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EXISTENCE OF SYMMETRIC POSITIVE SOLUTIONS FOR LIDSTONE TYPE INTEGRAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper establishes the existence of even number of symmetric positive solutions for the even order differential equation

$$(-1)^n u^{(2n)}(t) = f(t, u(t)),\ t \in (0, 1),$$

satisfying Lidstone type integral boundary conditions of the form

$$u^{(2i)}(0) = u^{(2i)}(1) = \int_0^1 a_{i+1}(x) u^{(2i)}(x) dx,\ \text{for } 0 \le i \le n-1,$$

where $n \ge 1$, by applying Avery–Henderson fixed point theorem.

Key words: Green's function, integral boundary conditions, cone, positive solution, fixed point theorem.

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1. Introduction

In many branches of applied mathematics, the goal is to formulate the mathematical model of the real world problems by analyzing the given situations. Most of these models involve the rate of change of the dependent variable, which will form differential equations. The theory of differential equations offers a broad mathematical basis to understand the problems of modern society, which are complex and interdisciplinary by nature.

The existence of positive solutions [1] of the boundary value problems (BVPs) have created a great deal of interest due to wide applicability in both theory and applications. Davis and Henderson [6], Wong and Agarwal [16], Davis, Henderson and Wong [7], Ehme and Henderson [8], Bai and Ge [3] and Zhang and Liu [19] considered Lidstone type BVPs associated with ordinary differential equations and established the existence of positive solutions to the boundary value problems by using various methods.

Recently, there is an increasing interest shown in establishing the existence of positive solutions for boundary value problems (BVPs) with integral boundary conditions, see [4, 5, 9, 10, 11, 12, 15, 20, 21, 22]. Recent results indicate that considerable achievement has been made in the existence of positive solutions of the boundary value problems. However they did not further provide characteristics of positive solutions such as symmetry.

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Symmetry has been widely used in science, engineering and technology. The reason is that the symmetry has not only its theoretical value in studying the metric manifolds and symmetric graph and so forth, but also its practical value, for example, we can apply this characteristic to study graph structures and chemistry structures. However, the existence of symmetric positive solutions for BVPs with integral boundary conditions are still very few, see [13, 14, 17, 18].

Motivated by the papers mentioned above, we extend the results to $2n^{\text{th}}$ order boundary value problem with integral boundary conditions of the form

$$(-1)^n u^{(2n)}(t) = f(t, u(t)),\ t \in (0, 1), \tag{1}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = \int_0^1 a_{i+1}(x) u^{(2i)}(x)dx,\ \text{for } 0 \le i \le n-1, \tag{2}$$

where $n \ge 1$, and establish the existence of even number symmetric positive solutions by applying Avery–Henderson fixed point theorem.

We assume the following conditions hold throughout this paper:

($A1$) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and $f(t, u)$ is symmetric on $[0, 1]$ for all $u \in [0, \infty)$, i.e., $f(1-t, u) = f(t, u)$ for all $t \in [0, 1]$ and $u \in [0, \infty)$,

($A2$) $a_j \in L^1[0, 1]$, $a_j(x) > 0$ and $d_j = \int_0^1 a_j(x)dx \in (0, 1)$ for $1 \le j \le n$.

This paper is organized as follows. In section 2.2, we derive Green's function for the homogeneous boundary value problem corresponding to (1)-(2) and obtain bounds for the Green's function. In section 2.3, we develop criteria for the existence of at least two symmetric positive solutions of the BVP (1)-(2) by using Avery–Henderson fixed point theorem. We also establish the existence of at least $2m$ symmetric positive solutions to the BVP (1)-(2) for an arbitrary positive integer m. Finally, we give an example to illustrate our results.

2. Preliminary Results

In this section, we construct the Green function for the homogeneous boundary value problem corresponding to (1)-(2) and estimate the bounds for the Green's function. We prove certain lemmas which are needed in establishing further results of this paper.

First we compute the Green's function $G_j(t, s)$, $1 \le j \le n$, for the second order homogeneous BVP,

$$-u''(t) = 0,\ t \in (0, 1), \tag{3}$$

$$u(0) = u(1) = \int_0^1 a_j(x) u(x)dx,\ \text{for } 1 \le j \le n, \tag{4}$$

and then obtain the bounds for this Green's function. Using this Green's function, the Green's function for the homogeneous boundary value problem corresponding to (1)-(2) is constructed and bounds for the Green's function are estimated.

