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ON GENERALIZATION OF PACHPATTE TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRAL

F. USTA[1], M. Z. SARIKAYA[2], §

Abstract. The main target addressed in this article is presenting Pachpatte type inequalities for Katugampola conformable fractional integral. In accordance with this purpose we try to use more general type of function in order to make a generalization. Thus our results cover the previous published studies for Pachpatte type inequalities.

Keywords: Pachpatte inequality, conformable fractional integral.

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1. Introduction & Preliminaries

In light of recent developments in mathematics, fractional calculus is becoming extremely popular in a number of application areas such as control theory, computational analysis and engineering [10], see also [14]. Together with these developments a number of new definitions have been introduced to provide the best method for fractional calculus. For instance a new local, limit-based definition of a conformable derivative has been introduced in [1], [11], [9], with several follow-up papers [2], [3], [6]-[9], [17] in more recent times. In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon},\ D^{\alpha}(f)(0) = \lim_{t\to 0} D^{\alpha}(f)(t), \tag{1}$$

provided the limits exist (for detail see, [9]). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t). \tag{2}$$

A function f is $\alpha-$differentiable at a point $t \geq 0$ if the limit in (1) exists and is finite. This definition yields the following results;

Theorem 1.1. *Let $\alpha \in (0,1]$ and f,g be $\alpha-$differentiable at a point $t > 0$. Then*
i. $D^{\alpha}(af+bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, *for all* $a,b \in \mathbb{R}$,
ii. $D^{\alpha}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
iii. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.
e-mail: fuatusta@duzce.edu.tr; ORCID: https://orcid.org/0000-0002-7750-6910.
[2] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.
e-mail: sarikayamz@gmail.com; ORCID: http://orcid.org/0000-0002-6165-9242.
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iv. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{fD^{\alpha}(g) - gD^{\alpha}(f)}{g^2}$

v. $D^{\alpha}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$

vi. $D^{\alpha}(f \circ g)(t) = f'(g(t)) D^{\alpha}(g)(t)$ *for* f *is differentiable at* $g(t)$.

Definition 1.1 (Conformable fractional integral)**.** *Let* $\alpha \in (0,1]$ *and* $0 \le a < b$. *A function* $f : [a,b] \to \mathbb{R}$ *is* α*-fractional integrable on* $[a,b]$ *if the integral*

$$\int_a^b f(x)\, d_{\alpha}x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite. All α*-fractional integrable on* $[a,b]$ *is indicated by* $L_{\alpha}^1([a,b])$

Remark 1.1.

$$I_{\alpha}^a(f)(t) = I_1^a\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

When we are presenting the main findings in this paper we will also use the following important results, which can be derived from the results above.

Lemma 1.1. *Let the conformable differential operator* D^{α} *be given as in (1), where* $\alpha \in (0,1]$ *and* $t \ge 0$, *and assume the functions* f *and* g *are* α*-differentiable as needed. Then*

i. $D^{\alpha}(\ln t) = t^{-\alpha}$ *for* $t > 0$

ii. $D^{\alpha}\left[\int_a^t f(t,s)\, d_{\alpha}s\right] = f(t,t) + \int_a^t D^{\alpha}[f(t,s)]\, d_{\alpha}s$

iii. $\int_a^b f(x) D^{\alpha}(g)(x)\, d_{\alpha}x = fg|_a^b - \int_a^b g(x) D^{\alpha}(f)(x)\, d_{\alpha}x.$

The definition given in below is a generalization of the limit definition of the derivative for the case of a function with many variables.

Definition 1.2. *Let* f *be a function with* n *variables* $t_1, ..., t_n$ *and the conformable partial derivative of* f *of order* $\alpha \in (0,1]$ *in* x_i *is defined as follows*

$$\frac{\partial^{\alpha}}{\partial t_i^{\alpha}} f(t_1, ..., t_n) = \lim_{\varepsilon \to 0} \frac{f(t_1, ..., t_{i-1}, t_i e^{\varepsilon t_i^{-\alpha}}, ..., t_n) - f(t_1, ..., t_n)}{\varepsilon}. \tag{3}$$

The below theorem is the generalization of Theorem 2.10 of [3], where the proof can be found in [15].

