TWMS J. App. Eng. Math. V.8, N.1, 2018, pp. 83-93

SOLUTION TO TIME FRACTIONAL COUETTE FLOW

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Abstract. In this study, the Couette flow of a second grade fluid is discussed in a porous layer when the bottom plate moves suddenly. The Laplace transform method is implemented to derive the analytical solution.The main object of this paper is to demonstrate how we can make significant progress in treating a variety of problems in the theory of partial fractional differential equations by combining theory of special functions and operational methods. In this article, it has been shown that the combined use of integral transforms and exponential operators methods provides a powerful tool to solve certain system of KdV. Constructive examples are also provided.

Keywords: Fractional partial differential equations, Riemann Liouville fractional derivative, Time fractional Couette flow, Laplace transforms, Fourier transforms

AMS Subject Classification: 26A33; 34A08; 34K37; 35R11.

1. Introduction

In this work, we present a general method of operational nature to obtain solutions for several types of partial fractional differential equations.

Definition 1.1: The Laplace transform of function $f(t)$ is defined as follows

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s). \qquad (1.1)$$

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds, \qquad (1.2)$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Definition 1.2: The Fourier transform of function $f(t)$ is defined as follows

$$\mathcal{F}\{f(t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{i\omega t} f(t)dt := F(\omega). \qquad (1.3)$$

If $\mathcal{F}\{f(t)\} = F(\omega)$, then $\mathcal{F}^{-1}\{F(\omega)\}$ is given by

$$\mathcal{F}^{-1}\{F(\omega)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-it\omega} F(\omega)d\omega = f(t). \qquad (1.4)$$

Definition 1.3: If the function $\Phi(t)$ belongs to $C[a,b]$ and $a < t < b$,

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§ Manuscript received: August 27, 2016; accepted: January 12, 2107.
TWMS Journal of Applied and Engineering Mathematics, Vol.8, No.1

then the left Riemann-Liouville fractional integral of order $0 < \alpha < 1$ is defined as [11]

$$I_a^{RL,\alpha}\{\Phi(t)\} = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\Phi(\xi)}{(t-\xi)^{1-\alpha}} d\xi. \qquad (1.5)$$

The left Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined as follows [11]

$$D_a^{RL,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{\Phi(\xi)}{(t-\xi)^{\alpha}} d\xi. \qquad (1.6)$$

It follows that $D_a^{RL,\alpha}\phi(x)$ exists for all $\Phi(t)$ belongs to $C[a,b]$ and $a < t < b$.
Note: A very useful fact about the R - L operators is that they satisfy semi group properties of fractional integrals.The special case of fractional derivative when $\alpha = 0.5$ is called semi - derivative.
Definition 1.4: The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\phi(t)$ is as follows[9]

$$D_a^{c,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{1}{(t-\xi)^{\alpha}} \phi'(\xi) d\xi. \qquad (1.7)$$

Lemma 1.1. Let $L\{f(t)\} = F(s)$ then the following identities hold true.

(1) $\mathcal{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})} \int_0^\infty e^{-t\xi - \frac{k^2}{4\xi}} d\xi$,
(2) $e^{-\omega s^\beta} = \frac{1}{\pi} \int_0^\infty e^{-r^\beta(\omega cos\beta\pi)} sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau - r\tau} d\tau) dr$,
(3) $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi} \int_0^\infty f(u) \int_0^\infty e^{-tr - ur^\alpha cos\alpha\pi} sin(ur^\alpha sin\alpha\pi) dr du$,
(4) $\mathcal{L}^{-1}(F(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty u e^{-\frac{u^2}{4t}} f(u) du$,
(5) $\mathcal{L}^{-1}(\frac{1}{\sqrt{s}(\sqrt{s}+a)}) = e^{a^2 t} Erfc(a\sqrt{t})$,
(6) $\mathcal{L}^{-1}(\frac{a}{s(\sqrt{s}+a)}) = 1 - e^{a^2 t} Erfc(a\sqrt{t})$.

Proof. See [1][3].
Lemma 1.2. The following exponential identities hold true.

(1) $\exp(\pm\lambda \frac{d}{dt})\Phi(t) = \Phi(t \pm \lambda)$,
(2) $\exp(\pm\lambda t \frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda})$,
(3) $\exp(\lambda q(t) \frac{d}{dt})\Phi(t) = \Phi(Q(F(t) + \lambda))$,

where $F(t)$ is the primitive function of $(q(t))^{-1}$ and $Q(t)$ is the inverse function of $F(t)$.
Proof. See[2] [5][6].
Lemma 1.3. The following exponential identity holds true.