Lemma 2.1. *Suppose that* $d_j = \int_0^1 a_j(x)dx \in (0, 1)$, *for* $1 \le j \le n$. *If* $h(t) \in (C[0, 1], \mathbb{R}^+)$, *then the BVP,*

$$u'' + h(t) = 0,\ t \in (0, 1), \tag{5}$$

satisfying (4) *has a unique solution*

$$u(t) = \int_0^1 G_j(t, s)h(s)ds,\ \text{for } 1 \le j \le n,$$

where

$$G_j(t, s) = G(t, s) + \frac{1}{(1-d_j)} \int_0^1 G(x, s) a_j(x)dx,\ \text{for } 1 \le j \le n, \tag{6}$$

and

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases} \tag{7}$$

Proof. Integrating both sides of (5) from 0 to t, we have

$$u'(t)=-\int_0^t h(s)ds+B, \tag{8}$$

where $B=u'(0)$. Again integrating (8) from 0 to t, we get

$$u(t)=-\int_0^t\left(\int_0^x h(s)ds\right)dx+Bt+A,$$

which gives that

$$u(t)=-\int_0^t (t-s)h(s)ds+Bt+A, \tag{9}$$

where $A=u(0)$. In particular, $u(1)=-\int_0^1(1-s)h(s)ds+Bt+A$. Using the boundary conditions (4), we get

$$B=\int_0^1(1-s)h(s)ds \tag{10}$$

and

$$\begin{aligned}
A&=\int_0^1 a_j(x)u(x)dx\\
&=\int_0^1 a_j(x)\Big[-\int_0^x(x-s)h(s)ds+Bx+A\Big]dx\\
&=\int_0^1 a_j(x)\Big[-\int_0^x(x-s)h(s)ds+x\int_0^1(1-s)h(s)ds\Big]dx+Ad_j\\
&=\int_0^1 a_j(x)\Big[-\int_0^x(x-s)h(s)ds\\
&\quad+x\Big(\int_0^x(1-s)h(s)ds+\int_x^1(1-s)h(s)ds)\Big)\Big]dx+Ad_j\\
&=\int_0^1 a_j(x)\Big[\int_0^x s(1-x)h(s)ds+\int_x^1 x(1-s)h(s)ds\Big]dx+Ad_j\\
&=\int_0^1 a_j(x)\Big[\int_0^1 G(x,s)h(s)ds\Big]dx+Ad_j\\
&=\int_0^1\Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds+Ad_j,
\end{aligned}$$

which implies that

$$A=\frac{1}{(1-d_j)}\int_0^1\Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds. \tag{11}$$

From (9), (10) and (11), the solution of boundary value problem (5), (4) is

$$\begin{aligned}
u(t)=&-\int_0^t(t-s)h(s)ds+t\int_0^1(1-s)h(s)ds\\
&+\frac{1}{(1-d_j)}\int_0^1\Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t (t-s)h(s)ds + t\Big[\int_0^t \big[(1-s)h(s)ds \\
&+ \int_t^1 (1-s)h(s)ds\Big] + \frac{1}{(1-d_j)}\int_0^1 \Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds \\
&= \int_0^t s(1-t)h(s)ds + \int_t^1 t(1-s)h(s)ds \\
&+ \frac{1}{(1-d_j)}\int_0^1 \Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds + \frac{1}{(1-d_j)}\int_0^1 \Big[\int_0^1 G(x,s)a_j(x)dx\Big]h(s)ds \\
&= \int_0^1 G_j(t,s)h(s)ds.
\end{aligned}
$$

□

Lemma 2.2. *Assume that the conditions (A2) is satisfied. Then $G(t,s)$ and $G_j(t,s)$ $(1 \le j \le n)$ satisfies the following inequalities:*

(i) *$G(t,s) > 0$ and $G_j(t,s) > 0$, for all $t,s \in (0,1)$,*
(ii) *$G(1-t,1-s) = G(t,s)$ and $G(s,s)G(t,t) \le G(t,s) \le G(s,s)$, for all $t,s \in [0,1]$,*
(iii) *$\xi_j G_j(s,s) \le G_j(t,s) \le G_j(s,s)$, for all $t,s \in [0,1]$,*

where

$$\xi_j = \frac{\eta_j}{(1-d_j+\eta_j)} \in (0,1), \tag{12}$$

and

$$\eta_j = \int_0^1 G(x,x)a_j(x)dx.$$

Proof. We can easily establish the inequalities (i) and (ii). For the inequality (iii), let

$$E_j(s) = \frac{1}{(1-d_j)}\int_0^1 G(x,s)a_j(x)dx, \text{ for } 1 \le j \le n.$$

