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ON ARITHMETIC-GEOMETRIC INDEX (GA) AND EDGE GA INDEX

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Abstract. Let $G(V(G), E(G))$ be a simple connected graph and $d_G(u)$ be the degree of the vertex u. Topological indices are numerical parameters of a graph which are invariant under graph isomorphisms. Recently, people are studying various topological measures such as the arithmetic-geometric index and the edge version of arithmetic-geometric index of a graph G. Topological index based on the ratios of geometrical and arithmetical means of end vertex degrees of edges. In this paper, exact values for the arithmetic-geometric index and the edge version of arithmetic-geometric index of wheel related graphs namely gear, helm, sunflower and friendship graph are obtained.

Keywords: Arithmetic-Geometric Index; Edge Arithmetic-Geometric Index; Network Design and Communication; Gear graph; Helm graph; Sunflower graph; Friendship graph

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1. Introduction

Graph theoretical applications in chemistry underwent a dramatic revival lately. Molecules and molecular compounds are often represented by graphs, which are identified by their vertices and edges where vertices are atom types and edges are bonds. Graph theory has successfully provided chemists with a variety of very useful tools, namely, the topological index. Topological indices of molecular graph are one of the oldest and most widely used descriptors in QSPR/QSAR research. Topological indices are numerical parameters of a graph which are invariant under graph isomorphisms. Recently, people are studying various topological measures, like the wiener index [18], the braching index [11], the randic connectivity index [5, 8], the zagreb indices [4, 10], the edge eccentric connectivity index [15], and so on. For a graph G, with vertex set $V(G)$ and edge set $E(G)$, Vukicevic and Furtula defined a new topological index the *arithmetic-geometric index* of a graph G [16], denoted by $GA(G)$ and defined by

$$GA = GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_G(u)d_G(v)}}{d_G(u)+d_G(v)},$$

where uv is an edge of the graph G connecting the vertices u and v, also $d_G(u)$ stands for the degree of the vertex u, and where the summation goes over all edges of G. Needless to say that $GA(G)$ is one more vertex-degree-based graph invariant [3, 16].

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The *edge version of arithmetic-geometric index* based on the end vertex degree of edges in a line graph G as follows:

$$GA_e = GA_e(G) = \sum_{ef \in E(L(G))} \frac{2\sqrt{d_G(e)d_G(f)}}{d_G(e)+d_G(f)},$$

where $d_G(e)$ denotes the degree of the edge e in G [9, 12]. The line graph $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their edges are incident, meaning they share a common end vertex, in graph G [1, 2, 17].

Now, some notation and terminology is introduced. We consider only simple finite undirected graphs without loops and multiple edges. Let $G(V(G), E(G))$ be a simple connected graph with vertex and edge sets $V(G)$ and $E(G)$, where $V(G) = \{v_1, v_2, ..., v_n\}$, $|V(G)| = n$, and $|E(G)| = m$. For a vertex u of a graph G, the *open neighborhood* of u is $N(u) = \{v \in V(G) | (u, v) \in E(G)\}$. We define analogously for any $S \subseteq V(G)$ the open neighborhood $N(S) = \cup_{u \in S} N(u)$. The degree of a vertex u of G is denoted by $d_G(u) = |N(u)|$. The covering and independence number, the maximum and the minimum degree of a graph G are denoted by $\alpha(G)$, $\beta(G)$, $\Delta(G)$, and $\delta(G)$, respectively [1, 2, 17]. A notation is used in order to make the proof of the given theorems understandable. Let u and v be any two vertices of the graph G. If these two vertices are adjacent in the graph G, then the edge between these two vertices is denoted by e_{uv} in the graph G.

In this paper, we focus our attention to these topological indices: the *arithmetic-geometric index* and edge version of *arithmetic-geometric index*. This paper organized as follows. In Section 2, known results for the *arithmetic-geometric index* and *edge version of arithmetic-geometric index* are given. In Section 3, definitions of wheel and related graphs namely gear, helm, sunflower and friendship graph are given and exact values for the *arithmetic-geometric index* of those graphs are determined. In Section 4, we give exact values for the *edge version of arithmetic-geometric index* of wheel and related graphs. Finally, concluding remarks of this paper are given in Section 5.

2. Basic results

In this section well known basic results are given with regard to the arithmetic-geometric index and edge version of arithmetic-geometric index.

