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STREAMLINE UPWIND/PETROV GALERKIN SOLUTION OF OPTIMAL CONTROL PROBLEMS GOVERNED BY TIME DEPENDENT DIFFUSION-CONVECTION-REACTION EQUATIONS

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Abstract. The streamline upwind/Petrov Galerkin (SUPG) finite element method is studied for distributed optimal control problems governed by unsteady diffusion-convection-reaction equations with control constraints. We derive stability and convergence estimates for fully-discrete state, adjoint and control and discuss the choice of the stabilization parameter by applying backward Euler method in time. We show that by balancing the error terms in the convection dominated regime, optimal convergence rates can be obtained. The numerical results confirm the theoretically observed convergence rates.

Keywords: optimal control problems, unsteady diffusion-convection-reaction equations, finite element elements, a priori error estimates.

AMS Subject Classification: 49N10,49K20, 65M60,65M15.

1. Introduction

Optimal control problems (OCPs) governed by diffusion-convection-reaction equations arise in environmental control problems, like air and water pollution, optimal control of fluid flow, steel formation and in many other industrial applications. It is well known that the standard Galerkin finite element discretization causes nonphysical oscillations in the solution when convection dominates. Stable and accurate numerical solutions can be achieved by various effective stabilization techniques such as the streamline upwind/Petrov Galerkin (SUPG) finite element method [6], the local projection stabilization [3], the edge stabilization [11] and the symmetric stabilization [4].

In the recent years, most of the research is concentrated on parabolic OCPs. There are few publications dealing with the OCPs governed by non-stationary diffusion-convection-reaction equations. For example, the local discontinuous Galerkin (dG) approximation and the characteristic finite element solution of the control constraint OCP are discussed in [10, 17], respectively. The symmetric interior penalty Galerkin method with backward Euler time discretization is studied in [1]. SUPG discretization of a time-dependent diffusion-convection-reaction equation and a priori error analysis are given in [12].

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The choice of the stabilization parameter for time-dependent diffusion-convection-reaction equation is discussed in [12] for SUPG discretization in space, backward Euler and Crank-Nicolson method in time. When the stabilization parameter is chosen proportional to the mesh size h, i.e. $\tau = \mathcal{O}(h)$, for all cells, the discrete solution converges for the time-continuous case. When discretization is performed first in time, then the stabilization parameter can be chosen proportional to the time step k. However, this leads to large spurious oscillations. When the time and space grids are comparable, i.e. $k \sim h$, then the stabilization parameter can be chosen as for the steady-state case, whereas the spatial and temporal errors have to be balanced.

Our work is motivated by the study [12] where the SUPG-backward Euler discretization is studied for a single parabolic partial differential equation (PDE). We have discretized the OCP using SUPG in space and backward Euler method in time by extending the error analysis for evolutionary convection-diffusion-reaction equations provided in [12] to OCPs governed by time-dependent convection-diffusion-reaction equations. According to [9, 10], the characteristic finite element method combined with backward Euler discretization leads to the first order of convergence with the choice of $h = k$. Here, we choose the stabilization parameter depending on the length of the time step to balance the error terms for the convection-dominated regime. It turns out that the SUPG improves the convergence rates up to the order $\mathcal{O}(h^{4/3})$ with $k \cong h^{4/3}$ and the oscillations in the solutions disappear. The theoretically observed convergence rates are confirmed by the numerical results.

The rest of the paper is organized as follows. In Section 2, we define the model problem and derive the optimality system. In Section 3, we present the SUPG finite element method and state the semi-discrete optimality system. In Section 4, stability and convergence estimates for the fully discrete optimality system are presented and the choice of the stabilization parameter is discussed. In Section 5, numerical results are presented for different choices of stabilization parameters. The paper ends with some conclusions.