From (ii), the second inequality of (iii) is obvious, we prove the first inequality of (iii). Using the inequality $G(s,s)G(t,t) \le G(t,s)$, then for $t,s \in [0,1]$, we have

$$
\begin{aligned}
E_j(s) &\ge \frac{1}{(1-d_j)}\int_0^1 G(s,s)G(x,x)a_j(x)dx \\
&= \frac{\eta_j}{(1-d_j)}G(s,s),
\end{aligned}
$$

which implies that

$$(1-d_j)E_j(s) \ge \eta_j G(s,s).$$

So,

$$
\begin{aligned}
(1-d_j+\eta_j)E_j(s) &\ge \eta_j[G(s,s)+E_j(s)] \\
&= \eta_j G_j(s,s).
\end{aligned}
$$

Subsequently,

$$
\begin{aligned}
E_j(s) &\ge \frac{\eta_j}{(1-d_j+\eta_j)}G_j(s,s) \\
&= \xi_j G_j(s,s).
\end{aligned}
$$

Therefore,

$$\begin{aligned} G_j(t,s) &= G(t,s) + E_j(s) \\ &\geq E_j(s) \\ &\geq \xi_j G_j(s,s). \end{aligned}$$

□

Lemma 2.3. *Assume that the condition (A2) is satisfied. Let $G_1(t,s) = H_1(t,s)$ and recursively define*

$$H_j(t,s) = \int_0^1 H_{j-1}(t,r) G_j(r,s) dr, \ \text{for } 2 \leq j \leq n. \tag{13}$$

Then the Green's function for the homogeneous boundary value problem corresponding to (1)-(2) is $H_n(t,s)$, where $G_j(t,s)(1 \leq j \leq n)$ is given in (6).

Lemma 2.4. *Assume that the condition (A2) is satisfied. If we define $K = \prod_{j=1}^{n-1} K_j$ and $L = \prod_{j=1}^{n-1} \xi_j K_j$, then the Green's function $H_n(t,s)$ in (13) satisfies the following inequalities:*

(*i*) $0 \leq H_n(t,s) \leq K G_n(s,s)$, *for all* $t, s \in [0,1]$, *and*

(*ii*) $H_n(t,s) \geq \xi_n L G_n(s,s)$, *for all* $t, s \in [0,1]$,

where ξ_n is given in (12) and $K_j = \int_0^1 G_j(s,s) ds$, for $1 \leq j \leq n$.

Lemma 2.5. *Assume that the condition (A2) is satisfied. Then the Green's function $H_j(t,s)$ $(1 \leq j \leq n)$ satisfies the symmetric property,*

$$H_j(t,s) = H_j(1-t, 1-s), \ \text{for all } t, s \in [0,1]. \tag{14}$$

Proof. By the definition of $H_j(t,s), (2 \leq j \leq n)$,

$$H_j(t,s) = \int_0^1 H_{j-1}(t,r) G_j(r,s) dr$$

The proof is by induction. First, for $j = 1$, the equation (14) is obvious. Next, we assume that the equation (14) is true for fixed $j \geq 2$. Then from (13) and using the transformation $r_1 = 1 - r$, we have

$$\begin{aligned} H_{j+1}(t,s) &= \int_0^1 H_j(t,r) G_{j+1}(r,s) dr \\ &= \int_0^1 H_j(1-t, 1-r) G_{j+1}(1-r, 1-s) dr \\ &= \int_0^1 H_j(1-t, r_1) G_{j+1}(r_1, 1-s) dr_1 \\ &= H_{j+1}(1-t, 1-s). \end{aligned}$$

The proof is complete. □

3. Even Number of Positive Solutions

In this section, we establish the existence of at least two symmetric positive solutions for the boundary value problem (1)-(2) by Avery–Henderson fixed point theorem . And then, we establish the existence of at least $2m$ symmetric positive solutions to the boundary value problem (1)-(2) for an arbitrary positive integer m.

Let B be a real Banach space. A nonempty closed convex set $P \subset B$ is called a cone, if it satisfies the following conditions:

(i) $y \in P$, $\lambda \geq 0$ implies $\lambda y \in P$, and
(ii) $y \in P$ and $-y \in P$ implies $y = 0$.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for nonnegative real numbers a' and b', we define the sets

$$P(\psi, a') = \{y \in P : \psi(y) < a'\}$$

and

$$P_{b'} = \{y \in P : \|y\| < b'\}.$$

In obtaining multiple symmetric positive solutions of the boundary value problem (1)-(2), the following Avery–Henderson functional fixed point theorem [2] will be the fundamental tool.