Theorem 2.1. [16] *Let G be a simple connected graph with n vertices, then*

$$\frac{2(n-1)^{3/2}}{n} \leq GA(G) \leq \binom{n}{2}.$$

Lower bound is achieved if and only if $G \cong K_{1,n-1}$ and upper bound is achieved if and only if $G \cong K_n$.

Theorem 2.2. [16] *Let T be a tree with $n > 2$ vertices, then*

$$\frac{2(n-1)^{3/2}}{n} \leq GA(T) \leq \frac{4\sqrt{2}}{3} + n - 3.$$

Lower bound is achieved if and only if $T \cong K_{1,n-1}$ and upper bound is achieved if and only if $T \cong P_n$.

Theorem 2.3. [3] *Let G be a simple connected graph of m edges with the maximum vertex degree $\Delta(G)$ and minimum vertex degree $\delta(G)$. Then,*

$$GA(G) \geq \frac{2m\sqrt{\Delta(G)\delta(G)}}{\Delta(G)+\delta(G)}.$$

Theorem 2.4. [3] *Let G be a simple connected graph on n vertices with a connected $\overline{G}$, then the Nordhaus-Gaddum-type result for GA index of G is*

$$GA(G) + GA(\overline{G}) \leq \binom{n}{2}$$

with equality holding if and only if G is isomorphic to a regular graph.

Theorem 2.5. [9] *The GA and GA_e of*
- **(a)** *the complete graph K_n with n vertices is $GA(K_n) = GA_e(K_n) = n(n-1)^2/2$;*
- **(b)** *the star graph $K_{1,n-1}$ with n vertices is $GA(K_{n-1}) = GA_e(K_{1,n-1}) = \binom{n-1}{2}$;*
- **(c)** *the path graph P_n with n vertices is $GA(P_{n-1}) = GA_e(P_n) = 4\sqrt{2}/3 + (n-4)$;*
- **(d)** *the cycle graph C_n with n vertices is $GA(C_n) = GA_e(C_n) = n$.*

Theorem 2.6. [9] *Let G be a simple connected graph with n vertices, then*

$$0 \leq GA_e(G) \leq n(n-1)^2/2.$$

Lower bound is achieved if and only if G is an empty graph and upper bound is achieved if and only if $G \cong K_n$.

Theorem 2.7. [9] *Let G be a simple connected graph with n vertices, then*

$$(n-4) + 4\sqrt{2}/3 \leq GA_e(G) \leq n(n-1)^2/2.$$

Lower bound is achieved if and only if $G \cong P_n$ and upper bound is achieved if and only if $G \cong K_n$.

Theorem 2.8. [9] *Let G be a simple connected graph on n vertices with a connected $\overline{G}$, then the Nordhaus-Gaddum-type result for $GA_e(G)$ index of G is*

$$(3(n^2 - n - 4) + 8\sqrt{2})/6 \leq GA_e(G) + GA_e(\overline{G}) \leq (n-2)(n-1)n(n+1)/8.$$

3. Arithmetic-Geometric index of wheel related graphs

Definition 3.1. [6, 7, 13] *The wheel W_n with n ($n \geq 3$) spokes is a graph that contains an n-cycle and one additional central vertex v_c that is adjacent to all vertices of the cycle. The central vertex v_c of the wheel W_n has a vertex degree of n. Wheel graph has $(n+1)$ vertices and $2n$ edges.*

Theorem 3.1. *Let W_n be a wheel graph of order $n+1$. Then, $GA(W_n) = \frac{2n\sqrt{3n}}{n+3} + n$.*

Proof. Let the vertex and edge sets of W_n be $V(W_n) = V_1 \cup V_2$ and $E(W_n) = E_1 \cup E_2$, respectively; where:
V_1: The set contains the center vertex v_c whose degree n of the graph W_n.
V_2: The set contains vertices of $V(W_n) \setminus \{v_c\}$, and we show that these vertices with v_i, where $i = \overline{1, n}$.
Similarly, $E_1 = \{e_{v_c v_i} \in E(W_n) | v_c \in V_1, v_i \in V_2\}$ and $E_2 = \{e_{v_i v_j} \in E(W_n) | v_i, v_j \in V_2\}$.
We have two cases depending on the edges of graph W_n.