(1) $\exp(\lambda \frac{d^3}{dt^3})\Phi(t) = \int_{-\infty}^\infty \Phi(t + \xi\sqrt[3]{3\lambda}) Ai(\xi) d\xi. \qquad (1.8)$

Proof. It is well known that $\mathcal{F}\{Ai(t)\} = \frac{1}{\sqrt{2\pi}} exp(i\frac{(\omega)^3}{3}), \qquad (1.9)$
in other words, we have the following relation

$$\mathcal{F}\{Ai(t)\} = (\frac{1}{\sqrt{2\pi}}) \int_{-\infty}^{+\infty} e^{i\omega t} Ai(t) dt := \frac{1}{\sqrt{2\pi}} exp(i\frac{(\omega)^3}{3}). \qquad (1.10)$$

Let us introduce a change of parameter as follows

$$(\sqrt[3]{3\lambda})\beta = iw, \qquad (1.11)$$

after substitution of (1.11) in (1.10) and simplifying, we obtain

$$\mathcal{F}\{Ai(t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{\sqrt[3]{3\lambda}\beta t} Ai(t)dt := \tfrac{1}{\sqrt{2\pi}} exp(\lambda\beta^3), \qquad (1.12)$$

in relation (1.12), if we set $\beta = \frac{d}{dt}$, we get the following operational identity,

$$exp(\lambda\tfrac{d^3}{dt^3})\Phi(t) = \int_{-\infty}^{+\infty}(e^{\sqrt[3]{3\lambda}\xi\frac{d}{dt}}\Phi(t))Ai(\xi)d\xi. \qquad (1.13)$$

In view of the first part of the Lemma 1.1, we obtain

$$exp(\lambda\tfrac{d^3}{dt^3})\Phi(t) = \int_{-\infty}^{+\infty}\Phi(t+\xi\sqrt[3]{3\lambda}))Ai(\xi)d\xi. \qquad (1.14)$$

Note: In the above identity, Ai(.) stands for the Airy function (see[12]for details).
The Laplace transform is useful tool in applied mathematics, for instance for solving singular integral equations, partial differential equations, and in automatic control, where it defines a transfer function.

Example 1.1. Let us consider the following nonlinear impulsive differential equation

$$(\sqrt{D_t - a})y(t) = (t-\lambda)^k\delta^{(k)}(t-\lambda). \qquad (1.15)$$

Solution. Direct use of part 3 of the Lemma 1.1, the above differential equation can be written as below

$$y(t) = \frac{1}{(\sqrt{D_t - a})}(t-\lambda)^k\delta^{(k)}(t-\lambda),$$

from which we deduce

$$y(t) = \int_0^\infty d\xi\frac{e^{-a\xi}}{\sqrt{a+\xi}}e^{-\xi D_t}(t-\lambda)^k\delta^{(k)}(t-\lambda),$$

finally, using elementary properties of Dirac delta function leads to the following solution

$$y(t) = \int_0^\infty d\xi\frac{e^{-a\xi}}{\sqrt{a+\xi}}(-1)^k k!\delta(t-\xi-\lambda) = \frac{(-1)^k k! e^{-a(t-\lambda)}}{\sqrt{\pi(a+t-\lambda)}}.$$

Example 1.2. Show that the following exponential identities hold true.

(1) $\exp(\pm\frac{k^m}{mt^{m-1}}\frac{d}{dt})\Phi(t) = \Phi(\sqrt[m]{t^m \pm k^m})$,
(2) $\exp(-kt^2\frac{d}{dt})\Phi(t) = \Phi(\frac{t}{k+t})$.