Theorem 3.1. [2] *Let P be a cone in a real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \leq \theta(y) \leq \alpha(y)$ and $\|y\| \leq k\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \leq \lambda\theta(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial P(\theta, b')$. Further, let $T : \overline{P(\gamma, c')} \to P$ be a completely continuous operator such that*

$(B1)$ $\gamma(Ty) > c'$, for all $y \in \partial P(\gamma, c')$,
$(B2)$ $\theta(Ty) < b'$, for all $y \in \partial P(\theta, b')$,
$(B3)$ $P(\alpha, a') \neq \emptyset$ and $\alpha(Ty) > a'$, for all $y \in \partial P(\alpha, a')$.

Then T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

Let

$$M = \prod_{j=1}^{n} \xi_j. \tag{15}$$

Let $B = \{u : u \in C[0, 1]\}$ be the Banach space equipped with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$. Define the cone $P \subset B$ by

$$P = \big\{u \in B : u(t) \geq 0, u(t) \text{ is symmetric on } [0, 1] \text{ and } \min_{t \in [0,1]} u(t) \geq M\|u\|\big\},$$

where M is given in (15).

Define the nonnegative increasing continuous functionals γ, θ and α on the cone P by

$$\gamma(u) = \min_{t \in [0,1]} u(t), \ \theta(u) = \max_{t \in [0,1]} u(t) \text{ and } \alpha(u) = \max_{t \in [0,1]} u(t).$$

We observe that for any P,

$$\gamma(u) \leq \theta(u) = \alpha(u) \tag{16}$$

and

$$\|u\| \leq \frac{1}{M} \min_{t \in [0,1]} u(t) = \frac{1}{M}\gamma(u) \leq \frac{1}{M}\alpha(u). \tag{17}$$

Theorem 3.2. *Assume that the conditions $(A1)$-$(A2)$ are satisfied. Suppose that there exist real numbers a', b' and c' with $0 < a' < b' < c'$ such that f satisfies the following conditions:*

$(D1)$ $f(t, u) > \frac{c'}{\prod_{j=1}^{n} \xi_j K_j}$, for $t \in [0, 1]$ and $u \in [c', \frac{c'}{M}]$,
$(D2)$ $f(t, u) < \frac{b'}{\prod_{j=1}^{n} K_j}$, for $t \in [0, 1]$ and $u \in [0, \frac{b'}{M}]$,
$(D3)$ $f(t, u) > \frac{a'}{\prod_{j=1}^{n} \xi_j K_j}$, for $t \in [0, 1]$ and $u \in [a', \frac{a'}{M}]$.

Then the boundary value problem (1)-(2) *has at least two symmetric positive solutions.*

Proof. Define the operator $T : P \to B$ by

$$Tu(t) = \int_0^1 H_n(t,s)f(s,u(s))ds. \tag{18}$$

It is obvious that a fixed point of T is the solution of the boundary value problem (1)-(2). We seek two fixed points $u_1, u_2 \in P$ of T. First we show that $T : P \to P$. Let $u \in P$. Clearly, $Tu(t) \geq 0$ on $[0,1]$. Noticing that $f(t,u)$ is symmetric on $[0,1]$, we have

$$\begin{aligned} Tu(1-t) &= \int_0^1 H_n(1-t,s)f(s,u(s))ds \\ &= \int_1^0 H_n(1-t,1-s)f(1-s,u(1-s))d(1-s) \\ &= \int_0^1 H_n(t,s)f(s,u(s))ds \\ &= Tu(t). \end{aligned}$$

Therefore, T is symmetric on $[0,1]$.
On the other hand, by Lemma 2.4, we obtain

$$\begin{aligned} Tu(t) &= \int_0^1 H_n(t,s)f(s,u(s))ds \\ &\leq K\int_0^1 G_n(s,s)f(s,u(s))ds \end{aligned}$$

so that

$$\|Tu(t)\| \leq K\int_0^1 G_n(s,s)f(s,u(s))ds. \tag{19}$$

Next, if $u \in P$, then from Lemma 2.4 and (19), we have

$$\begin{aligned} \min_{t\in[0,1]} Tu(t) &= \min_{t\in[0,1]} \int_0^1 H_n(t,s)f(s,u(s))ds \\ &\geq \xi_n L\int_0^1 G_n(s,s)f(s,u(s))ds \\ &\geq \xi_n\left(\prod_{j=1}^{n-1}\xi_j\right)\|Tu\| \\ &= M\|Tu\|. \end{aligned}$$

