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G-CALCULUS

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Abstract. Based on M. Grossman in [13] and Grossman an Katz [12], in this paper we prove geometric Rolle's theorem, Taylor's theorem, Mean value theorem. Also, we discuss about the properties and applications of bigeometric calculus.

Keywords: Geometric real numbers; geometric arithmetic; bigeometric derivative.

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1. Introduction

Non-Newtonian calculus also called as multiplicative calculus, introduced by Grossman and Katz [12]. The operations of multiplicative calculus are called as multiplicative derivative and multiplicative integral. We refer to Grossman and Katz [12], Stanley [20], Campbell [10], Grossman [13, 14], Jane Grossman [15, 16] for different types of Non-Newtonian calculus and its applications. Bashirov et al. [3] gaven the complete mathematical description of multiplicative calculus. An extension of multiplicative calculus to functions of complex variables found in [1, 2, 21, 22, 23]. The generalized Runge-Kutta method with respect to non-Newtonian calculus studied by Kadak and Özlük [17]. Çakmak and Başar [7] constructed the field $\mathbb{C}^*$ of $*$-complex numbers. Çakmak and Başar [8], the line and double integrals in the sense of $*$-calculus are given. Moreover, in the sense of $*$-calculus, the fundamental theorems of calculus for line integrals and double integrals are stated with some applications. Çakmak and Başar [9], characterized matrix transformations in sequence spaces based on multiplicative calculus. Riza and Aktöre [18] discussed Runge-Kutta method in term of geometric multiplicative calculus.

Bigeometric-calculus is one of the family of non-Newton calculus. It provides differentiation and integration tools based on multiplication instead of addition. Generally, in growth related problems, price elasticity, numerical approximations problems Bigeometric-calculus can be advocated instead of a traditional Newtonian one. We refer [4, 6] to know basics of $\alpha - generator$ and geometric arithmetic $(\mathbb{R}(G), \oplus, \ominus, \odot, \oslash)$.

Türkmen and Başar [22] defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows:

$$\mathbb{Z}(G) = \{e^x : x \in \mathbb{Z}\}, \qquad \mathbb{R}(G) = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \backslash \{0\},$$
$$\mathbb{C}(G) = \{e^z : z \in \mathbb{C}\} = \mathbb{C} \backslash \{0\}.$$

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If we take extended real number line, then $\mathbb{R}(G) = [0, \infty]$.

Remark 1.1. *$(\mathbb{R}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e. But $(\mathbb{C}(G), \oplus, \odot)$ is not a field, however, geometric binary operation $\odot$ is not associative in $\mathbb{C}(G)$. For, we take $x = e^{1/4}, y = e^4$ and $z = e^{(1+i\pi/2)} = ie$. Then $(x \odot y) \odot z = e \odot z = z = ie$ but $x \odot (y \odot z) = x \odot e^4 = e$.*

Geometric positive real numbers and negative real numbers are defined respectively as

$$\mathbb{R}^+(G) = \{x \in \mathbb{R}(G) : x > 1\} \quad \text{and} \quad \mathbb{R}^-(G) = \{x \in \mathbb{R}(G) : x < 1\}.$$

1.1. **Useful relations between geometric and ordinary arithmetic operations.** For all $x, y \in \mathbb{R}(G)$

- $x \oplus y = xy$
- $x \ominus y = x/y$
- $x \odot y = x^{\ln y} = y^{\ln x}$
- $x \oslash y$ or $\frac{x}{y}_G = x^{\frac{1}{\ln y}}, y \neq 1$
- $x^{2_G} = x \odot x = x^{\ln x}$
- $x^{p_G} = x^{\ln^{p-1} x}$
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$
- $x^{-1_G} = e^{\frac{1}{\ln x}}$
- $x \odot e = x$ and $x \oplus 1 = x$
- $e^n \odot x = x^n$
- $$|x|^G = \begin{cases} x, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } 0 < x < 1 \end{cases}$$
- $\sqrt{x^{2_G}}^G = |x|^G$
- $|e^y|^G = e^{|y|}$
- $|x \odot y|^G = |x|^G \odot |y|^G$
- $|x \oplus y|^G \leq |x|^G \oplus |y|^G$
- $|x \oslash y|^G = |x|^G \oslash |y|^G$
- $|x \ominus y|^G \geq |x|^G \ominus |y|^G$
- $0_G \ominus 1_G \odot (x \ominus y) = y \ominus x$.

2. Main Results

2.1. **Geometric Real Number Line.** Consecutive geometric integers are geometrically equidistant as $e^{n+1} \ominus e^n = e^{n+1-n} = e$. Since $(\mathbb{R}(\mathbb{G}), \oplus, \odot)$ is a complete field with geometric identity e and geometric zero 1, so, we can consider a new number line with respect to geometric arithmetic which will be called geometric real number line.

2.2. **Geometric Co-ordinate System.** We consider two mutually perpendicular geometric real number lines which intersect each other at $(1, 1)$ as shown in FIGURE 1 to form geometric co-ordinate system.