2. The Optimal Control Problem

We adopt the standard notations for Sobolev spaces on computational domains and their norms. Ω and Ω_U are bounded convex polygonal domains in $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$, respectively. We consider the following distributed optimal control problem governed by the unsteady diffusion-convection-reaction equation with control constraints

$$\underset{u\in U^{ad}\subseteq L^2(0,T;L^2(\Omega_U))}{\text{minimize}} \; J(y,u) := \frac{1}{2}\int_0^T \left(\|y-y_d\|^2_{L^2(\Omega)} + \alpha\, \|u\|^2_{L^2(\Omega_U)} \right) dt, \tag{1a}$$

$$\text{subject to } \partial_t y - \epsilon\Delta y + \beta\cdot\nabla y + \sigma y = f + Bu, \quad (x,t)\in\Omega\times(0,T], \tag{1b}$$

$$y(x,t) = 0, \qquad (x,t)\in\partial\Omega\times(0,T], \tag{1c}$$

$$y(x,0) = y_0(x), \quad x\in\Omega, \tag{1d}$$

$$U^{ad} = \left\{ u\in L^2(0,T;L^2(\Omega_U)) \;:\; u_a \le u \le u_b \text{ a.e. in } \Omega_U\times(0,T] \right\}, \tag{2}$$

For well-posedness of the optimal control problem (1) we refer to [1, 9, 10].

We use the Hilbert space $X := \{\varphi\in L^2(0,T;V);\; \varphi_t\in L^2(0,T;V^*)\}$, where $V = H^1_0(\Omega)$ be Hilbert spaces and V^* denotes the dual space of V.

The variational formulation corresponding to (1) is given by

$$\underset{u\in U^{ad}}{\text{minimize}} \quad J(y,u) := \frac{1}{2}\int_0^T \left(\|y-y_d\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega_U)}\right) dt, \tag{3a}$$

$$\text{subject to} \quad (\partial_t y, v) + a(y,v) + b(u,v) = (f,v) \qquad \forall v \in V, \quad t \in (0,T], \tag{3b}$$

$$y(x,0) = y_0,$$

$$a(y,v) = \int_\Omega (\epsilon\nabla y\nabla v + \beta\cdot\nabla y v + \sigma y v)dx, \quad b(u,v) = -\int_\Omega B u v dx, \quad (f,v) = \int_\Omega f v dx.$$

It is well known that the triple (y,u,p) is the unique solution of (3) if and only if there is an adjoint $p(x,t)$ such that (y,p,u) satisfies the following optimality system [1, 2]:

$$(\partial_t y, v) + a(y,v) + b(u,v) = (f,v), \qquad \forall v \in V, \quad y(x,0) = y_0, \tag{4a}$$

$$-(\partial_t p, \psi) + a(\psi, p) = -(y - y_d, \psi), \qquad \forall \psi \in V, \quad p(x,T) = 0, \tag{4b}$$

$$\int_0^T (\alpha u - B^* p, w - u)_U \, dt \geq 0, \qquad \forall w \in U^{ad}, \tag{4c}$$

where B^* denotes the adjoint of B.

3. Streamline Upwind/Petrov Galerkin(SUPG) Finite Element Method for Optimal Control Problem

Let $\{\mathfrak{T}_h\}$ be a triangulation of Ω such that $\overline{\Omega} = \cup_{K\in\mathfrak{T}_h}\overline{K}$, $K_i \cap K_j = \emptyset$ for $K_i, K_j \in \mathfrak{T}_h$, $i \neq j$. The diameter of an element K and the length of an edge E are denoted by h_K and h_E, respectively. In addition, the maximum value of element diameter is denoted by $h = \max_{K\in\mathfrak{T}_h} h_K$. We note that the subindex U denotes the associated triangularization for the control. In general, the sizes of the elements in $\{(\mathfrak{T}_h)_U\}_h$ are smaller than those in $\{\mathfrak{T}_h\}_h$, so we assume that $h_U/h \leq C$ throughout this paper [9, Sec.3].

We use piecewise continuous linear finite element space to define the discrete spaces of the state, the adjoint and the control

$$V_h = \left\{v \in H^1_0(\Omega) \;:\; v\,|_K \in \mathbb{P}^1(K) \quad \forall K \in \mathfrak{T}_h\right\},$$

$$U_h = \left\{u \in L^2(\Omega_U) \;:\; u\,|_{K_U} \in \mathbb{P}^1(K_U) \quad \forall K_U \in (\mathfrak{T}_h)_U\right\}.$$

Finite element approximations of the state, the adjoint and the control are given as

$$y_h(x,t) = \sum_{i=1}^{m-1} y_{h,i}(t)\varphi_i(x), \; p_h(x,t) = \sum_{i=1}^{m-1} p_{h,i}(t)\varphi_i(x), \; u_h(x,t) = \sum_{i=0}^{m_u} u_{h,i}(t)\phi_i(x),$$

$$y_h(t) = (y_{h,1}(t),\ldots,y_{h,m-1}(t))^T, p_h(t) = (p_{h,1}(t),\ldots,p_{h,m-1}(t))^T, u_h(t) = (u_{h,0}(t),\ldots,u_{h,m_u}(t))^T.$$