Hence $Tu \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (16) and (17), for each $u \in P$, we have $\gamma(u) \leq \theta(u) \leq \alpha(u)$ and $\|u\| \leq \frac{1}{M}\gamma(u)$. Also, for any $0 \leq \lambda \leq 1$ and $u \in P$, we have $\theta(\lambda u) = \max_{t\in[0,1]}(\lambda u)(t) = \lambda \max_{t\in[0,1]} u(t) = \lambda\theta(u)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of the Theorem 3.1 are satisfied.

Firstly, we shall verify that the condition $(B1)$ of Theorem 3.1 is satisfied. Since $u \in \partial P(\gamma, c')$, from (17) we have that $c' = \min_{t\in[0,1]} u(t) \le \|u\| \le \frac{c'}{M}$, for $t \in [0,1]$. Then,

$$\begin{aligned}
\gamma(Tu) &= \min_{t\in[0,1]} \int_0^1 H_n(t,s) f(s,u(s)) ds \\
&\ge \xi_n L \int_0^1 G_n(s,s) f(s,u(s)) ds \\
&> \frac{c'}{\prod_{j=1}^n \xi_j K_j} \xi_n L \int_0^1 G_n(s,s) ds \\
&= c',
\end{aligned}$$

using hypothesis $(D1)$.

Now, we shall show that condition $(B2)$ of Theorem 3.1 is satisfied. Since $u \in \partial P(\theta, b')$, from (17) we have that $0 \le u(t) \le \|u\| \le \frac{b'}{M}$, for $t \in [0,1]$. Thus,

$$\begin{aligned}
\theta(Tu) &= \max_{t\in[0,1]} \int_0^1 H_n(t,s) f(s,u(s)) ds \\
&\le K \int_0^1 G_n(s,s) f(s,u(s)) ds \\
&< \frac{b'}{\prod_{j=1}^n K_j} K \int_0^1 G_n(s,s) ds \\
&= b',
\end{aligned}$$

by hypothesis $(D2)$.

Finally, using hypothesis $(D3)$, we shall show that condition $(B3)$ of Theorem 3.1 is satisfied. Since $0 \in P$ and $a' > 0, P(\alpha, a') \neq \emptyset$. Since $u \in \partial P(\alpha, a')$, $a' = \max_{t\in[0,1]} u(t) \le \|u\| \le \frac{a'}{M}$, for $t \in [0,1]$. Therefore,

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,1]} \int_0^1 H_n(t,s) f(s,u(s)) ds \\
&\ge \int_0^1 H_n(t,s) f(s,u(s)) ds \\
&\ge \xi_n L \int_0^1 G_n(s,s) f(s,u(s)) ds \\
&> \frac{a'}{\prod_{j=1}^n \xi_j K_j} \xi_n L \int_0^1 G_n(s,s) ds \\
&= a'.
\end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. So there exist at least two symmetric positive solutions $u_1, u_2 \in \overline{P(\gamma, c')}$ for the boundary value problem $(1)-(2)$. This completes the proof of the theorem.

□

Theorem 3.3. *Let m be an arbitrary positive integer. Suppose there exist real numbers $a_r (r = 1, 2, \ldots, m+1)$ and $b_s (s = 1, 2, \ldots, m)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m <$*

$b_m < a_{m+1}$ *such that* f *satisfies following conditions:*

$$f(t,u) > \frac{a_r}{\prod_{j=1}^{n} \xi_j K_j}, \text{ for } t \in [0,1] \text{ and } u \in \left[a_r, \frac{a_r}{M}\right], \quad r = 1,2,\ldots,m+1. \tag{20}$$

$$f(t,u) < \frac{b_s}{\prod_{j=1}^{n} K_j}, \text{ for } t \in [0,1] \text{ and } u \in \left[0, \frac{b_s}{M}\right], \quad s = 1,2,\ldots,m. \tag{21}$$

Then the boundary value problem (1)-(2) *has at least* $2m$ *symmetric positive solutions in* $\bar{P}_{a_{m+1}}$.