Case1. Let $e_{v_c v_i}$ be any edge of E_1. Since the structure of graph W_n, we have $d_{W_n}(v_c) = n$ and $d_{W_n}(v_i) = 3$, where $i = \overline{1, n}$. Thus, we get $\frac{2\sqrt{3n}}{n+3}$ for every edge $e_{v_c v_i}$. Due to $|E_1| = n$, $n(\frac{2\sqrt{3n}}{n+3}) = \frac{2n\sqrt{3n}}{n+3}$ is obtained for edges of E_1.

Case2. Let $e_{v_iv_j} \in E_2$. It is clear that $d_{W_n}(v_i) = d_{W_n}(v_j) = 3$. Thus, we get $\frac{2\sqrt{3.3}}{3+3}$ for every edge $e_{v_iv_j}$. Since there are n edges in E_2, $n(\frac{2\sqrt{3.3}}{3+3}) = n$ is obtained for every edges in E_2.
By summing up the Cases 1 and 2, it is clear that $GA(W_n) = \frac{2n\sqrt{3n}}{n+3} + n$.
The proof is completed. □

Theorem 3.2. *Let W_{2n} be a wheel graph of order $2n+1$. Then, $GA(W_{2n}) = \frac{4n\sqrt{6n}}{2n+3} + 2n$.*

Proof. By the Theorem 3.1, it is clear. □

Definition 3.2. [6, 7, 13] *The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. Gear graph G_n includes an even cycle C_{2n}. The vertices of C_{2n} in G_n are of two kinds: vertices of degree two and three, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree three to as major vertices. Let the central vertex of gear graph G_n be v_c. The central vertex v_c has a vertex degree of n. Gear graph G_n has $(2n+1)$ vertices and $3n$ edges.*

Theorem 3.3. *Let G_n be a gear graph of order $2n+1$. Then, $GA(G_n) = \frac{2n\sqrt{3n}}{n+3} + \frac{4n\sqrt{6}}{5}$.*

Proof. We partition the vertices of graph G_n into three subsets V_1, V_2 and V_3, as follows:
$V_1 = \{v_c \in V(G_n) | d_{G_n}(v_c) = n\}$, $V_2 = \{v_i \in V(G_n) | d_{G_n}(v_i) = 3, i = \overline{1,n}\}$ and
$V_3 = \{v_i \in V(G_n) | d_{G_n}(v_i) = 2, i = \overline{n+1, 2n}\}$.
Similarly, we partition the edges of graph G_n into two subsets E_1 and E_2, as follows:
$E_1 = \{e_{v_cv_i} \in E(G_n) | v_c \in V_1, v_i \in V_2\}$ and $E_2 = \{e_{v_iv_j} \in E(G_n) | v_i \in V_2, v_j \in V_3\}$.
We have two cases depending on the edges of G_n.

Case1. This case is very similar to the Case 1 of the proof of Theorem 3.1. As a result, we have $\frac{2n\sqrt{3n}}{n+3}$ for all edges of E_1.

Case2. Let $e_{v_iv_j}$ be any edge of the set E_2. For $v_i \in V_2$, $v_j \in V_3$ and $|E_2| = 2n$, the value of the arithmetic-geometric index $2n(\frac{2\sqrt{3.2}}{3+2}) = \frac{4n\sqrt{6}}{5}$ is obtained for all edges of E_2.
By summing up the Cases 1 and 2, it is clear that $GA(G_n) = \frac{2n\sqrt{3n}}{n+3} + \frac{4n\sqrt{6}}{5}$.
Thus, the proof is completed. □

Definition 3.3. [6, 7, 14] *is collection of n triangles with a common vertex. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternate edges of the cycle. Another way of obtaining friendship graph is addition of K_1 and n copies of K_2. Let the central vertex of a friendship graph f_n be v_c. The central vertex v_c has a vertex degree of $2n$. Friendship graph f_n has $(2n+1)$ vertices and $3n$ edges.*

Theorem 3.4. *Let f_n be a friendship graph of order $2n+1$. Then, $GA(f_n) = \frac{4n\sqrt{n}}{n+1} + n$.*

Proof. Let the vertex and edge sets of f_n be $V(f_n) = V_1 \cup V_2$ and $E(f_n) = E_1 \cup E_2$, respectively; where:
$V_1 = \{v_c \in V(f_n) | d_{f_n}(v_c) = 2n\}$ and $V_2 = \{v_i \in V(f_n) | d_{f_n}(v_i) = 2, i = \overline{1, 2n}\}$,
$E_1 = \{e_{v_cv_i} \in E(f_n) | v_c \in V_1, v_i \in V_2\}$ and $E_2 = \{e_{v_iv_j} \in E(f_n) | v_i, v_j \in V_2\}$.
We have two cases for the all edges of the graph f_n.