Solution.(part1).
Let us take $\pm\frac{k^m}{m} = \lambda$ and $q(t) = \frac{1}{t^{m-1}}$, then we get $\frac{1}{q(t)} = t^{m-1}$ and $F(t) = \frac{t^m}{m}$ where $F(t)$ is the primitive function of $\frac{1}{q(t)}$. $Q(t)$ the inverse function of $F(t)$ is $Q(t) = \sqrt[m]{mt}$. Now, direct application of part 4 of the Lemma 1.2 leads to the following

$$\exp(\pm\frac{k^m}{mt^{m-1}}\frac{d}{dt})\Phi(t) = \Phi(\sqrt[m]{t^m + m\lambda}) = \Phi(\sqrt[m]{t^m \pm k^m}).$$

Solution.(part2).Let us take $-t^2 = q(t)$ then$\frac{1}{q(t)} = -\frac{1}{t^2}$ from which we get $F(t) = \frac{1}{t}$ where $F(t)$ is the primitive function of $\frac{1}{q(t)}$. $Q(t)$ the inverse function of $F(t)$, therefore, we get $Q(t) = \frac{1}{t}$.In view of part 4 of the Lemma 1.2, we have

$$\exp(-kt^2\frac{d}{dt})\Phi(t) = \Phi(\frac{t}{k+t}).$$

Example 1.3. Let us solve the following fractional Volterra integral equation of convolution type. The Laplace transform provides a useful technique for the solution of such fractional singular integro- differential equations.

$$\lambda \int_0^t \exp(\beta(t-\xi))D^\alpha\phi(\xi)d\xi = (\tfrac{t}{a})^{\frac{\mu}{2}} J_\mu(2\sqrt{at}), \qquad \phi(0)=0.$$

Solution. Upon taking the Laplace transform of the given integral equation, yields

$$s^\alpha \Phi(s)\tfrac{\lambda}{(s-\beta)} = \tfrac{e^{-\frac{a}{s}}}{s^{1+\mu}},$$

solving the above equation, leads to

$$\Phi(s) = \tfrac{(s-\beta)e^{-\frac{a}{s}}}{(\lambda)s^{1+\alpha+\mu}},$$

or equivalently

$$\Phi(s) = \tfrac{(se^{-\frac{a}{s}}-\beta e^{-\frac{a}{s}})}{(\lambda)s^{1+\alpha+\mu}},$$

at this stage, taking the inverse Laplace transform term wise, after simplifing we obtain

$$\phi(t) = \tfrac{1}{\lambda}(\tfrac{t}{a})^{\frac{\alpha+\mu}{2}} J_{\alpha+\mu}(2\sqrt{at}) - \tfrac{\beta}{\lambda}(\tfrac{t}{a})^{\frac{(\alpha+\mu+1)}{2}} J_{\alpha+\mu+1}(2\sqrt{at}).$$

Note: $J_\eta(.)$, stands for the Bessel's function of the first kind of order η[12].
Lemma1.4. The following second order exponential operator relation holds true.

$$1. \exp(r(\tfrac{\partial}{\partial x})^2)\Phi(x) = \tfrac{1}{(2r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\Phi(x+iu)+\Phi(x-iu))du. \qquad (1.16)$$

Proof. Let us consider the following elementary integral

$$r\sqrt{\pi}\exp(-r(b^2-a^2)) = \int_0^\infty e^{-\frac{u^2}{4r}}\cos(au)\cosh(bu)du. \qquad (1.17)$$

By integration by parts, we can easily find the value of the integral and after some algebra, we obtain

$$\exp(-r(b^2-a^2)) = \tfrac{1}{(2r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\exp(iau)+exp(-iau))\cosh(bu)du. \quad (1.18)$$

1 . In the above integral relation, we set $a=(\frac{\partial}{\partial x})$, $b=0$ to obtain

$$\exp(r(\tfrac{\partial}{\partial x})^2)\Phi(x) = \tfrac{1}{(2r\sqrt{\pi})}\int_0^\infty du(e^{-\frac{u^2}{4r}}(\exp(iu)(\tfrac{\partial}{\partial x})+\exp(-iu)(\tfrac{\partial}{\partial x}))\Phi(x), \quad (1.19)$$

in view of the Lemma 1.1, we get finally

$$\exp(r(\tfrac{\partial}{\partial x})^2)\Phi(x) = \tfrac{1}{(2r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\Phi(x+iu)+\Phi(x-iu))du.$$

Lemma 1.5. Let us assume that

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\} = \int_0^\infty e^{-st}\frac{J_\mu(\lambda t)}{t}dt := \frac{((\sqrt{s^2+\lambda^2})-s)^\mu}{\mu\lambda^\mu}, \qquad (1.20)$$

then, we have the following integral identities

$$\int_0^\infty \frac{J^*_\mu(\lambda t)}{t}dt := -\frac{1}{\mu^2}, \qquad (1.21)$$

$$\int_0^\infty \frac{J_{\frac{1}{3}}(\lambda t)-J_{-\frac{1}{3}}(\lambda t}{t}dt = \int_0^\infty \frac{-\sqrt{3}Bi(-\lambda(1.5t)^{\frac{2}{3}})}{t(1.5t)^{\frac{1}{3}}}dt = 9, \qquad (1.22)$$

and

$$\int_0^\infty \frac{J_{\frac{2}{3}}(\lambda t)+J_{-\frac{2}{3}}(\lambda t}{t}dt=\int_0^\infty \frac{\sqrt{3}Bi'(-\lambda(1.5t)^{\frac{2}{3}})}{t(1.5t)^{\frac{2}{3}}}dt=3. \qquad (1.23)$$