2.3. **Geometric Factorial.** In [4], we defined geometric factorial notation $!_G$ as

$$n!_G = e^n \odot e^{n-1} \odot e^{n-2} \odot \cdots \odot e^2 \odot e = e^{n!}.$$

2.4. **Geometric Pythagorean Triplets.** Three numbers $x, y, z \in \mathbb{R}(G)$ are said to be formed a geometric Pythagorean triplet(GPT) if

$$x^{2_G} = y^{2_G} \oplus z^{2_G}. \tag{1}$$

Or, equivalently

$$x^{\ln x} = y^{\ln y}.z^{\ln z} \text{ or } (\ln x)^2 = (\ln y)^2 + (\ln z)^2.$$

Thus, if $\{x, y, z\} \subset \mathbb{R}(G)$ is a GPT, then $\{\ln x, \ln y, \ln z\}$ forms an ordinary Pythagorean triplet(OPT). Conversely, if $\{a, b, c\}$ is a positive OPT, then $\{e^a, e^b, e^c\}$ forms a GPT.

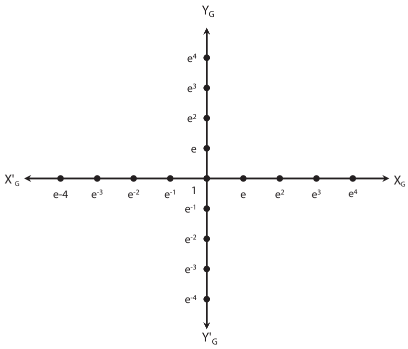

FIGURE 1. Geometric Co-ordinate System

Definition 2.1 (Geometric Right Triangle)**.** *In the geometric co-ordinate system, if geometric lengths of the three sides of a triangle represent a GPT, then the triangle will be called geometric right triangle.*

It is to be noted that a GPT does not form a triangle in ordinary sense.

Definition 2.2. *Area of geometric right triangle* $= \ln\left(\sqrt{base \odot altitude}\right)$.

2.5. **Geometric Trigonometric Ratios.** Let θ be an acute angle of a geometric right triangle and length of the sides be $h, p, b \in \mathbb{R}(G)$ with usual meaning. Then we define

$$\operatorname{sing}\theta = \frac{p}{h}\,{}_G = p^{\frac{1}{\ln h}} \qquad \operatorname{cscg}\theta = \frac{h}{p}\,{}_G = h^{\frac{1}{\ln p}}$$

$$\operatorname{cosg}\theta = \frac{b}{h}\,{}_G = b^{\frac{1}{\ln h}} \qquad \operatorname{secg}\theta = \frac{h}{b}\,{}_G = h^{\frac{1}{\ln b}}$$

$$\operatorname{tang}\theta = \frac{p}{b}\,{}_G = p^{\frac{1}{\ln b}} \qquad \operatorname{cotg}\theta = \frac{b}{p}\,{}_G = b^{\frac{1}{\ln p}}$$

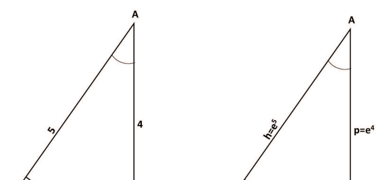

FIGURE 2. Geometric Right Triangle

2.6. **Relation between geometric and ordinary trigonometry.** Since n unit length in ordinary coordinate system is equal to e^n unit in geometric coordinate system. So properties of the geometric right triangle having sides $h, p, b \in \mathbb{R}(G)$ will be same to the ordinary right triangle having sides $h' = \ln(h), p' = \ln(p)$ and $b' = \ln(b)$, respectively. An example is given in FIGURE 2. Here, area of the both the triangles $= 6$ square unit, $\angle A = 36.87°, \angle B = 53.13°$ and $\angle C = 90°$.

It can be proved that $\operatorname{sing}\theta = e^{\sin\theta}, \operatorname{cosg}\theta = e^{\cos\theta}, \operatorname{tang}\theta = e^{\tan\theta}$ and $\frac{\operatorname{sing}\theta}{\operatorname{cosg}\theta}\,{}_G = \operatorname{tang}\theta$.

2.7. **Geometric Trigonometric Identities.** We can verify that

$$\operatorname{sing} A \odot \operatorname{cscg} A = e, \qquad \operatorname{sing}^{2_G} A \oplus \operatorname{cosg}^{2_G} A = e$$
$$\cos A \odot \sec A = e, \qquad \operatorname{tang}^{2_G} A \oplus e = \operatorname{secg}^{2_G} A$$
$$\operatorname{tang} A \odot \operatorname{cotg} A = e, \qquad \operatorname{cotg}^{2_G} A \oplus e = \operatorname{cscg}^{2_G} A.$$

$$\operatorname{sing}(A+B) = \operatorname{sing} A \odot \operatorname{cosg} B \oplus \operatorname{cosg} A \odot \operatorname{sing} B.$$
$$\operatorname{cosg}(A+B) = \operatorname{cosg} A \odot \operatorname{cosg} B \ominus \operatorname{sing} A \odot \operatorname{sing} B.$$