Case1. Let $e_{v_cv_i}$ be any edge of E_1. For $v_c \in V_1$, $v_i \in V_2$ and $|E_1| = 2n$, the value of the arithmetic-geometric index $2n(\frac{2\sqrt{2n.2}}{2n+2}) = \frac{4n\sqrt{n}}{n+1}$ is obtained.

Case2. Let $e_{v_iv_j}$ be any edge of E_2. Due to $d_{f_n}(v_i) = d_{f_n}(v_j) = 2$ and $|E_2| = n$, we obtain $n(\frac{2\sqrt{2.2}}{2+2}) = n$ for all edges of E_2.
By summing up the Case 1 and 2, it is clear that $GA(f_n) = \frac{4n\sqrt{n}}{n+1} + n$.
Thus, the proof is completed. □

Definition 3.4. [6, 7, 13] *Helm H_n is a graph of order $2n+1$ obtained from a wheel W_n with cycle C_n having a pendant edge attached to each vertex of the cycle. Helm H_n consists of the vertex set $V(H_n) = \{v_i|0 \le i \le n-1\} \cup \{a_i|0 \le i \le n-1\} \cup \{v_c\}$ and edge set $E(H_n) = \{e_{v_iv_{i+1}}|0 \le i \le n-1\} \cup \{e_{v_ia_i}|0 \le i \le n-1\} \cup \{e_{v_iv_c}|0 \le i \le n-1\}$, where $i+1$ is taken modulo n. Let v_c be the central vertex of H_n. The central vertex v_c has a vertex degree with n. The vertices of $H_n \setminus \{v_c\}$ are of two kinds: vertices of degree four and one, respectively. The vertices of degree one will be referred to as minor vertices and vertices of degree four to as major vertices . Helm graph has $(2n+1)$ vertices and $3n$ edges.*

Theorem 3.5. *Let H_n be a helm graph of order $2n+1$. Then, $GA(H_n) = \frac{4n\sqrt{n}}{n+4} + \frac{9n}{5}$.*

Proof. We partition the vertices of the graph H_n into three subsets V_1, V_2 and V_3, as follows:
$V_1 = \{v_c \in V(H_n)|d_{H_n}(v_c) = n\}$, $V_2 = \{v_i \in V(H_n)|d_{H_n}(v_i) = 4, i = \overline{1,n}\}$ and
$V_3 = \{v_i \in V(H_n)|d_{H_n}(v_i) = 1, i = \overline{n+1, 2n}\}$.
Similarly, we partition the edges of graph H_n into three subsets E_1, E_2 and E_3 as follows:
$E_1 = \{e_{v_cv_i} \in E(H_n)|v_c \in V_1, v_i \in V_2\}$, $E_2 = \{e_{v_iv_j} \in E(H_n)|v_i, v_j \in V_2\}$ and
$E_3 = \{e_{v_iv_j} \in E(H_n)|v_i \in V_2, v_j \in V_3\}$.
Then the proof proceeds in the following three cases:

Case1. This case is very similar to the Case 1 of the proof of Theorem 3.1. Then, clearly $d_{H_n}(v_i) = 4$, where $i = \overline{1,n}$ in H_n. Thus, $n(\frac{2\sqrt{4.n}}{n+4}) = \frac{4n\sqrt{n}}{n+4}$.

Case2. This case is very similar to the Case 2 of the proof of Theorem 3.4. Then, $n(\frac{2\sqrt{4.4}}{4+4}) = n$ is obtained for all edges of E_2.