Where$J_\nu(.)$ stands for the Bessel's function of the first kind of order ν and Ai(.), Bi(.) are the Airy functions of first and secon kind (see[11] for details).

Proof. By definition of the Laplace transform, we have

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\}=\int_0^\infty \frac{J_\mu(\lambda t)e^{-st}}{t}dt=\frac{((\sqrt{s^2+\lambda^2})-s)^\mu}{\mu\lambda^\mu}. \qquad (1.24)$$

Let us introduce a change of parameter $s=\lambda\sinh\phi$, after simplifying in the above integral, we obtain

$$\int_0^\infty \frac{J_\mu(\lambda t)e^{-(\lambda\sinh\phi)t}}{t}dt:=\frac{e^{-\phi\mu}}{\mu}. \qquad (1.25)$$

In relation (1.25), let us first choose $\phi=0$ after simplifying, we have

$$\int_0^\infty \frac{J_\mu(\lambda t)}{t}dt=\frac{1}{\mu}, \qquad (1.26)$$

at this point, if we differentiate with respect to order μ, we get

$$\int_0^\infty \frac{J^*_\mu(\lambda t)}{t}dt=-\frac{1}{\mu^2}. \qquad (1.27)$$

In relation (1.26), let us first choose $\mu=\frac{1}{3}$, $\mu=-\frac{1}{3}$ and $\mu=\frac{2}{3}$, $\mu=-\frac{2}{3}$ after subtracting and adding the relations respectively, we obtain

$$\int_0^\infty \frac{J_{\frac{1}{3}}(\lambda t)-J_{-\frac{1}{3}}(\lambda t}{t}dt=\int_0^\infty \frac{-\sqrt{3}Bi(-\lambda(1.5t)^{\frac{2}{3}})}{t(1.5t)^{\frac{1}{3}}}dt=9, \qquad (1.28)$$

and

$$\int_0^\infty \frac{J_{\frac{2}{3}}(\lambda t)+J_{-\frac{2}{3}}(\lambda t}{t}dt=\int_0^\infty \frac{\sqrt{3}Bi'(-\lambda(1.5t)^{\frac{2}{3}})}{t(1.5t)^{\frac{2}{3}}}dt=3. \qquad (1.29)$$

2. Linearized Korteweg. de - vries

Solution to Time Fractional Non- Homogeneous Linearized KdV

The KdV equations are attracting many researchers, and a great deal of works has already been done in some of these equations. In this section, we will implement the operational method to construct exact solution for a variant of the KdV equation.

Problem 2.1. Let us consider the following linearized KdV.We implement integral transform method to obtain a formal solution to the above mentioned linearized KdV.

$$\frac{(\partial)^{0.5}u(x,t)}{\partial t^{0.5}}+\frac{\partial^3 u(x,t)}{\partial x^3}=k, \qquad (1)$$

where $-\infty<x<\infty$, $t>0$ and subject to the initial condition

$$u(x,0)=\phi(x), -\infty<x<\infty.$$

Note: Fractional derivative is in the Caputo sense.The constant k is the source term.Though much simplified compared with the nonlinear KdV, the above equation itself has many

applications. As an example, it is used to study the propagation of fairly long waves in the shallow water. Linearty property makes possible to use many effective analytical methods such as the Laplace transform.
Solution: Let us define the joint Laplace - Fourier transform as follows

$$\mathcal{F}\{L\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{i\omega x}\int_0^{\infty} e^{-st}u(x,t)dt)dx := U(\omega,s),$$

taking the joint Laplace - Fourier transform of PDE term wise and Fourier transform of boundary condition leads to the following relationship

$$U(\omega,s) = \tfrac{s^{-0.5}\Phi(\omega)}{s^{0.5}+(i\omega)^3} + \tfrac{ks^{-1}\delta(\omega)}{s^{0.5}+(i\omega)^3},$$

