r))} \lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

is valid.

Putting estimates $\|F_1\|_{L_p(w,B(x_0,r))}$, $\|F_2\|_{L_p(w,B(x_0,r))}$, $\|F_2\|_{L_p(w,B(x_0,r))}$ together, we get the desired conclusion

$$\left\|T_\Omega^{A,2} f_2\right\|_{L_p(w,B(x_0,r))} \lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

Combining all the estimates for $\left\|T_\Omega^{A,2} f_1\right\|_{L_p(w,B(x_0,r))}$ and $\left\|T_\Omega^{A,2} f_2\right\|_{L_p(w,B(x_0,r))}$, we get

$$\left\|T_\Omega^{A,2} f\right\|_{L_p(w^p,B(x_0,r))} \lesssim \|A\|_*^2 \|w\|_{L_1(B(x_0,r))}^{\frac{1}{p}} \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|f\|_{L_p(w,B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-\frac{1}{p}} \frac{1}{t} dt.$$

Therefore, Theorem 1.1 is completely proved. □

Proof of Theorem 1.2.

Proof. We consider (13) firstly. By the proof of Theorem 2.2. in [6],

$$\frac{\|f\|_{L_p(w,B(x_0,t))}}{\operatorname*{essinf}_{0<t<\tau<\infty}\varphi_1(x_0,\tau)\|w\|^{1/p}_{L_1(B(x_0,\tau))}} \lesssim \|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)} \tag{20}$$

is valid. Later, for $s' < p < \infty$ and $w \in A_{\frac{p}{s'}}$, since (φ_1, φ_2) satisfies (12) and by (20), we have

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \|f\|_{L_p(w,B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_1(B(x_0,t))}\frac{1}{t}dt$$

$$\lesssim \int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \frac{\|f\|_{L_p(w,B(x_0,t))}}{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\|w\|^{\frac{1}{p}}_{L_1(B(x_0,\tau))}}\frac{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\|w\|^{\frac{1}{p}}_{L_1(B(x_0,\tau))}}{\|w\|^{\frac{1}{p}}_{L_1(B(x_0,t))}}\frac{dt}{t}$$

$$\lesssim \|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)}\int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \frac{\operatorname*{essinf}_{t<\tau<\infty}\varphi_1(x_0,\tau)\|w\|^{\frac{1}{p}}_{L_1(B(x_0,\tau))}}{\|w\|^{\frac{1}{p}}_{L_1(B(x_0,t))}}\frac{dt}{t}$$

$$\lesssim \|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)}\,\varphi_2(x_0,r). \tag{21}$$

At last, by (11) and (21), we get

$$\begin{aligned}\left\|T^{A,m}_\Omega f\right\|_{M_{p,\varphi_2}(w,\mathbb{R}^n)} &= \sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}\,\|w\|^{-\frac{1}{p}}_{L_1(B(x_0,r))}\left\|T^{A,m}_\Omega f\right\|_{L_p(w,B(x_0,r))}\\ &\lesssim \|A\|^m_*\sup_{x_0\in\mathbb{R}^n,r>0}\varphi_2(x_0,r)^{-1}\\ &\times\int_r^\infty\left(1+\ln\frac{t}{r}\right)^m\|f\|_{L_p(w,B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_1(B(x_0,t))}\frac{1}{t}dt\\ &\lesssim \|A\|^m_*\,\|f\|_{M_{p,\varphi_1}(w,\mathbb{R}^n)}.\end{aligned}$$

Hence, we have completed the proof of (13). □

We are now in a place of proving (14) in Theorem 1.2. The conclusion of (14) is a direct consequence of (13) and (4). Indeed, from the process proving (13), it is easy to see that the conclusions of (13) also hold for $\widetilde{T}^{A,m}_{|\Omega|}$ defined as (5). Combining this with (4), we can immediately obtain (14), which completes the proof of (14).

3. Conclusions

In this work, we study the boundedness of the higher order commutators of singular integral and maximal operators with rough kernels. Under the conditions that the rough kernels belong to $L_s(S^{n-1})\,(s>1)$, some bounds for the above operators on the generalized weighted Morrey spaces were established. As applications, the boundedness of these operators on weighted Morrey spaces are also obtained.

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