2.8. ***G*-Limit.** According to Grossman and Katz [12], geometric limit of a positive valued function defined in a positive interval is same to the ordinary limit. Here, we define G-limit of a function with the help of geometric arithmetic as follows:

A function f, which is positive in a given positive interval, is said to tend to the limit $l > 0$ as x tends to $a \in \mathbb{R}$, if, corresponding to any arbitrarily chosen number $\epsilon > 1$, however small(but greater than 1), there exists a positive number $\delta > 1$, such that

$$1 < |f(x) \ominus l|^G < \epsilon$$

for all values of x for which $1 < |x \ominus a|^G < \delta$. We write ${}_G\lim_{x\to a} f(x) = l$ or $f(x) \xrightarrow{G} l$. Here,

$$|x \ominus a|^G < \delta \Rightarrow \frac{a}{\delta} < x < a\delta \quad \text{and} \quad |f(x) \ominus l|^G < \epsilon \Rightarrow \frac{l}{\epsilon} < f(x) < l\epsilon.$$

A function f is said to tend to limit l as x tends to a from the left, if for each $\epsilon > 1$ (however small), there exists $\delta > 1$ such that $|f(x) \ominus l|^G < \epsilon$ when $a/\delta < x < a$. In symbols

$${}_G\lim_{x\to a-} f(x) = l \text{ or } f(a-1) = l.$$

Similarly, a function f is said to tend to limit l as x tends to a from the right, if for each $\epsilon > 1$, there exists $\delta > 1$ such that $|f(x) \ominus l|^G < \epsilon$ when $a < x < a\delta$. In symbols

$${}_G\lim_{x\to a+} f(x) = l \text{ or } f(a+1) = l.$$

2.9. ***G*-Continuity.** A function f is said to be G-continuous at $x = a$ if

(i) $f(a)$ i.e., the value of $f(x)$ at $x = a$, is a definite number,

(ii) the G-limit of the function $f(x)$ as $x \xrightarrow{G} a$ exists and is equal to $f(a)$.

Alternatively, a function f is said to be G-continuous at $x = a$, if for arbitrarily chosen $\epsilon > 1$, however small, there exists a number $\delta > 1$ such that $|f(x) \ominus f(a)|^G < \epsilon$ for all values of x for which, $|x \ominus a|^G < \delta$.

It is seen that a function f is G-continuous at $x = a$ if $\lim_{x\to a} \frac{f(x)}{f(a)} = 1$.

3. Basic Properties of G-Calculus

3.1. ***G*-Derivative and its Interpretation.** In [5] we defined the G-differentiation of $f(x)$ as

$$\frac{d^G f}{dx^G} = f^G(x) = {}_G\lim_{h\to 1} \frac{f(x \oplus h) \ominus f(x)}{h}{}_G = \lim_{h\to 1} \left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}} \quad \text{for } h \in \mathbb{R}(\mathbb{G}). \tag{2}$$

The G-derivative of a positive valued function f at a point c belonging to a positive interval can be defined as

$$f^G(c) = {}_G\lim_{x\to c} \frac{f(x) \ominus f(c)}{x \ominus c}{}_G = \lim_{x\to c} \left[\frac{f(x)}{f(c)}\right]^{\frac{1}{\ln(\frac{x}{c})}}. \tag{3}$$

Equation (3) is the bigeometric slope defined by Grossman in [13]. Depending on Grossman [13], Grossman and Katz [14], different researchers have been developing the bigeometric calculus taking arithmetic increment to the independent variable. But we are trying to develop their work with the help of geometric increments. So, to remove the confusion among the readers, instead of the phase "bigeometric calculus" the term "G-calculus" is used throughout the paper.

From (3), it is clear that G-derivative exists if both $f(x)$ and $f(c)$ takes same sign and at the same time x and c takes same sign.

$x+h$ is arithmetic change and $x \oplus h = xh$ is geometric change to the independent variable x. Now, as x changes to xh, value of the function changes from $f(x)$ to $f(x \oplus h) = f(xh)$. Geometric changes to x and y are given by

$$\Delta x = x \oplus h \ominus x = \frac{xh}{x} = h \text{ and } \Delta y = f(x \oplus h) \ominus f(x) = \frac{f(xh)}{f(x)}.$$

In case of ordinary derivative $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$ gives the average additive change in $f(x)$ per unit change in x over the interval $[x, x+\Delta x] = [x, x+h]$. Here in G-calculus,

$$\frac{\Delta y}{\Delta x}{}_G = (\Delta y)^{\frac{1}{\ln(\Delta x)}} = \left[\frac{f(xh)}{f(x)}\right]^{\frac{1}{\ln h}}$$

gives the average geometric change in $f(x)$ per unit geometric change in x over the interval $[x, xh]$. Now taking the limit as Δx(i.e. h) tends to 1, we get

$$\frac{d^G y}{d^G x} = {}^G\lim_{\Delta x \to 1} \frac{\Delta y}{\Delta x}{}_G = {}^G\lim_{\Delta x \to 1} (\Delta y)^{\frac{1}{\ln(\Delta x)}} = \lim_{h \to 1} \left[\frac{f(xh)}{f(x)}\right]^{\frac{1}{\ln h}}.$$

It is to be noted that G-derivative exists if $f(x) \neq 0$ and $f(x), f(hx)$ are both positive or both negative.