upon inverting the joint Laplace - Fourier transform leads to

$$\mathcal{F}^{-1}\{L^{-1}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}(\int_{c-i\infty}^{c+i\infty} \tfrac{s^{-0.5}\Phi(\omega)e^{st}+ks^{-1}\delta(\omega)e^{st}}{s^{0.5}+(i\omega)^3}ds)d\omega := u(x,t),$$

or, equivalently

$$u(x,t) = (\tfrac{1}{\sqrt{2\pi}})(\int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)(\int_{c-i\infty}^{c+i\infty} \tfrac{e^{st}ds}{\sqrt{s}(\sqrt{s}+(i\omega)^3)})d\omega + ...$$

$$......... + k\int_{-\infty}^{+\infty} e^{-i\omega x}\delta(\omega)(\int_{c-i\infty}^{c+i\infty}(\tfrac{e^{st}ds}{s(\sqrt{s}+(i\omega)^3)})d\omega)$$

in view of the Lemma 1.1 and after calculation of the inner integrals we obtain

$$u(x,t) = \tfrac{1}{\sqrt{2\pi}}(\int_{-\infty}^{+\infty} e^{-i\omega x-\omega^6 t}\Phi(\omega)(Erfc((i\omega)^3\sqrt{t})d\omega + ...$$

$$.......... + \int_{-\infty}^{+\infty} e^{-i\omega x-\omega^6 t}\tfrac{\delta(\omega)}{(i\omega)^3}(1 - Erfc((i\omega)^3\sqrt{t}))d\omega,$$

finally,

$$u(x,t) = \tfrac{1}{\sqrt{2\pi}}(\int_{-\infty}^{+\infty} e^{-i\omega x-\omega^6 t}\Phi(\omega)Erfc((i\omega)^3\sqrt{t})d\omega - k\sqrt{t}),$$

obviously, we have

$$u(x,0) = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)d\omega = \phi(x).$$

Problem 2.2. Let us consider the following system of linear KdV. We implement the exponential operator scheme to obtain a formal solution to the system.

$$\frac{\partial u(x,t)}{\partial t} - \beta v(x,t) + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \tag{2}$$

$$\frac{\partial v(x,t)}{\partial t} + \beta u(x,t) + \frac{\partial^3 v(x,t)}{\partial x^3} = 0, \tag{3}$$

where $-\infty < x < \infty$, $t > 0$ and subject to the boundary conditions and the initial condition

$$u(x,0) = \Phi(x), v(x,0) = \Psi(x), -\infty < x < \infty.$$

Solution: Let us define the function $w(x,t) = u(x,t) + iv(x,t)$ and the initial condition $w(x,0) = \Omega(x)$ we get the following KdV equation

$$\frac{\partial w(x,t)}{\partial t} + i\beta w(x,t) + \frac{\partial^3 w(x,t)}{\partial x^3} = 0, \tag{4}$$

with the initial condition $w(x,0) = \Omega(x)$. At this point, in order to solve the above linear KdV, we may rewrite the equation in the following form

$$\frac{\partial w(x,t)}{\partial t} = -(i\beta + \frac{\partial^3}{\partial x^3})w(x,t). \tag{5}$$

In order to obtain a solution for equation (4), first by solving the first order PDE with respect to t, and applying the initial condition, we get the following

$$w(x,t) = \exp(-i\beta t)\exp(-t\tfrac{\partial^3}{\partial x^3})\Omega(x),$$

by virtue of the Lemma 1.2, we have

$$w(x,t) = \exp(-i\beta t)\int_{-\infty}^{\infty}\Omega(x-\xi\sqrt[3]{3t})Ai(\xi)d\xi,$$

finally, we obtain the solution to the system of KdV as below

$$u(x,t) = \cos(\beta t)\int_{-\infty}^{\infty}\Phi(x-\xi\sqrt[3]{3t})Ai(\xi)d\xi + \sin(\beta t)\int_{-\infty}^{\infty}\Psi(x-\xi\sqrt[3]{3t})Ai(\xi)d\xi,$$

and

$$v(x,t) = \cos(\beta t)\int_{-\infty}^{\infty}\Psi(x-\xi\sqrt[3]{3t})Ai(\xi)d\xi - \sin(\beta t)\int_{-\infty}^{\infty}\Phi(x-\xi\sqrt[3]{3t})Ai(\xi)d\xi.$$

Note: It is easy to verify that $u(x,0) = \Phi(x)$, $v(x,0) = \Psi(x)$.