The semi-discrete approximation of the optimal control problem (3) is defined as follows:

$$\underset{u_h\in U_h^{ad}}{\text{minimize}} \quad \int_0^T \Big(\frac{1}{2}\sum_{K\in\mathfrak{T}_h}\|y_h - y_{d,h}\|^2_{L^2(K)} + \frac{\alpha}{2}\sum_{K_U\in\mathfrak{T}_h^U}\|u_h\|^2_{L^2(K_U)}\Big)\, dt, \tag{5a}$$

$$\text{subject to} \quad (\partial_t y_h, v_h) + \sum_{K\in\mathfrak{T}_h}\tau(\partial_t y_h, \beta\cdot\nabla v_h)_K + a_h^s(y_h,v_h) + b_h^s(u_h,v_h) = (f,v_h)_h^s, \tag{5b}$$

$$(y_h^0, \varphi) = (y_h(0,x),\varphi) \quad \text{and} \quad (y_h,u_h) \in V_h \times U_h^{ad},$$

$$a_h^s(y, v_h) = a(y, v_h) + \sum_{K \in \mathfrak{T}_h} \tau(-\epsilon \Delta y + \beta \cdot \nabla y + \sigma y, \beta \cdot \nabla v_h)_K, \tag{6a}$$

$$b_h^s(u, v_h) = b(u, v_h) - \sum_{K \in \mathfrak{T}_h} \tau(Bu, \beta \cdot \nabla v_h)_K, \tag{6b}$$

$$(f, v_h)_h^s = (f, v_h) + \sum_{K \in \mathfrak{T}_h} \tau(f, \beta \cdot \nabla v_h)_K. \tag{6c}$$

The stabilization parameter τ is chosen depending on a priori error estimates in Section 4. We use *discretize-then-optimize approach* to solve the OCP. We derive the fully-discrete optimality system by differentiating the discrete Lagrangian with respect to the state, adjoint and control variables. The semi-discrete optimality system is discretized in time with the backward Euler method and resulting fully discrete optimality system is solved using all all at once approach [15] with the MINRES. The fully discrete optimality system is given as:

$$\underset{u_h^n \in U_h^{ad}}{\text{minimize}} \quad \left(\frac{k}{2} \sum_{K \in \mathfrak{T}_h} \|y_h^n - y_{d,h}^n\|_{L^2(K)}^2 + \alpha \frac{k}{2} \sum_{K_U \in \mathfrak{T}_h^u} \|u_h^n\|_{L^2(K_U)}^2 \right) \tag{7}$$

$$(y_h^n - y_h^{n-1}, \varphi) + k a_h^s (y_h^n, \varphi) = k (f^n + B u_h^n, \varphi) + k \left[\sum_{K \in \mathfrak{T}_h} \tau \left(f^n + B u_h^n, \beta \cdot \nabla \varphi\right)_K \right]$$
$$- \left[\sum_{K \in \mathfrak{T}_h} \tau(y_h^n - y_h^{n-1}, \beta \cdot \nabla \varphi)_K \right], \quad \forall \varphi \in V_h, \quad n = 1, \ldots, N+1, \tag{8a}$$

$$(\psi, p_h^{n-1} - p_h^n) + k a_h^s \left(\psi, p_h^{n-1}\right)$$
$$= -k \left((y_h^{n-1} - y_{d,h}^{n-1}), \psi \right) - k \left[\sum_{K \in \mathfrak{T}_h} \tau(\psi, \beta \cdot \nabla (p_h^{n-1} - p_h^n))_K \right], \forall \psi \in V_h, n = N+1, \ldots, 2, \tag{8b}$$

$$(\alpha u_h^n - B^* p_h^{n-1} - \tau \beta \cdot \nabla B^* p_h^{n-1}, w_h - u_h^n)_U \geq 0, \quad \forall w_h \in U_h^{ad}, \quad n = 1, \ldots, N+1. \tag{8c}$$