$y = m \odot x \oplus c$ i.e. $y = c.x^{\ln m}$ represents a straight line with slope m in geometric co-ordinate system as well as in log-log paper. Then, $y^G = m$. i.e. G-derivative is the slope of the geometric straight line.

Note: We'll denote n^{th} geometric derivative by $f^{[n]}$. We call that left hand G-derivative and right hand G-derivative exist at $x = c$ if

$$\lim_{x \to c-} \left(\frac{f(c.h)}{f(c)}\right)^{\frac{1}{\ln(\frac{x}{c})}} \text{ and } \lim_{x \to c+} \left(\frac{f(c.h)}{f(c)}\right)^{\frac{1}{\ln(\frac{x}{c})}}$$

exist, respectively.

Theorem 3.1. *If a function f is G-differentiable and is positive, then it is both G-continuous and ordinary continuous.*

Proposition 3.1. *A continuous function f is not necessarily G-derivable.*

Proof. Let us consider the function

$$f(x) = |x|^G = \begin{cases} x, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } 0 < x < 1. \end{cases}$$

Then, obviously it is continuous at $x = 1$. But it is not G-differentiable at $x = 1$ as $Lf^G(1) = e$ but $Rf^G(1) = \frac{1}{e}$. ☐

Example 3.1. *If $f(x) = x^{n_G}$, then $f^G(x) = e^n \odot x^{(n-1)_G}$ and $f^{(n_G)} = e^{n!}$.*

Remark 3.1. *$f(x) = x^n$ is a polynomial of degree n in ordinary sense, but geometrically it is a polynomial of degree one as $x^n = e^n \odot x$. So, its G-derivative is constant.*

3.2. **Relation between G-derivative and ordinary derivative.** By definition, G-derivative of a positive valued function $f(x)$ is given by

$$f^G(x) = {}_G\lim_{h\to 1} \frac{f(x \oplus h) \ominus f(x)}{h}{}_G$$
$$= \lim_{h\to 1} \left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}}, \text{ which is in } 1^\infty \text{ indeterminate form.}$$

Using logarithm, to transform it to $\frac{0}{0}$ indeterminate form and then applying L' Hospital rule, we can make a relation between G-derivative and ordinary derivative as follows:

$$f^G(x) = \lim_{h\to 1} e^{\ln\left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}}} = \lim_{h\to 1} e^{\frac{\ln f(hx) - \ln f(x)}{\ln h}} = e^{\lim_{h\to 1} \frac{hxf'(hx)}{f(hx)}} = e^{\frac{xf'(x)}{f(x)}}. \tag{4}$$

3.3. **G-derivatives of some standard functions.**

- **G-derivative of a constant:** If $f(x) = c$, then $f^G(x) = 1$
- **G-derivative of ordinary product of a constant and a function:**
$$\frac{d^G}{dx^G}(cf(x)) = e^{x\frac{cf'(x)}{cf(x)}} = e^{x\frac{f'(x)}{f(x)}} = \frac{d^G}{dx^G}(f(x)).$$
- **G-derivative of ordinary product of two functions:**
$$\frac{d^G}{dx^G}(f(x).g(x)) = \frac{d^G}{dx^G}(f(x)).\frac{d^G}{dx^G}(g(x)). \tag{5}$$
- **G-derivative of quotient of two functions:**
$$\frac{d^G}{dx^G}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d^G}{dx^G}(f(x))}{\frac{d^G}{dx^G}(g(x))}. \tag{6}$$
- **G-derivative of trigonometric functions:**
$$\frac{d^G}{dx^G}(\sin x) = e^{x\cot x}, \qquad \frac{d^G}{dx^G}(\cot x) = e^{-x\sec x\csc x}$$
$$\frac{d^G}{dx^G}(\cos x) = e^{-x\tan x}, \qquad \frac{d^G}{dx^G}(\sec x) = e^{x\tan x}$$
$$\frac{d^G}{dx^G}(\tan x) = e^{x\sec x\csc x}, \qquad \frac{d^G}{dx^G}(\csc x) = e^{-x\cot x}.$$

Theorem 3.2. *If $f : (a, b) :\longrightarrow \mathbb{R}(G)$ is G-differentiable, then*
(i) *f is increasing, if $f^G \geq 1$.*
(ii) *f is decreasing, if $f^G \leq 1$.*

Proof. Let c be an interior point of the domain $[a, b]$ of a function f and $f^G(c)$ exists and be positive, i.e. $f^G(c) > 1$. Then, $f^G(c)$ is the limit of $\left[\frac{f(x)}{f(c)}\right]^{\frac{1}{\ln(x/c)}}$. Then for given $\epsilon > 1, \exists\delta > 1$ such that

$$\frac{f^G(c)}{\epsilon} < \left[\frac{f(x)}{f(c)}\right]^{\frac{1}{\ln(x/c)}} < \epsilon.f^G(c) \text{ where } x \in]c/\delta, c\delta[.$$