3. Evaluation of Certain Integrals and Solution to Singular Integral Equation

The main purpose of this section is to introduce the use of exponential differential operator technique for evaluation of certain integrals.
where $E_{\alpha,\beta}(.:.)$ stands for the Mittag-Leffler function with parameters α,β.
Note:The special function of the form defined by the following series representation
$E_{\alpha;\beta,\gamma}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma^{\alpha+1}(n\beta+\gamma)}$

is known as α- Mittag - Leffler function with three papameters. It has a wide application in the problem of physics, chemistry, engineering, applied mathematical sciences. An extension of Mittag - Leffler function of two parameters has given by H.M.Srivastava(see [13] for details).
Lemma 3.1. Considering the integral

$$I_0 = I(x,\alpha;\beta,\gamma) = \int_0^{\infty}E_{\alpha;\beta,\gamma}(\tfrac{x}{(k^2+t^2)^{\mu}})dt, \qquad (3.1)$$

as a function with parameters α ;β,γ, show that$I(x,\alpha;\beta,\gamma)$ satisfies the following relationship

$$I_0 = \int_0^{\infty}E_{\alpha;\beta,\gamma}(\tfrac{x}{(k^2+t^2)^{\mu}})dt = \sum_{n=0}^{\infty}\frac{k\sqrt{\pi}}{2\Gamma^{\alpha+1}(n\beta+\gamma)}\frac{\Gamma(\mu(n+\nu)-0.5)}{\Gamma(\mu(n+\nu))}(k^{-2\mu}x)^n.(3.2)$$

Proof. By making a change of variable $t = ky$ and letting $x = k^{2\mu}r$, we get

$$I_0 = k\int_0^{\infty}E_{\alpha;\beta,\gamma}(\tfrac{r}{(1+y^2)^{\mu}})dy, \qquad (3.3)$$

The above integral can be written in the following operational form

$$I_0 = k\int_0^{\infty}E_{\alpha;\beta,\gamma}(\tfrac{r}{(1+y^2)^{\mu}})dy = k(\int_0^{\infty}(\tfrac{1}{1+y^2})^{\mu rD_r}dy)E_{\alpha;\beta,\gamma}(r), \qquad (3.4)$$

after evaluation and simplifying the right hand side integral, this last result leads to

$$I_0 = k\int_0^{\infty}E_{\alpha;\beta,\gamma}(\tfrac{r}{(1+y^2)^{\mu}})dy = \frac{k\sqrt{\pi}}{2}\frac{\Gamma(\mu rD_r-0.5)}{\Gamma(\mu rD_r)}E_{\alpha;\beta,\gamma}(r). \qquad (3.5)$$

By using Taylor expansion of the α - Mittag-Leffler function with parameters α,β,γ , we have

$$I_0 = \frac{k\sqrt{\pi}}{2}\sum_{n=0}^{\infty}\frac{1}{\Gamma^{\alpha+1}(n\beta+\gamma)}\frac{\Gamma(\mu rD_r-0.5)}{\Gamma(\mu rD_r)}(r)^n, \qquad (3.6)$$

finally,

$$I_0 = \frac{k\sqrt{\pi}}{2}\sum_{n=0}^{\infty}\frac{1}{\Gamma^{\alpha+1}(n\beta+\gamma)}\frac{\Gamma(\mu(n+\nu)-0.5)}{\Gamma(\mu(n+\nu))}(k^{-2\mu}x)^n. \qquad (3.7)$$

Corollary 3.1. Let us consider the following Fredholm singular integral equation

$$\int_{-\infty}^{\infty} e^{-\xi^2}\phi(x+2\xi\sqrt{\lambda})d\xi = \frac{k}{x+\omega}, \tag{3.8}$$

the above integral equation has the following formal solution

$$\Phi(x) = \frac{k}{2\lambda}e^{\frac{(x+\omega)^2}{4\lambda}}. \tag{3.9}$$

Proof. Let us rewrite the right hand side of the above equation as below

$$\int_{-\infty}^{\infty} d\xi e^{-\xi^2}e^{-2\sqrt{\lambda}\xi D_x}\Phi(x) = \frac{k}{x+\omega}, \tag{3.10}$$

and treating the derivative operator as a constant, the evaluation of the integral yields

$$\Phi(x) = \frac{1}{\sqrt{\pi}}e^{\lambda D_x^2}\frac{k}{x+\omega}, \tag{3.11}$$