4. A Priori Error Estimates

In this section, we shall derive the stability and convergence estimates for the fully-discrete OCP. We start with the stability estimates following the approach in [12] for time-dependent diffusion-convection-reaction equations. In this section, r denotes the degree of local polynomials and $\| \cdot \|_r$ denotes the norm in $H^r(\Omega)$ with $H^0(\Omega) = L^2(\Omega)$. To prove the a priori error estimate of the fully-discrete scheme, we need the discrete time-dependent norm for $1 \leq q < \infty$ by [9],

$$\|v\|_{L^q(0,T;L^2(\Omega))} = \left(\sum_{n=1}^{N+1} k \|v_n\|_{L^2(\Omega)}^q \right)^{1/q}.$$

4.1. **Stability Estimates.** We take a fixed time step k, and we denote the fully discrete state, adjoint and control solution at time $t_n = nk$ by y_h^n, p_h^n and u_h^n, respectively. Moreover, the exact solutions of the state, the adjoint and the control at time t_n are defined as y^n, p^n and u^n, respectively. We give first some useful inequalities which are needed.

The elliptic projection $\pi_h : V \to V_h$ is defined by $(\nabla(y - \pi_h y), \nabla v_h) = 0$ for all $v_h \in V_h$ and

$$(\pi_h y)_t = \pi_h(y_t) = \pi_h y_t. \tag{9}$$

The following inverse inequality holds for each $v_h \in V_h$ with the assumption of a quasi uniform mesh (see, e.g., [5]):

$$\|v_h\|_{W^{m,q}(K)} \leq c_{inv} h_K^{l-m-d(\frac{1}{q'}-\frac{1}{q})} \|v_h\|_{W^{l,q'}(K)}, \tag{10}$$

where $0 \leq l \leq m \leq 1,\ 1 \leq q' \leq q \leq \infty$, h_K is the mesh size diameter of $K \in \mathfrak{T}_h$. We note that we take the same step size $h_K = h$ for all mesh cell K. The interpolation error estimate for $y \in V \cap H^{r+1}$ given in [5] is

$$\|y - \pi_h y\|_{L^2(\Omega)} + h\|y - \pi_h y\|_{H^1(\Omega)} \leq Ch^{r+1}\|y\|_{H^{r+1}(\Omega)}. \tag{11}$$

We introduce an element integral averaging operator $\tilde{\Pi}_h$ from U to U_h such that

$$\tilde{\Pi}_h v|_{K_U} = \frac{1}{|K_U|}\int_{K_U} v, \quad \forall K_U \in \mathfrak{T}_h^U,$$

where $|K_U|$ denotes the measure of K_U [10, Sec.3].

There is a positive constant C independent of h_U such that the following estimate holds [5]: $|v - \tilde{\Pi}_h v|_{0,p,K_U} \leq Ch_U|v|_{1,p,K_U}$, for $v \in W^{1,p}(\Omega_U)$ and $1 \leq p \leq \infty$.

The coercivity condition for the bilinear form $a_h^s(\cdot,\cdot)$ given in [16, Lemma 10.3].

Lemma 4.1. *Let μ_0 be a positive constant satisfying $\sigma - \frac{1}{2}\nabla\cdot\beta \geq \mu_0$ holds. If the SUPG parameter τ is chosen such that*

$$\tau \leq \frac{\mu_0}{2\|\sigma\|^2_{L^\infty(K)}}, \tag{12}$$

then the bilinear form $a_h^s(\cdot,\cdot)$ associated with SUPG method satisfies

$$a_h^s(y_h, y_h) \geq \frac{1}{2}\|y_h\|_s^2, \tag{13}$$

$$\|y_h\|_s^2 := \epsilon\|\nabla y_h\|^2_{L^2(\Omega)} + \sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h\|^2_{L^2(K)} + \|\mu_0^{1/2} y_h\|^2_{L^2(\Omega)}. \tag{14}$$

Lemma 4.2. *Let y_h^n be a solution of discrete OCP and (12) be fulfilled and $\tau \leq \frac{4k}{5}$,*

$$\|y_h^n\|^2_{L^2(\Omega)} + \frac{3k}{40}\sum_{j=1}^n \|y_h^j\|_s^2 \leq \|y_h^0\|^2_{L^2(\Omega)} + 8k\left(\frac{1}{\mu_0} + \frac{4k}{5}\right)\sum_{j=1}^n(\|f^j\|^2_{L^2(\Omega)} + \|Bu_h^j\|^2_{L^2(\Omega)}). \tag{15}$$