*Case3.*Let $e_{v_iv_j}$ be any edge of E_3. Due to $d_{H_n}(v_i) = 4$ and $d_{H_n}(v_j) = 1$, we obtain $n(\frac{2\sqrt{4.1}}{4+1}) = \frac{4n}{5}$ for all edges of E_3.
By summing up the Cases 1, 2 and 3, it is clear that $GA(H_n) = \frac{4n\sqrt{n}}{n+4} + n + \frac{4n}{5} = \frac{4n\sqrt{n}}{n+4} + \frac{9n}{5}$.
Thus, the proof is completed. □

Definition 3.5. [6, 7, 13] *Sunflower graph SF_n consists of a wheel with central vertex v_c and an n-cycle $v_0, v_1, v_2, ..., v_{n-1}$ and additional n vertices $w_0, w_1, w_2, ..., w_{n-1}$ where w_i is joined by edges to $e_{v_iv_{i+1}}$ for $i = \{0, 1, ..., n-1\}$ where $i+1$ is taken modulo n. Let v_c be the central vertex of SF_n. The central vertex v_c has a vertex degree of n. The vertices of $SF_n \setminus \{v_c\}$ are of two kinds: vertices of degree five and two, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree five to as major vertices. Sunflower graph has $(2n+1)$ vertices and $4n$ edges.*

Theorem 3.6. *Let SF_n be a sunflower graph of order $2n+1$. Then,*

$$GA(SF_n) = \frac{2n\sqrt{5n}}{n+5} + \frac{7n + 4n\sqrt{10}}{7}.$$

Proof. Let the vertex and edge sets of SF_n be $V(SF_n) = V_1 \cup V_2 \cup V_3$ and $E(SF_n) = E_1 \cup E_2 \cup E_3$, respectively; where:

$V_1 = \{v_c \in V(SF_n) | d_{SF_n}(v_c) = n\}$, $V_2 = \{v_i \in V(SF_n) | d_{SF_n}(v_i) = 5, i = \overline{1,n}\}$ and $V_3 = \{v_i \in V(SF_n) | d_{SF_n}(v_i) = 2, i = \overline{n+1, 2n}\}$.
Similarly, we partition the edges of graph SF_n into three subsets E_1, E_2 and E_3 as follows: $E_1 = \{e_{v_c v_i} \in E(SF_n) | v_c \in V_1, v_i \in V_2\}$, $E_2 = \{e_{v_i v_j} \in E(SF_n) | v_i, v_j \in V_2\}$ and $E_3 = \{e_{v_i v_j} \in E(SF_n) | v_i \in V_2, v_j \in V_3\}$.
We have three cases depending on the edges of SF_n. The proof is similar to above mentioned theorems. When we calculated these three cases, we obtain $\frac{2n\sqrt{5n}}{n+5}$, n and $\frac{4n\sqrt{10}}{7}$ for all edges of the sets E_1, E_2 and E_3, respectively.
By summing these values, we have $GA(SF_n) = \frac{2n\sqrt{5n}}{n+5} + \frac{7n+4n\sqrt{10}}{7}$.
Thus, the proof is completed. □

4. The edge GA of wheels and related graphs

Theorem 4.1. *Let W_n be a wheel graph of order $n+1$. Then,*

$$GA_e(W_n) = \frac{n^2+n}{2} + \frac{8n\sqrt{n+1}}{n+5}.$$

Proof. Since the structure of $L(W_n)$, we have $|V(L(W_n))| = 2n$ and $|E(L(W_n))| = \frac{n^2+5n}{2}$.
Let the vertex set of the graph $L(W_n)$ be $V(L(W_n)) = V_1 \cup V_2$, where:
$V_1 = \{v_i \in V(L(W_n)) | d_{L(W_n)}(v_i) = n+1, i = \overline{1,n}\}$ and
$V_2 = \{v_i \in V(L(W_n)) | d_{L(W_n)}(v_i) = 4, i = \overline{n+1, 2n}\}$.
Similarly, let the edge set of the graph $L(W_n)$ be $E(L(W_n)) = E_1 \cup E_2 \cup E_3$, where:
$E_1 = \{e_{v_i v_j} \in E(L(W_n)) | v_i, v_j \in V_1\}$, $E_2 = \{e_{v_i v_j} \in E(L(W_n)) | v_i \in V_1, v_j \in V_2\}$ and
$E_3 = \{e_{v_i v_j} \in E(L(W_n)) | v_i, v_j \in V_2\}$.
It is not difficult to see that $|E_1| = \frac{n(n-1)}{2}$, $|E_2| = 2n$ and $|E_3| = n$. Thus, we have three cases for computing the $GA_e(W_n)$.