at this point, using relation (1.23) , leads to

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\frac{1}{(2\lambda\sqrt{\pi})}\int_0^{\infty} e^{-\frac{u^2}{4\lambda}}\left(\frac{k}{x+iu+\omega}+\frac{k}{x-iu+\omega}\right)du, \tag{3.12}$$

from which and after some easy calculations, we arrive at

$$\Phi(x) = \frac{k(x+\omega)}{2\lambda\pi}\int_{-\infty}^{\infty}\frac{e^{-\frac{u^2}{4\lambda}}}{u^2+(x+\omega)^2}du, \tag{3.13}$$

in order to evaluate the above integral, we may use calculus of residues to obtain

$$\Phi(x) = \frac{k}{2\lambda}e^{\frac{(x+\omega)^2}{4\lambda}}. \tag{3.14}$$

4. Time Fractional Couette Flow

Fractional calculus has been used to model physical and engineering processes which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer order derivatives, including nonlinear models do not work adequately in many cases. In this section, the author implemented the operational method for solving certain time fractional partial differential equations. The study of non-Newtonian fluids has generated much interest in recent years in view of their numerous industrial applications [7], especially in polymer and chemical industries. The examples of such fluids includes various suspensions such as coal-water or coal-oil slurries, molten plastics, polymer solutions, food products, glues, paints, printing inks, soaps, shampoos, toothpastes, clay coating, grease, cosmetic products, custard, blood, etc. Couette flow is an important type of flow in the history of fluid mechanics. Researchers have deep interest in this flow and they study it in many ways. Some important studies about this flow are as follows: Fang [8] studied Couette flow problem for unsteady incompressible viscous fluid bounded by porous walls. Khaled and Vafai [10] considered Stokes and Couette flows due to an oscillating wall. Consequently, considerable attention has been given to the solution of fractional partial differential equations of physical interest. In fluid dynamics Couette flow is the laminar flow of a viscous fluid in the space between two parallel plates, one of which is moving relative to the other. The flow is driven by virtue of viscous drag force acting on the fluid and the applied pressure gradient parallel to the plates. This type of flow is named in honor of Maurice Marie Alfred Couette. Couette flow is frequently used in physics and engineering to illustrate shear driven fluid motion[4]. The simplest conceptual configuration finds two infinite, parallel plates separated by a distance h. One plate, say the bottom one, translates with a constant velocity u_0 in its own plane. This equation reflects the assumption that the flow is uni-directional. That is, only one of the three velocity components is non-trivial.

Problem 4.1: Solving the Couette Flow with sudden motion of bottom plate

$$(1+l^2\beta^2)D_t^\alpha u(y,t)=\frac{\partial^2 u}{\partial y^2}+l^2\frac{\partial^2}{\partial y^2}D_t^\alpha u(y,t)+\beta^2 u, 0\le\alpha<1, \quad (4.1)$$

$$u(0,t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad u(1,t)=0, \qquad u(y,0)=0, 0<y<1. \quad (4.2)$$

Solution: The Laplace transform applied to (4.1) yields

$$\frac{\partial^2 U}{\partial y^2}-\left(\frac{\beta^2+(1+l^2\beta^2)s^\alpha}{1+s^\alpha l^2}\right)U=0, \quad (4.3)$$

the solution of which is

$$U(y,s)=c_1 e^{-y\sqrt{\frac{\beta^2+(1+l^2\beta^2)s^\alpha}{1+s^\alpha l^2}}}+c_2 e^{+y\sqrt{\frac{\beta^2+(1+l^2\beta^2)s^\alpha}{1+s^\alpha l^2}}}. \quad (4.4)$$

Taking into account the Laplace transform of the boundary conditions

$$U(0,s)=\frac{1}{s^\alpha}, U(1,s)=0, \quad (4.5)$$

and therefore, through use of the relation (4.5), we arrive at the result

$$U(y,s)=\frac{sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s^\alpha}{1+s^\alpha l^2}})}{s^\alpha sinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s^\alpha}{1+s^\alpha l^2}})}. \quad (4.6)$$

At this stage, in order to invert (4.6), we use the method of the residues.
Let us define the function $G(y,s)$ as follows

$$G(y,s)=\frac{sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}{ssinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}, \quad (4.7)$$

it is easy to verify that $G(y,s^\alpha)=U(y,s)$. (4.8)