Proof. Let us take $\varphi = y_h^n$ in (8a). By the coercivity estimate (13) and using the following equality

$$\left(y_h^n - y_h^{n-1}, y_h^n\right) = \frac{1}{2}\left(\|y_h^n\|^2_{L^2(\Omega)} - \|y_h^{n-1}\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)},$$

we obtain

$$\frac{1}{2}\left(\|y_h^n\|^2_{L^2(\Omega)} - \|y_h^{n-1}\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + \frac{k}{2}\|y_h^n\|_s^2$$
$$\leq \underbrace{\left|k\left(f^n + Bu_h^n, y_h^n\right)\right|}_{A_1} + \underbrace{\left|k\left[\sum_{K\in\mathfrak{T}_h}\tau\left(f^n + Bu_h^n, \beta\cdot\nabla(y_h^n)\right)_K\right]\right|}_{A_2} + \underbrace{\left|\sum_{K\in\mathfrak{T}_h}\tau(y_h^n - y_h^{n-1}, \beta\cdot\nabla(y_h^n)_K\right|}_{A_3}. \tag{16}$$

We estimate A_1, A_2 by using the Cauchy-Schwarz and Young's inequalities as in [12].

$$A_1 \leq \frac{4k}{\mu_0}\|f^n + Bu_h^n\|^2_{L^2(\Omega)} + \frac{k}{16}\|\mu_0^{1/2}y_h^n\|^2_{L^2(\Omega)},$$
$$A_2 \leq 4k\sum_{K\in\mathfrak{T}_h}\tau\|f^n + Bu_h^n\|^2_{L^2(K)} + \frac{k}{16}\sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h^n\|^2_{L^2(K)}.$$

The term A_3 can be estimated under the condition $\tau \leq \frac{4k}{5}$ using the condition (14):

$$A_3 \leq \frac{5}{8k}\sum_{K\in\mathfrak{T}_h}\tau\|y_h^n - y_h^{n-1}\|^2_{L^2(K)} + \frac{2k}{5}\sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h^n\|^2_{L^2(K)}$$
$$\leq \frac{1}{2}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + \frac{2k}{5}\sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h^n\|^2_{L^2(K)}.$$

$$A_1 + A_2 + A_3 \leq \frac{4k}{\mu_0}\|f^n + Bu_h^n\|^2_{L^2(\Omega)} + 4k\sum_{K\in\mathfrak{T}_h}\tau\|f^n + Bu_h^n\|^2_{L^2(K)}$$
$$+ \underbrace{\frac{k}{16}\|\mu_0^{1/2}y_h^n\|^2_{L^2(\Omega)} + \frac{k}{16}\sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h^n\|^2_{L^2(K)}}_{\leq\frac{k}{16}\|y_h^n\|_s^2}$$
$$+ \frac{1}{2}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + \underbrace{\frac{2k}{5}\sum_{K\in\mathfrak{T}_h}\tau\|\beta\cdot\nabla y_h^n\|^2_{L^2(K)}}_{\leq\frac{2k}{5}\|y_h^n\|_s^2}.$$

and inserting all these estimates into (16) we obtain

$$\|y_h^n\|^2_{L^2(\Omega)} + \frac{3k}{40}\|y_h^n\|_s^2 \leq \|y_h^{n-1}\|^2_{L^2(\Omega)} + \frac{8k}{\mu_0}(\|f^n\|^2_{L^2(\Omega)} + \|Bu_h^n\|^2_{L^2(\Omega)}) + 8k\sum_{K\in\mathfrak{T}_h}\tau\|f^n + Bu_h^n\|^2_{L^2(K)}.$$

We sum the resulting inequality over $j = 1, 2, ..., n$ with the condition $\tau \leq \frac{4k}{5}$ to arrive at (15). □

Lemma 4.3. *Let p_h^n be in* (8b) *and (12) be fulfilled and $\tau \leq \frac{4k}{5}$,*

$$\|p_h^n\|^2_{L^2(\Omega)} + \frac{3k}{40}\sum_{j=1}^n\|p_h^{j-1}\|_s^2 \leq \|p_h^N\|^2_{L^2(\Omega)} + 8k\left(\frac{1}{\mu_0} + \frac{4k}{5}\right)\sum_{j=1}^n\|y_h^j - y_{h,d}^j\|^2_{L^2(\Omega)}. \tag{17}$$