Case 1. Let $e_{v_i v_j} \in E_1$. It is easy to see that $d_{L(W_n)}(v_i) = d_{L(W_n)}(v_j) = n+1$. By the definition of the edge GA , we have $(\frac{2\sqrt{(n+1)(n+1)}}{2(n+1)})$ for every edges of the set E_1. Thus, we get $\frac{n(n-1)}{2}(\frac{2\sqrt{(n+1)(n+1)}}{2(n+1)}) = \frac{n(n-1)}{2}$ for this case.

Case 2. Let $e_{v_i v_j} \in E_2$. Due to $d_{L(W_n)}(v_i) = n+1$ and $d_{L(W_n)}(v_j) = 4$, then we arrive the result $\frac{2\sqrt{4(n+1)}}{n+5}$ that for every edges of the set E_2. Thus, we get $2n(\frac{2\sqrt{4(n+1)}}{n+5}) = \frac{8n\sqrt{4(n+1)}}{n+5}$.

Case 3. Let $e_{v_i v_j} \in E_3$. It is easy to see that $d_{L(W_n)}(v_i) = d_{L(W_n)}(v_j) = 4$. By the definition of the edge GA , we have $(\frac{2\sqrt{4.4}}{4+4}) = 1$ for every edges of the set E_3. Thus, we get $n(\frac{2\sqrt{4.4}}{4+4}) = n$ for this case.
By summing up the Cases 1, 2 and 3, it is clear that $GA_e(W_n) = \frac{n^2+n}{2} + \frac{8n\sqrt{n+1}}{n+5}$.
Thus, the proof is completed. □

Theorem 4.2. *Let W_{2n} be a wheel graph of order $2n+1$. Then,*

$$GA_e(W_{2n}) = 2n^2 + n + \frac{16n\sqrt{2n+1}}{2n+5}.$$

Proof. By the Theorem 4.1, it is clear. □

Theorem 4.3. *Let G_n be a gear graph of order $2n+1$. Then,*

$$GA_e(G_n) = \frac{n^2+3n}{2} + \frac{4n\sqrt{3n+3}}{n+4}.$$

Proof. Let the vertex and edge set of the graph $L(G_n)$ be $V(L(G_n)) = V_1 \cup V_2$ and $E(L(G_n)) = E_1 \cup E_2 \cup E_3$, respectively. It is clear that $|V(L(G_n))| = 2n$ and $|E(L(G_n))| = \frac{n^2+7n}{2}$. Furthermore, $V_1 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = n+1, i = \overline{1,n}\}$ and $V_2 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = 3, i = \overline{n+1, 3n}\}$.
Moreover, the sets E_1, E_2 and E_3 are the same sets of the Theorem 4.1. We have three cases depending on the edges of graph $L(G_n)$. The proof is very similar to the proof of Theorem 4.1. Thus when computing $GA_e(G_n)$, we receive the three values: $\frac{n(n-1)}{2}$, $\frac{4n\sqrt{3n+3}}{n+4}$ and n for all edges of the sets E_1, E_2 and E_3, respectively.
By summing up the Cases 1, 2 and 3, it is clear that $GA_e(G_n) = \frac{n^2+3n}{2} + \frac{4n\sqrt{3n+3}}{n+4}$.
Thus, the proof is completed. □

Theorem 4.4. *Let f_n be a friendship graph of order $2n+1$. Then,*

$$GA_e(f_n) = 2n^2 - n + \frac{8n\sqrt{n}}{n+1}.$$

Proof. It is clear that $|V(L(f_n))| = 3n$ and $|E(L(f_n))| = 2n^2+n$. Let $V(L(f_n)) = V_1 \cup V_2$ and $E(L(f_n)) = E_1 \cup E_2$, then we have as follows:
$V_1 = \{v_i \in V(L(f_n)) | d_{L(f_n)}(v_i) = 2n, i = \overline{1, 2n}\}$,
$V_2 = \{v_i \in V(L(f_n)) | d_{L(f_n)}(v_i) = 2, i = \overline{2n+1, 3n}\}$,
$E_1 = \{e_{v_iv_j} \in E(L(f_n)) | v_i, v_j \in V_1\}$ and
$E_2 = \{e_{v_iv_j} \in E(L(f_n)) | v_i \in V_1, v_j \in V_2\}$.
The edges of the graph f_n in two cases should be examined.