If $\epsilon > 1$ is so chosen that $\epsilon < f^G(c)$, then $\left[\frac{f(x)}{f(c)}\right]^{\frac{1}{\ln(x/c)}} > \frac{f^G(c)}{\epsilon} > 1$. Then

(i) $\frac{f(x)}{f(c)} > 1$, i.e. $f(x) > f(c)$ if $x \in]c, c\delta[$,

(ii) $\frac{f(x)}{f(c)} < 1$, i.e. $f(x) < f(c)$ if $x \in]c/\delta, c[$.

Thus from (i) and (ii) $f(x)$ is increasing at $x = c$. Hence the function is increasing at $x = c$ if $f^G(c) > 1$. Similarly, it can be proved that the function is decreasing at $x = c$ if $f^G(c) < 1$. □

Theorem 3.3 (Geometric Darboux's Theorem)**.** *If a function f is G-derivable on a closed interval $[a, b]$ and $f^G(a), f^G(b)$ are of opposite signs (i.e. one is > 1, other is < 1) then there exists at least one point c between a and b such that $f^G(c) = 0$.*

Proof. Let $f^G(a) < 1$ and $f^G(b) > 1$. Since, G-derivative exists $\Rightarrow$ ordinary derivative exists, so, $f'(a)$ and $f'(b)$ exist. Now

$$f^G(a) < 1 \Rightarrow f'(a) < 0 \text{ and } f^G(b) > 1 \Rightarrow f'(b) > 0.$$

From Newtonian calculus, there exists $c \in [a, b]$ s.t. $f'(c) = 0$. So $f^G(c) = e^{c\frac{f'(c)}{f(c)}} = 1$. □

Theorem 3.4 (Geometric Intermediate value theorem for derivatives)**.** *If a function f is G-derivable on a closed interval $[a, b]$ and $f^G(a) \neq f^G(b)$ and k be a number lying between $f^G(a)$ and $f^G(b)$, then $\exists$ at least one point $c \in]a, b[$ such that $f^G(c) = k$.*

Proof. Let $g(x) = \frac{f(x)}{x^{\ln k}}$. Then $g^G(a) = \frac{f^G(a)}{k}$ and $g^G(b) = \frac{f^G(b)}{k}$. Since $f^G(a) < k < f^G(b)$, so $\frac{f^G(a)}{k}$ and $\frac{f^G(b)}{k}$ can not be greater than 1 at the same time. Therefore, if $g^G(a) > 1$ then $g^G(b) < 1$. Hence, $g(x)$ satisfies the conditions of Darboux's theorem. Thus, there exists at least one point $c \in]a, b[$ such that $g^G(c) = 1$, i.e. $f^G(c) = k$. □

Theorem 3.5 (Geometric Rolle's Theorem)**.** *If a function f defined on $[a, b]$ is*

(i) *G-continuous on $[a, b]$,*
(ii) *G-derivable on $]a, b[$,*
(iii) $f(a) = f(b)$,

then there exists at least one number c between a and b such that $f^G(c) = 1$.

Proof. Since G-continuous functions are ordinary continuous and $f'(x)$ exists if $f^G(x)$ exists. So, f satisfies the conditions of G ordinary Rulle's theorem. So, there exists $c \in]a, b[$ such that $f'(c) = 0$. Hence $f^G(c) = e^{\frac{cf'(c)}{f(c)}} = 1$. □

Theorem 3.6 (Lagrange's Mean Value Theorem)**.** *If a function f defined on $[a, b]$ is*

(i) *G-continuous on $[a, b]$,*
(ii) *G-derivable on $]a, b[$,*

then there exists at least one $c \in]a, b[$ such that

$$f^G(c) = \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{\ln(\frac{b}{a})}}$$

Proof. Let us define a function

$$\phi(x) = x^{\ln k}.f(x)$$

where the constant k is so determined that $\phi(a) = \phi(b)$.

$$\phi(a) = \phi(b) \Rightarrow a^{\ln k}.f(a) = b^{\ln k}.f(b) \Rightarrow \left[\frac{a}{b}\right]^{\ln k} = \frac{f(b)}{f(a).}$$

Using natural logarithm to both sides we get

$$k = \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{\ln(\frac{a}{b})}} = \left[\frac{f(b)}{f(a)}\right]^{\frac{-1}{\ln(\frac{b}{a})}}.$$

Now, $\phi(x)$, the product of two G-derivable and G-continuous functions, is itself

(i) G-continuous on $[a, b]$,
(ii) G-derivable on $]a, b[$, and
(iii) $\phi(a) = \phi(b)$.