The function $G(y,s)$ has a simple pole at $s=0$ and simple poles at $s_k=-\frac{\beta^2+(\pi k)^2}{1+l^2(\beta^2+(\pi k)^2)}$, $k=1,2,3,...$, therefore, we find that

$$\mathcal{L}^{-1}G(y,s)=(Res\frac{sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}{ssinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}:s=0)+$$

$$+\sum_{k=1}^{k=+\infty}(Res\frac{sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}{ssinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}:s_k=-\frac{\beta^2+(\pi k)^2}{1+l^2(\beta^2+(\pi k)^2)}). \quad (4.9)$$

Let us evaluate the residues at simple poles

$$s=0, s_k=-\frac{\beta^2+(\pi k)^2}{1+l^2(\beta^2+(\pi k)^2)}, k=1,2,3,...,. \quad (4.10)$$

The residue at simple pole $s=0$ is as follows

$$b_0=lim_{s->0}\frac{sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}{sinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}=\frac{sinh(1-y)\beta}{sinh\beta}. \quad (4.11)$$

The residue at simple poles $s_k = -\frac{\beta^2+(\pi k)^2}{1+l^2(\beta^2+(\pi k)^2)}, k = 1, 2, 3, ...$
after simplifying is as follows

$$b_k = lim_{s->s_k} \frac{(s-s_k)sinh((1-y)\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})}{ssinh(\sqrt{\frac{\beta^2+(1+l^2\beta^2)s}{1+sl^2}})} = -\frac{(1-y)siny}{(\beta^2+(\pi k)^2)(1+l^2(\beta^2+(\pi k)^2)}, (4.12)$$

from which we deduce

$$\mathcal{L}^{-1}G(y,s) = g(y,t) = h(t)(\frac{sinh(1-y)\beta}{sinh\beta} + 2\pi \sum_{k=1}^{+\infty} \frac{(-1)^k k e^{s_k t} sin(1-y)k\pi}{(\beta^2+(\pi k)^2)(1+l^2(\beta^2+(\pi k)^2)}),$$

finally, by using part three of the Lemma 1.1, we arrive at the solution to (4.1)-(4.2)

$$\mathcal{L}^{-1}U(y,s) = \mathcal{L}^{-1}G(y,s^\alpha) = \frac{1}{\pi}\int_0^\infty g(y,w)(\int_0^\infty e^{-tr-wr^\alpha cos\alpha\pi} sin(wr^\alpha sin\alpha\pi)dr)dw. (4.13)$$

Let us consider the special case $\alpha = 0.5$, using part four of the Lemma 1.1, we get the following

$$(1 + l^2\beta^2)D_t^{0.5}u(y,t) = \frac{\partial^2 u}{\partial y^2} + l^2\frac{\partial^2}{\partial y^2}D_t^{0.5}u(y,t) + \beta^2 u, \quad (4.14)$$
$$u(0,t) = \frac{1}{\sqrt{\pi t}}, \quad u(1,t) = 0, \quad u(y,0) = 0, 0 < y < 1, \quad .(4.15)$$

with formal solution as below

$$\mathcal{L}^{-1}U(y,s) = \mathcal{L}^{-1}G(y,\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty re^{-\frac{r^2}{4t}}g(y,r)dr, \quad (4.16)$$

equivalently,

$$u(y,t) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty re^{-\frac{r^2}{4t}}h(r)(\frac{sinh(1-y)\beta}{sinh\beta} + 2\pi\sum_{k=1}^{+\infty}\frac{(-1)^k k e^{rs_k} sin(1-y)k\pi}{(\beta^2+(\pi k)^2)(1+l^2(\beta^2+(\pi k)^2)})dr, (4.17)$$

it is easy to check the boundary conditions;

$u(y,0) = 0, u(1,t) = 0, u(0,t) = \frac{1}{\sqrt{\pi t}}$.
Note. In relation (4.17) $h(.)$ stands for the Heaviside unit step function.

5. Conclusion

Operational methods provide fast and universal mathematical tool for obtaining solution of PDEs or even FPDEs. Combination of integral transforms, operational methods and special functions give more powerful analytical instrument for solving a wide range of engineering and physical problems. The paper is devoted to study exponential operators and their applications in solving certain boundary value problems. The main purpose of this work is to develop methods for solving certain linear time fractional Couette flow.

Acknowledgement. The author would like to express sincere thanks to the referees for valuable comments and suggestions that lead to a vast improvement in the paper.

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