Case 1. Let $e_{v_iv_j}$ be any edge of the set E_1. By the definition of the edge GA, we have the following value GA_e for every edges of the set E_1:

$$GA_e = (2n^2 - n)(\frac{2\sqrt{2n.2n}}{2n+2n}) = 2n^2 - n.$$

Case 2. Let $e_{v_iv_j}$ be any edge of the set E_2. Due to $d_{L(f_n)}(v_i) = 2n$ and $d_{L(f_n)}(v_j) = 2$, we get the following value GA_e for every edges of the set E_2:

$$GA_e = (2n)(\frac{2\sqrt{2n.2}}{2n+2}) = \frac{8n\sqrt{n}}{2n+2}.$$

By summing up the Cases 1 and 2, it is clear that $GA_e(f_n) = 2n^2 - n + \frac{8n\sqrt{n}}{n+1}$.
Thus, the proof is completed. □

Theorem 4.5. *Let H_n be a helm graph of order $2n+1$. Then,*

$$GA_e(H_n) = \frac{n^2+n}{2} + \frac{4n\sqrt{2}}{3} + \frac{4n\sqrt{6n+12}}{n+8} + \frac{2n\sqrt{3n+6}}{n+5}.$$

Proof. Since the structure of the graph $L(H_n)$, we have $|V(L(H_n))| = 3n$ and $|E(L(H_n))| = \frac{n^2+11}{2}$. Furthermore, we have three types of vertices depending on degrees. They can be showed as follows:
$V_1 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = n+2, i = \overline{1,n}\}$,
$V_2 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = 6, i = \overline{n+1, 2n}\}$ and

$V_3 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = 3, i = \overline{2n+1, 3n}\}$.
Similarly, we have five types for edges of $L(H_n)$ as follows:
$E_1 = \{e_{v_iv_j} \in E(L(H_n)) | v_i, v_j \in V_1\}$, $E_2 = \{e_{v_iv_j} \in E(L(H_n)) | v_i \in V_1, v_j \in V_2\}$,
$E_3 = \{e_{v_iv_j} \in E(L(H_n)) | v_i, v_j \in V_2\}$, $E_4 = \{e_{v_iv_j} \in E(L(H_n)) | v_i \in V_2, v_j \in V_3\}$ and
$E_5 = \{e_{v_iv_j} \in E(L(H_n)) | v_i \in V_1, v_j \in V_3\}$.
It is easy to see that $|E_1| = \frac{n(n-1)}{2}$, $|E_2| = |E_4| = 2n$ and $|E_3| = |E_5| = n$. Thus, we have five cases for computing the $GA_e(H_n)$. When we calculated the these five cases, we obtained $\frac{n(n-1)}{2}$, $\frac{4n\sqrt{6n+12}}{n+8}$, n, $\frac{4n\sqrt{2}}{3}$ and $\frac{2n\sqrt{3n+6}}{n+5}$ for all edges of the sets E_1, E_2, E_3, E_4 and E_5, respectively.
By summing up the these values, we receive

$$GA_e(H_n) = \frac{n^2+n}{2} + \frac{4n\sqrt{2}}{3} + \frac{4n\sqrt{6n+12}}{n+8} + \frac{2n\sqrt{3n+6}}{n+5}.$$

Thus, the proof is completed. □

Theorem 4.6. *Let SF_n be a sunflower graph of order $2n+1$. Then,*

$$GA_e(SF_n) = \frac{n^2+5n}{2} + \frac{4\sqrt{2n+6}}{n+11} + \frac{16n\sqrt{10}}{13} + \frac{12n\sqrt{5n+15}}{n+8}.$$