Therefore by Geometric Rolle's theorem $\exists c \in]a, b[$ such that $\phi^G(c) = 1$. But

$$\phi^G(x) = \frac{d^G}{dx^G}(x^{\ln k}).\frac{d^G}{dx^G}(f(x)) = k.f^G(x).$$

$$\Rightarrow 1 = \phi^G(c) = k.f^G(c) \Rightarrow f^G(c) = \frac{1}{k} = \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{\ln(\frac{b}{a})}}.$$

□

Note: If we replace b by ah, where $h > 1$, then $c \in]a, b[$ may be taken as $a.h^{\ln\theta}$ for $1 < \theta < e$. Thus

$$f^G(a.h^{\ln\theta}) = \left[\frac{f(ah)}{f(a)}\right]^{\frac{1}{\ln(\frac{ah}{a})}}$$

$$\Rightarrow \qquad f(ah) = f(a).\left[f^G(a.h^{\ln\theta})\right]^{\ln h}, \text{ where } 1 < \theta < e.$$

Now we deduce geometric Taylor's expansion for $f(ah)$ with the help of Geometric Rolle's Theorem. Firstly, we have to find G-derivative of two important functions as follows.

Lemma 3.1. *If* $y = \left[f^{[n]}(x)\right]^{\frac{\ln^n(\frac{ah}{x})}{n!}}$ *then* $y^G = \dfrac{\left[f^{[n+1]}(x)\right]^{\frac{\ln^n(\frac{ah}{x})}{n!}}}{\left[f^{[n]}(x)\right]^{\frac{\ln^{(n-1)}(\frac{ah}{x})}{(n-1)!}}}$

Proof. Taking logarithm to both sides of y and differentiating, we get

$$\Rightarrow \frac{y'}{y} = \frac{\frac{d}{dx}\left(f^{[n]}(x)\right)}{f^{[n]}(x)}.\frac{\ln^n(\frac{ah}{x})}{n!} + \ln f^{[n]}(x).\frac{\ln^{n-1)}(\frac{ah}{x})}{(n-1)!}.\frac{\frac{-ah}{x^2}}{\frac{ah}{x}}$$

$$\Rightarrow e^{x\frac{y'}{y}} = e^{x\frac{f'^{[n]}(x)}{f^{[n]}(x)}.\frac{\ln^n(\frac{ah}{x})}{n!}}.e^{x\ln f^{[n]}(x).\frac{\ln^{(n-1)}(\frac{ah}{x})}{(n-1)!}.\frac{-1}{x}}$$

$$\Rightarrow y^G = \left[f^{[n+1]}(x)\right]^{\frac{\ln^n(\frac{ah}{x})}{n!}}.\left[f^{[n]}(x)\right]^{-\frac{\ln^{(n-1)}(\frac{ah}{x})}{(n-1)!}} = \frac{\left[f^{[n+1]}(x)\right]^{\frac{\ln^n(\frac{ah}{x})}{n!}}}{\left[f^{[n]}(x)\right]^{\frac{\ln^{(n-1)}(\frac{ah}{x})}{(n-1)!}}}.$$

□

Lemma 3.2. *If* $y = k^{\ln^p(\frac{ah}{x})}$ *where* k *is a constant and* p *is a positive integer, then* $y^G = k^{-p\ln^{(p-1)}(\frac{ah}{x})}$.

Proof. Taking logarithm on the both sides, and then differentiating we get the result. □

Theorem 3.7 (Geometric Taylor's Theorem)**.** *A function f defined on* $[a, ah]$ *is such that*

(i) *the* $(n-1)^{th}$ *G-derivative of* f*, i.e.* $f^{[n-1]}$ *is G-continuous on* $[a, ah]$*, and*
(ii) *the* n^{th} *G-derivative,* $f^{[n]}$ *exists on* $[a, ah]$

then there exists at least one number θ *between* 1 *and* e *such that*

$$f(ah) = f(a).\left[f^{[1]}(a)\right]^{\ln h}.\left[f^{[2]}(a)\right]^{\frac{\ln^2 h}{2!}}.\left[f^{[3]}(a)\right]^{\frac{\ln^3 h}{3!}}...$$

$$...\left[f^{[n-1]}(a)\right]^{\frac{\ln^{n-1} h}{(n-1)!}}.\left[f^{[n]}(a.h^{\ln\theta})\right]^{\frac{(1-\ln\theta)^{(n-p)}\ln^n h}{(n-1)!p}} \tag{7}$$

Proof. Condition (i) in the statement implies that $f^{[1]}, f^{[2]}, f^{[3]}, ..., f^{[n-1]}$ exists and are continuous on $[a, ah]$. Let us consider the function

$$\phi(x) \quad = \quad f(x).\left[f^{[1]}(x)\right]^{\ln(\frac{ah}{x})}.\left[f^{[2]}(x)\right]^{\frac{\ln^2(\frac{ah}{x})}{2!}}....\left[f^{[n-1]}(x)\right]^{\frac{\ln^{n-1}(\frac{ah}{x})}{(n-1)!}}.A^{\ln^p(\frac{ah}{x})} \quad (8)$$

where A is a constant to be determined such that $\phi(ah) = \phi(a)$.
But, putting $x = ah$ and $x = a$ in (8), respectively, we get