Theorem 1.2. *Assume that* $f(t,s)$ *is function for which* $\partial_t^{\alpha}\left[\partial_s^{\beta} f(t,s)\right]$ *and* $\partial_s^{\beta}\left[\partial_t^{\alpha} f(t,s)\right]$ *exist and are continuous over the domain* $D \subset \mathbb{R}^2$, *then*

$$\partial_t^{\alpha}\left[\partial_s^{\beta} f(t,s)\right] = \partial_s^{\beta}\left[\partial_t^{\alpha} f(t,s)\right]. \tag{4}$$

Theorem 1.3. *Let* $f, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *and assume that* r *is non-decreasing with* $r(t) \le t$ *for* $t \ge 0$. *If* $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$u(t) \le u_0 + \int_0^{r(t)} f(s)u(s)d_{\alpha}s + \int_0^{r(t)} f(s)\left[\int_0^s g(n)u(n)d_{\alpha}n\right] d_{\alpha}s, \quad t \ge 0, \tag{5}$$

then

$$u(t) \le u_0 + u_0 \int_0^t f(s) e^{\int_0^s [f(n)+g(n)] d_{\alpha}n} d_{\alpha}s, \quad t \ge 0. \tag{6}$$

Proof. The proof can be found in [16]. □

In addition to these, integral inequalities play a significant role in the theory of differential equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain application. One can refer to [4], [5], [12], [13], [16] and the references therein.

This prospective study was designed to investigate the Pachpatte type inequalities for conformable fractional integral. The established results are extensions of some existing the Pachpatte type inequalities in the literature.

2. Main Findings & Cumulative Results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^1(M,S)$ denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S.

Theorem 2.1. *Let $f,g \in C(\mathbb{R}^+,\mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$ and $k(t)$ be a positive and non-decreasing function over $\mathbb{R}$. If $u \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{r(t)} f(s)u(s)d_\alpha s + \int_0^{r(t)} f(s)\left[\int_0^s g(n)u(n)d_\alpha n\right] d_\alpha s, \quad t \geq 0, \tag{7}$$

then

$$u(t) \leq k(t) + k(t)\int_0^t f(s)e^{\int_0^s [f(n)+g(n)]d_\alpha n} d_\alpha s, \quad t \geq 0. \tag{8}$$

Proof. The proof is quite similar to Theorem 1.3. Because $k(t)$ is a positive and non-decreasing function over $\mathbb{R}$, we deduce from (7) that

$$\frac{u(t)}{k(t)} \leq 1 + \int_0^{r(t)} \frac{f(s)u(s)}{k(s)} d_\alpha s + \int_0^{r(t)} f(s)\left[\int_0^s \frac{g(n)u(n)}{k(n)} d_\alpha n\right] d_\alpha s, \quad t \geq 0. \tag{9}$$

By applying the Theorem 1.3, we obtain the desired result. □

Theorem 2.2. *Let $f,g,q,h \in C(\mathbb{R}^+,\mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies*

$$u(t) \leq u_0 + \int_0^{r(t)} [f(s)u(s)+q(s)]d_\alpha s + \int_0^{r(t)} f(s)\left[\int_0^s [g(n)u(n)+h(n)]d_\alpha n\right] d_\alpha s, \quad t \geq 0, \tag{10}$$

then

$$u(t) \leq u_0 + \int_0^t \left(q(s) + f(s)\Lambda(s)\right) d_\alpha s$$

where

$$\Lambda(s) = \left[u_0 e^{\int_0^s [f(\eta)+g(\eta)]d_\alpha \eta} + \int_0^s [m(n)+h(n)]e^{\int_n^s [f(\eta)+g(\eta)]d_\alpha \eta} d_\alpha n\right]$$