Proof. The graph SF_n has $4n$-vertices and $((n^2+21n)/2)$-edges. We partition the vertices of graph $L(SF_n)$ into three subsets V_1, V_2 and V_3 as follows:
$V_1 = \{v_i \in V(L(SF_n)) | d_{L(SF_n)}(v_i) = n+3, i = \overline{1, n}\}$,
$V_2 = \{v_i \in V(L(SF_n)) | d_{L(SF_n)}(v_i) = 8, i = \overline{n+1, 2n}\}$ and
$V_3 = \{v_i \in V(L(SF_n)) | d_{L(SF_n)}(v_i) = 5, i = \overline{2n+1, 4n}\}$.
Similarly, we partition the edges of graph $L(SF_n)$ into six subsets E_1, E_2, E_3, E_4, E_5 and E_6, as follows:
$E_1 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i, v_j \in V_1\}$, $E_2 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i \in V_1, v_j \in V_2\}$,
$E_3 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i, v_j \in V_2\}$, $E_4 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i \in V_2, v_j \in V_3\}$,
$E_5 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i, v_j \in V_3\}$ and $E_6 = \{e_{v_iv_j} \in E(L(SF_n)) | v_i \in V_1, v_j \in V_3\}$.
Clearly, $|E_1| = \frac{n(n-1)}{2}$, $|E_2| = |E_5| = |E_6| = 2n$, $|E_3| = n$ and $|E_4| = 4n$. The edges of graph $L(SF_n)$ in six cases should be examined. Remaining of the this proof is similar to Theorem 4.5.
By summing up the these values, we get

$$GA_e(SF_n) = \frac{n^2+5n}{2} + \frac{4\sqrt{2n+6}}{n+11} + \frac{16n\sqrt{10}}{13} + \frac{12n\sqrt{5n+15}}{n+8}.$$

Thus, the proof is completed. □

5. Conclusion

The graphics of values of GA and edge GA values of the graphs considered in the paper are given in the following Figure 1.

When evaluating GA values for the considered graphs, it can be easily seen that the number of vertices of each graph is same but the number of edges of W_{2n}, SF_n and the number of edges of f_n, G_n and H_n are same. Hence, making a comparison for the GA values of graphs in two groups is more accurate. Accordingly, we can say that due to the GA value, the architecture of SF_n is better than W_{2n}. Similarly, G_n is better than that of f_n and H_n.

If a similar interpretation is handled for the edge GA value, it is seen that the number of edges and vertices differs for graphs $L(G)$ which should be constructed for the definition of

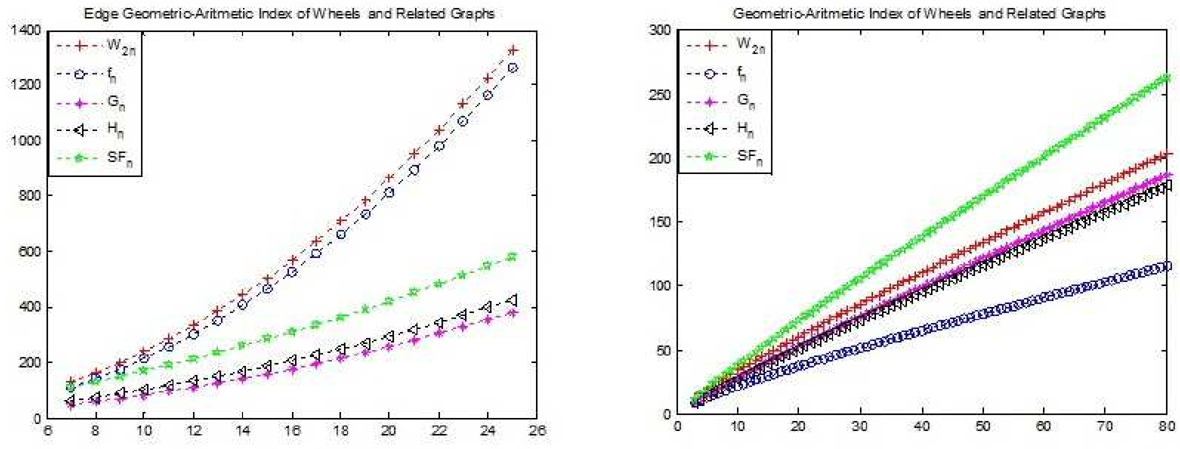

FIGURE 1. The values of the GA and edge GA indexes

parameter. Both the parameters considered in the paper are edge based. Therefore, when making a comparison, the number of edges should be taken into consideration. However, each constructed graph $L(G)$ has different number of edges. If a graphic based on the edge GA values of considered graphs is investigated, then it is observed that the results for the graphs $L(W_{2n})$ and $L(f_n)$, also $L(SF_n)$, $L(H_n)$ and $L(G_n)$ are close to each other. Hence, we can conclude that the graphs $L(W_{2n})$ and $L(SF_n)$ are more stable.

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