$$\phi(ah) = f(ah), \text{ and}$$

$$\phi(a) = f(a).\left[f^{[1]}(a)\right]^{\ln h}.\left[f^{[2]}(a)\right]^{\frac{\ln^2 h}{2!}}...\left[f^{[n-1]}(a)\right]^{\frac{\ln^{n-1} h}{(n-1)!}}.A^{\ln^p h}.$$

$$\therefore f(ah) = f(a).\left[f^{[1]}(a)\right]^{\ln h}.\left[f^{[2]}(a)\right]^{\frac{\ln^2 h}{2!}}...\left[f^{[n-1]}(a)\right]^{\frac{\ln^{n-1} h}{(n-1)!}}.A^{\ln^p h}. \quad (9)$$

Now

(i) $f, f^{[1]}, f^{[2]}, f^{[3]}, ..., f^{[n-1]}$ all being continuous on $[a, ah]$, the function $\phi(x)$ is continuous on $[a, ah]$;
(ii) the functions $f, f^{[1]}, f^{[2]}, f^{[3]}, ..., f^{[n-1]}$ and $\ln^r(\frac{ah}{x})$ for all r being derivable in $]a, ah[$, the function $\phi(x)$ is derivable in $]a, ah[$;
(iii) $\phi(ah) = \phi(a)$.

Hence, $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence there exists one real number $\theta \in]1, e[$ such that $\phi^G(a.h^{\ln\theta}) = 1$.

Now, using Lemma 3.1 and Lemma 3.2

$$\phi^G(x) \;=\; f^{[1]}(x).\frac{\left[f^{[2]}(x)\right]^{\ln(\frac{ah}{x})}}{f^{[1]}(x)}.\frac{\left[f^{[3]}(x)\right]^{\frac{\ln^2(\frac{ah}{x})}{2!}}}{\left[f^{[2]}(x)\right]^{\ln(\frac{ah}{x})}}....\frac{\left[f^{[n]}(x)\right]^{\frac{\ln^{(n-1)}(\frac{ah}{x})}{(n-1)!}}}{\left[f^{[n-1]}(x)\right]^{\frac{\ln^{(n-2)}(\frac{ah}{x})}{(n-2)!}}}.A^{-p\ln^{(p-1)}(\frac{ah}{x})}$$

which gives

$$A = \left[f^{[n]}(a.h^{\ln\theta})\right]^{\frac{(1-\ln\theta)^{(n-p)}\ln^{(n-p)} h}{(n-1)!p}}.$$

Now substituting the value of A in (9), we get

$$f(ah) = f(a).\left[f^{[1]}(a)\right]^{\ln h}...\left[f^{[n-1]}(a)\right]^{\frac{\ln^{n-1} h}{(n-1)!}}.\left[f^{[n]}(a.h^{\ln\theta})\right]^{\frac{(1-\ln\theta)^{(n-p)}\ln^n h}{(n-1)!p}}. \quad (10)$$

□

3.4. **Geometric Taylor's Series.** In (10), the term $R_n = \left[f^{[n]}(a.h^{\ln\theta})\right]^{\frac{(1-\ln\theta)^{(n-p)}\ln^n h}{(n-1)!p}}$ is called Taylor's remainder after n terms. Since, $0 < 1 - \ln\theta < 1$ as $1 < \theta < e$, so, $(1-\ln\theta)^{n-p} \to 0$ as $n \to \infty$. Therefore, if f possesses G-derivative of every order in $[a, ah]$ then $R_n \to 1$ as $n \to \infty$. Then Taylor's expansion becomes

$$f(ah) = f(a).\left[f^{[1]}(a)\right]^{\ln h}...\left[f^{[n]}(a)\right]^{\frac{\ln^n h}{n!}}... = \Pi_{n=0}^{\infty}\left[f^{[n]}(a)\right]^{\frac{\ln^n h}{n!}}. \quad (11)$$

This expression can be written in terms of geometric operations as

$$f(a \oplus h) = f(a) \oplus h \odot f^{[1]}(a) \oplus \frac{h^{2_G}}{2!_G}{}_G\odot f^{[2]}(a) \oplus ... = {}_G\sum_{n=0}^{\infty}\frac{h^{n_G}}{n!_G}{}_G\odot f^{[n]}(a), \quad (12)$$

where $h^{n_G} = h^{\ln^{(n-1)} h}$. The equivalent expressions (11) and (12) will be called respectively as Taylor's product and Geometric Taylor's series.