Proof. Let denote $z(t)$ the right hand side of inequality (10). Then $u(t) \leq z(t)$ and $z(0) = u_0$ and

$$\begin{aligned} D^{\alpha}z(t) &= [f(r(t))u(r(t)) + q(r(t))]D^{\alpha}r(t) \\ &+ f(r(t))D^{\alpha}r(t)\int_0^{r(t)}[g(n)u(n) + h(n)]d_{\alpha}n \\ &\leq q(r(t))D^{\alpha}r(t) + f(r(t))D^{\alpha}r(t)\left[z(t) + \int_0^{r(t)}[g(n)z(n) + h(n)]d_{\alpha}n\right]. \quad (11) \end{aligned}$$

Define a function $m(t)$ by

$$m(t) = z(t) + \int_0^{r(t)}[g(n)z(n) + h(n)]d_{\alpha}n, \tag{12}$$

then $m(0) = z(0) = u_0$, $D^{\alpha}z(t) \leq q(r(t))D^{\alpha}r(t) + f(r(t))D^{\alpha}r(t)m(t)$, from (11) and $z(t) \leq m(t)$ from (12) and

$$D^{\alpha}m(t) = D^{\alpha}z(t) + [g(r(t))z(r(t)) + h(r(t))]D^{\alpha}r(t).$$

So we get

$$D^{\alpha}m(t) \leq [q(r(t)) + h(r(t))]D^{\alpha}r(t) + [f(r(t)) + g(r(t))]D^{\alpha}r(t)m(t). \tag{13}$$

The inequality (13) implies the estimation of $m(t)$ such that

$$m(t) \leq u_0 e^{\int_0^{r(t)}[f(\eta)+g(\eta)]d_{\alpha}\eta} + \int_0^{r(t)}[q(n) + h(n)]e^{\int_n^{r(t)}[f(\eta)+g(\eta)]d_{\alpha}\eta}d_{\alpha}n. \tag{14}$$

Then using (14) and (11) we get

$$\begin{aligned} D^{\alpha}z(t) &\leq q(r(t))D^{\alpha}r(t) \\ &+ f(r(t))D^{\alpha}r(t)\left[u_0 e^{\int_0^{r(t)}[f(\eta)+g(\eta)]d_{\alpha}\eta} + \int_0^{r(t)}[m(n) + h(n)]e^{\int_n^{r(t)}[f(\eta)+g(\eta)]d_{\alpha}\eta}d_{\alpha}n\right]. \end{aligned}$$

Now by setting $r(t) = s$ in the above inequalities and integrating from 0 to t and substituting the bound $z(t)$ in $u(t) \leq z(t)$ we get

$$u(t) \leq u_0 + \int_0^t \left(q(s) + f(s)\Lambda(s)\right)d_{\alpha}s$$

where

$$\Lambda(s) = \left[u_0 e^{\int_0^s[f(\eta)+g(\eta)]d_{\alpha}\eta} + \int_0^s[m(n) + h(n)]e^{\int_n^s[f(\eta)+g(\eta)]d_{\alpha}\eta}d_{\alpha}n\right]$$

which this proves our claim. □

Theorem 2.3. *Let* $f, g, q, h \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *and assume that* r *is non-decreasing with* $r(t) \leq t$ *for* $t \geq 0$. *If* $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$u(t) \leq k(t) + q(t)\left(\int_0^{r(t)} f(s)u(s)d_{\alpha}s + \int_0^{r(t)} f(s)q(s)\left[\int_0^s g(n)u(n)d_{\alpha}n\right]d_{\alpha}s\right), \quad t \geq 0, \tag{15}$$

then

$$u(t) \leq k(t)+q(t) \left[\int_0^t f(s) \left(k(s) + q(s) \int_0^s k(n)[f(n) + g(n)]e^{\int_n^s q(\eta)[f(\eta)+g(\eta)]d_\alpha \eta} d_\alpha n\right) d_\alpha s\right].$$