Proof. We use induction on m. For $m = 1$, from (20), and (21), it is clear that $T : \bar{P}_{a_2} \to P_{a_2}$, then it follows from Avery–Henderson fixed point theorem that the boundary value problem (1)-(2) has at least two symmetric positive solutions in $\bar{P}_{a_2}$. Let us assume that this conclusion holds for $m = l$. In order to prove this conclusion holds for $m = l + 1$, we suppose that there exist real numbers $a_r(r = 1,2,\ldots,l+2)$ and $b_s(s = 1,2,\ldots,l+1)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} < a_{l+2}$ such that

$$f(t,u) > \frac{a_r}{\prod_{j=1}^{n} \xi_j K_j}, \text{ for } t \in [0,1] \text{ and } u \in \left[a_r, \frac{a_r}{M}\right], \quad r = 1,2,\ldots,l+2. \tag{22}$$

$$f(t,u) < \frac{b_s}{\prod_{j=1}^{n} K_j}, \text{ for } t \in [0,1] \text{ and } u \in \left[0, \frac{b_s}{M}\right], \quad s = 1,2,\ldots,l+1. \tag{23}$$

By assumption, the boundary value problem (1)-(2) has at least symmetric $2l$ positive solutions $u_i(i = 1,2,\ldots,2l)$ in $\bar{P}_{a_{l+1}}$. At the same time, it follows from Theorem 3.2, (22), and (23) that the boundary value problem (1)-(2) has at least two symmetric positive solutions u_1, u_2 in $\bar{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(u_1)$ with $\theta(u_1) < b_{l+1}$ and $b_{l+1} < \theta(u_2)$ with $\beta(u_2) < a_{l+2}$. Obviously u_1 and u_2 are different from $u_i(i = 1,2,\ldots,2l)$. Therefore, the boundary value problem (1), (2) has at least $2l + 2$ symmetric positive solutions in $\bar{P}_{a_{l+2}}$, which shows that conclusion holds for $m = l + 1$. $\square$

Example 3.1. *Let us consider an example to illustrate our established results. Let* $n = 2$ *and consider the boundary value problem*

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0,1), \tag{24}$$

satisfying

$$\left.\begin{aligned} u(0) = u(1) &= \int_0^1 a_1(x)u(x)dx, \\ u''(0) = u''(1) &= \int_0^1 a_2(x)u''(x)dx, \end{aligned}\right\} \tag{25}$$

where

$$f(t,u(t)) = \begin{cases} \frac{800(1+\sin \pi t)(u+1)^4}{3(u^2+999)}, & \text{for } t \in [0,1],\ u \in [0, 0.3], \\ \frac{228488(1+\sin \pi t)}{299727}, & \text{for } t \in [0,1],\ u \in [0.3, 9.03], \\ \frac{228488(1+\sin \pi t)e^{13(u-9.03)}}{299727}, & \text{for } t \in [0,1],\ u \in [9.03, \infty), \end{cases}$$

$a_1(x) = \frac{1}{2}$ *and* $a_2(x) = \frac{1}{8}$.
By direct calculations, we have

$$d_1 = \frac{1}{2},\ \eta_1 = \frac{1}{12},\ \xi_1 = \frac{1}{7},\ K_1 = \frac{1}{12},$$

$$d_2 = \frac{1}{8},\ \eta_2 = \frac{1}{48},\ \xi_2 = \frac{1}{43},\ K_2 = \frac{5}{28} \text{ and } M = \frac{1}{301}.$$

Clearly f is continuous on $[0,\infty)$ and symmetric on [0,1]. Choosing $a'=0.00001$, $b'=0.03$, $c'=10$, then $0<a'<b'<c'$ and f satisfies

(i) $f(t,u(t))>202272=\frac{c'}{\prod_{j=1}^{2}\xi_j K_j}$, *for* $t\in[0,1]$ *and* $u\in[10,3010]$,

(ii) $f(t,u(t))<2.016=\frac{b'}{\prod_{j=1}^{2}K_j}$, *for* $t\in[0,1]$ *and* $u\in[0,9.03]$,

(iii) $f(t,u(t))>0.202272=\frac{a'}{\prod_{j=1}^{2}\xi_j K_j}$, *for* $t\in[0,1]$ *and* $u\in[0.00001,0.00301]$.

Thus, all the conditions of Theorem 3.2 are satisfied and hence, the boundary value problem (24)-(25) *has at least two symmetric positive solutions.*

4. Conclusion

We derived sufficient conditions for the existence of at least two symmetric positive solutions to $2n^{\text{th}}$ order boundary value problem satisfying Lidstone type integral boundary conditions by using Avery–Henderson fixed point theorem. We also established the existence of at least $2m$ symmetric positive solutions to the boundary value problem for an arbitrary positive integer m.

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