If $x \in [a, ah]$ then it also satisfies the conditions in the interval $[a, x]$. Then replacing ah by x or h by x/a in (11), we get another form of Taylor's product as follows:

$$f(x) = f(a).\left[f^{[1]}(a)\right]^{\ln(\frac{x}{a})}.\left[f^{[2]}(a)\right]^{\frac{\ln^2(\frac{x}{a})}{2!}}...\left[f^{[n]}(a)\right]^{\frac{\ln^n(\frac{x}{a})}{n!}}... = \Pi_{n=0}^{\infty}\left[f^{[n]}(a)\right]^{\frac{\ln^n(\frac{x}{a})}{n!}}. \tag{13}$$

4. Some applications of G-calculus

4.1. **Expansion of some useful functions in Taylor's product. (i).** With the help of geometric Taylor's series, we can express different functions as a product. For example

$$e^x = e.e^{\ln x}.e^{\frac{\ln^2 x}{2!}}.e^{\frac{\ln^3 x}{3!}}... = e^{1+\ln x+\frac{\ln^2 x}{2!}+\frac{\ln^3 x}{3!}+...}$$

(ii). G-calculus gives better graphical and numerical approximations of functions than ordinary calculus. For, let $f(x) = \sin(x)$. In the FIGURE 3, we have given a comparison of linear approximation and exponential approximation respectively at $x = \frac{\pi}{6}$.

By ordinary Taylor's series, first order linear approximation is given by

$$L(x) = f(\frac{\pi}{6}) + (x - \frac{\pi}{6})f'(\frac{\pi}{6}) = \sin(\frac{\pi}{6}) + (x - \frac{\pi}{6})\cos(\frac{\pi}{6}) = \frac{1}{2} + (x - \frac{\pi}{6})\frac{\sqrt{3}}{2}.$$

By geometric Taylor's series, first order exponential approximation is given by

$$E(x) = f(\frac{\pi}{6}).\left[f^{[1]}(\frac{\pi}{6})\right]^{\ln\left(\frac{x}{\pi/6}\right)} = \sin(\frac{\pi}{6}).\left[e^{\frac{\pi}{6}\cot(\frac{\pi}{6})}\right]^{\ln(\frac{6x}{\pi})} = \frac{1}{2}.\left[e^{\frac{\pi}{2\sqrt{3}}}\right]^{\ln(\frac{6x}{\pi})}.$$

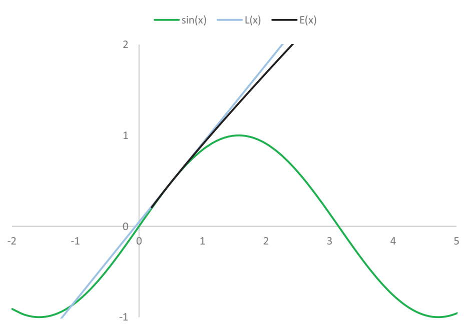

Figure 3. Exponential Approximation

From the FIGURE 3, it is clear that geometric Taylor's series gives better approximated value of the function $f(x) = \sin(x)$ at $x = \frac{\pi}{6}$ than Taylor's approximation given by Michael Coco in [11] with the help of multiplicative derivative.

(iii). G-derivative gives total growth of a growth function. For, let $y = a.b^x$, where a =initial amount> 0, b = growth(or decay) factor, x =time and y =total amount after time period x. Then, $\frac{d^G y}{dx^G} = b^x$, which is the total growth or total decay according to $b > 1$ or $0 < b < 1$ respectively.

(iv). It is easy to find ordinary derivative of complicated product or quotient functions with the help of G-derivative. For let, $f(x) = \frac{e^{-1/x^2}}{x^n \sin x}$. Then

$$f^G(x) = \frac{\frac{d^G}{dx^G}(e^{-1/x^2})}{\frac{d^G}{dx^G}(x^n).\frac{d^G}{dx^G}(\sin x)} = \frac{e^{2/x^2}}{e^n.e^{x\cot x}} = e^{\frac{2}{x^2}-n-x\cot x}$$

Therefore ordinary derivative is given by

$$f'(x) = \frac{f(x)\ln\left(f^G(x)\right)}{x} = \frac{e^{-1/x^2}}{x^{n+1}\sin x}\left(\frac{2}{x^2} - n - x\cot x\right).$$

(v). Price Elasticity: With the aid of G-derivative, we can find price elasticity to predict the impact of price changes on unit sales and to guide the firms profit-maximizing pricing decisions. According to [19, page no. 83], the price elasticity of demand is the ratio of the percentage change in quantity and the percentage change in the goods price, all other factors held constant. If x and y represents price and quantity respectively, then the price elasticity E_p is given by

$$E_p = \frac{\%\text{ change in } y}{\%\text{ change in } x} = \frac{\Delta y/y}{\Delta x/x} = x\frac{\frac{\Delta y}{\Delta x}}{y}$$

If price change is very small to the initially considered price, then making $\Delta x \to 0$, we get

$$E_p = x\frac{y'}{y} = \ln\left(e^{\frac{xy'}{y}}\right) = \ln(y^{[1]}).$$

$$\text{Resiliency} = e^{(\text{elasticity})} = e^{E_p} = y^{[1]}.$$

5. Conclusion

Bigeometric calculus is one of the most actively discussed Non-Newtonian Calculus having variety of applications. Some of such applications and advantages are discussed in our papers [4, 5]. Seeing its importance, here we discussed different properties of bigeometric calculus using geometric increments to the independent variable. We have formulated basic identities which can be expressed in terms of geometric arithmetic independently.

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