Proof. If we set

$$z(t) = \int_0^{r(t)} f(s)u(s)d_\alpha s + \int_0^{r(t)} f(s)q(s) \left[\int_0^s g(n)u(n)d_\alpha n\right] d_\alpha s,$$

then $z(0) = 0$ and $u(t) \leq k(t) + q(t)z(t)$ and

$$\begin{aligned} D^\alpha z(t) &= f(r(t))u(r(t))D^\alpha r(t) + f(r(t))q(r(t))D^\alpha r(t) \int_0^{r(t)} g(n)u(n)d_\alpha n \\ &\leq f(r(t))D^\alpha r(t) \left(k(r(t)) + q(r(t)) \left[z(t) + \int_0^{r(t)} g(n)\{k(n) + q(n)z(n)\}d_\alpha n\right]\right). \end{aligned}$$

Let define a function $m(t)$ by

$$m(t) = z(t) + \int_0^{r(t)} g(n)\{k(n) + q(n)z(n)\}d_\alpha n, \tag{16}$$

then $m(0) = z(0) = 0$, $D^\alpha z(t) \leq f(r(t))[k(r(t)) + q(r(t))m(t)]$ from (16) and $z(t) \leq m(t)$.

$$D^\alpha m(t) = D^\alpha z(t) + g(r(t))[k(r(t)) + q(r(t))z(r(t))]D^\alpha r(t).$$

Thus we have

$$D^\alpha m(t) \leq k(r(t))[f(r(t)) + g(r(t))]D^\alpha r(t) + q(r(t))m(r(t))[f(r(t)) + g(r(t))]D^\alpha r(t).$$

So the last inequality above implies that

$$m(t) \leq \int_0^{r(t)} k(n)[f(n) + g(n)]e^{\int_n^{r(t)} q(\eta)[f(\eta)+g(\eta)]d_\alpha \eta} d_\alpha n. \tag{17}$$

Then using (17) we get

$$D^\alpha z(t) \leq f(r(t))D^\alpha r(t) \left(k(r(t)) + q(r(t)) \int_0^{r(t)} k(n)[f(n) + g(n)]e^{\int_n^{r(t)} q(\eta)[f(\eta)+g(\eta)]d_\alpha \eta} d_\alpha n\right).$$

Now by setting $r(t) = s$ in the above inequalities and integrating from 0 to t and substituting the bound $z(t)$ in $u(t) \leq k(t) + q(t)z(t)$ we get the desired inequality. □

Theorem 2.4. *Let* $f, k, g, q \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *and assume that* r *is nondecreasing with* $r(t) \leq t$ *for* $t \geq 0$. *If* $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$u(t) \leq u_0 + \int_0^{r(t)} f(s)k(s)d_\alpha s + \int_0^{r(t)} f(s) \left(\int_0^s g(\eta) \left[\int_0^\eta q(n)u(n)d_\alpha n\right] d_\alpha \eta\right) d_\alpha s, \quad t \geq 0, \tag{18}$$

then

$$u(t) \leq \left[u_0 + \int_0^{r(t)} f(s)k(s)d_\alpha s\right] e^{\int_0^{r(t)} f(s) \int_0^s g(\eta)\left(\int_0^\eta q(n)d_\alpha n\right)d_\alpha \eta d_\alpha s}.$$

Proof. Let assume $u_0 > 0$. Then let define a function $z(t)$ by

$$z(t) = u_0 + \int_0^{r(t)} f(s)k(s)d_\alpha s. \tag{19}$$

Unambiguously $z(t)$ is a positive and non-decreasing function. Then by using (18) and (19), we get

$$\frac{u(t)}{z(t)} \leq 1 + \int_0^{r(t)} f(s) \left(\int_0^s g(\eta) \left[\int_0^\eta \frac{q(n)u(n)}{z(n)} d_\alpha n \right] d_\alpha \eta \right) d_\alpha s. \tag{20}$$

Now define another function $v(t)$ by the right hand side of inequality (20). Here $v(0) = 1$. Then we get,

$$D^\alpha v(t) \leq f(r(t)) \left[\int_0^{r(t)} g(\eta) \left(\int_0^\eta \frac{q(n)u(n)}{z(n)} d_\alpha n \right) d_\alpha \eta \right].$$

From the last inequality above, one can easily obtain that

$$D^\alpha \left[\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right) \right] = \frac{q(r(t))u(r(t))}{z(r(t))}.$$

Now using the fact that $\frac{u(t)}{z(t)} \leq v(t)$, we get

$$\frac{1}{v(r(t))} D^\alpha \left[\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right) \right] \leq q(r(t)).$$

Because of $\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right) \geq 0$, $D^\alpha v(t) \geq 0$ and $v(t) > 0$, we get that

$$\frac{1}{v(r(t))} D^\alpha \left[\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right) \right] \leq q(r(t)) + \frac{1}{v^2(r(t))} \left[\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right) D^\alpha v(r(t)) \right]$$

i.e.,

$$D^\alpha \left[\frac{\frac{1}{g(r(t))} D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right)}{v(r(t))} \right] \leq q(r(t)).$$

By setting $r(t) = n$ and integrating from 0 to $r(t)$ with respect to n, we get

$$\frac{D^\alpha \left(\frac{D^\alpha v(r(t))}{f(r(t))} \right)}{v(r(t))} \leq g(r(t)) \int_0^{r(t)} q(n) d_\alpha n.$$

Similarly, since $\frac{D^\alpha v(r(t))}{f(r(t))} \geq 0$, $D^\alpha v(t) \geq 0$ and $v(t) > 0$, we observe that

$$D^\alpha \left(\frac{\frac{D^\alpha v(r(t))}{f(r(t))}}{v(r(t))} \right) \leq g(r(t)) \int_0^{r(t)} q(n) d_\alpha n.$$

By taking $r(t) = \eta$ and integrating from 0 to $r(t)$ with respect to η, we get

$$\frac{D^\alpha v(r(t))}{v(r(t))} \leq f(r(t)) \int_0^{r(t)} g(\eta) \left(\int_0^\eta q(n) d_\alpha n \right) d_\alpha \eta$$

Finally the last inequality above implies the estimation that

$$v(t) \leq e^{\int_0^{r(t)} f(s) \int_0^s g(\eta) \left(\int_0^\eta q(n) d_\alpha n \right) d_\alpha \eta d_\alpha s}.$$

Now using the fact that $\frac{u(t)}{z(t)} \leq v(t)$, we get

$$u(t) \leq \left[u_0 + \int_0^{r(t)} f(s) k(s) d_\alpha s \right] e^{\int_0^{r(t)} f(s) \int_0^s g(\eta) \left(\int_0^\eta q(n) d_\alpha n \right) d_\alpha \eta d_\alpha s}$$

which this proves our claim □

3. Concluding Remark

The present study was designed to make the generalization of some inequalities for conformable differential equations. For this purpose we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0, 1]$. The findings of this investigation complement those of earlier studies. In other words the present study confirms previous findings and contributes additional evidence by making generalization.

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Fuat Usta received his BSc (Mathematical Engineering) degree from ITU, Turkey in 2009 and MSc (Mathematical Finance) from University of Birmingham, UK in 2011 and PhD (Applied Mathematics) from University of Leicester, UK in 2015. At present, he is working as an Asst. Professor in the Department of Mathematics at Düzce University. He is interested in Approximation Theory, Multivariate approximation using Quasi Interpolation, Radial Basis Functions and Hierarchical/Wavelet Bases, High-Dimensional Approximation using Sparse Grids, Financial Mathematics, Integral Equations, Fractional Calculus, Partial Differantial Equations.

Mehmet Zeki Sarıkaya received his BSc (Maths), MSc (Maths) and PhD (Maths) degree from Afyon Kocatepe University, Afyonkarahisar, Turkey in 2000, 2002 and 2007 respectively. At present, he is working as a Professor in the Department of Mathematics at Duzce University (Turkey) and as a Head of Department. Moreover, he is founder and Editor-in-Chief of Konuralp Journal of Mathematics (KJM). He is the author or coauthor of more than 200 papers in the field of Theory of Inequalities, Potential Theory, Integral Equations and Transforms, Special Functions, Time-Scales.