Proof. We choose $\psi = p_h^{n-1}$ in (8b) and follow the proof of Lemma 4.2 to obtain the desired result (17). □

4.2. **Convergence estimates.** We derive convergence estimates for the fully-discrete scheme. First, we shall use two auxiliary variables $y_h^n(u), p_h^n(u) \in V_h \times V_h$, $n = 1, 2, ..., N$, associated with the control variable to derive a priori error estimate of the fully discrete scheme as in [9]:

$$(y_h^n(u) - y_h^{n-1}(u), \varphi) + k a_h^s\,(y_h^n(u), \varphi)$$
$$= k\,(f^n + Bu^n, \varphi) + k \left[\sum_{K \in \mathfrak{T}_h} \tau\,(f^n + Bu^n, \beta \cdot \nabla \varphi)_K \right] - \left[\sum_{K \in \mathfrak{T}_h} \tau(y_h^n(u) - y_h^{n-1}(u), \beta \cdot \nabla \varphi)_K \right],$$
$$(y_h^0(u), \varphi) = (y_h^0, \varphi), \ \forall \varphi \in V_h, \tag{18a}$$

$$(\psi, p_h^{N+1}(u)) + k a_h^s \left(\psi, p_h^{N+1}(u)\right) = -k \left[\sum_{K \in \mathfrak{T}_h} \tau(\psi, \beta \cdot \nabla p_h^{N+1}(u))_K \right] - k \left((y_h^{N+1}(u) - y_{d,h}^{N+1}(u)), \psi \right), \quad \forall \psi \in$$
$$(\psi, p_h^{n-1}(u) - p_h^n(u)) + k a_h^s\left(\psi, p_h^{n-1}(u)\right)$$
$$= -k \left((y_h^{n-1}(u) - y_{d,h}^{n-1}(u)), \psi \right) - k \left[\sum_{K \in \mathfrak{T}_h} \tau(\psi, \beta \cdot \nabla (p_h^{n-1}(u) - p_h^n(u)))_K \right], \ \forall \psi \in V_h,$$

$$(\psi, p_h^0(u)) = (\psi, p_h^1(u)) - k \left[\sum_{K \in \mathfrak{T}_h} \tau(\psi, \beta \cdot \nabla p_h^1(u))_K \right], \quad \forall \psi \in V_h. \tag{18b}$$

The approximation solution (y_h^n, p_h^n) and the auxiliary solution $(y_h^n(u), p_h^n(u))$ connected as $\theta^n = y_h^n - y_h^n(u), \quad \zeta^n = p_h^n - p_h^n(u)$.

Lemma 4.4. *Let* (y_h, p_h) *and* $(y_h(u), p_h(u))$ *be the solutions of* (8a)-(8b) *and (18a-18b), respectively. Then, there exists a constant* C *independent of* h *and* k *such that the following estimate holds*

$$\|y_h - y_h(u)\|_{L^\infty(I;L^2(\Omega))} + \|p_h - p_h(u)\|_{L^\infty(I;L^2(\Omega))} \le C\|u - u_h\|_{L^2(I;L^2(\Omega_U))}. \tag{19}$$

Proof. As in [9] we subtract (8a) from (18a), and we obtain the following equation

$$(\theta^n - \theta^{n-1}, \varphi) + k a_h^s\,(\theta^n, \varphi)$$
$$= k\,(Bu_h^n - Bu^n, \varphi) + k \left[\sum_{K \in \mathfrak{T}_h} \tau\,(Bu_h^n - Bu^n, \beta \cdot \nabla \varphi)_K \right] - \sum_{K \in \mathfrak{T}_h} \tau(\theta^n - \theta^{n-1}, \beta \cdot \nabla \varphi)_K. \tag{20}$$

As in the proof of the Lemma 4.2, we choose $\varphi = \theta^n$ as a test function. By following the steps in the proof of Lemma 4.2, we get

$$\|\theta^n\|^2_{L^2(\Omega)} + \frac{3k}{40} \sum_{j=1}^n \|\theta^j\|_s^2 \le C \sum_{j=1}^n k(\|\theta^j\|^2_{L^2(\Omega)} + \|\theta^{j-1}\|^2_{L^2(\Omega)}) + C \sum_{j=1}^n k\|u^j - u_h^j\|^2_{L^2(\Omega_U)}. \tag{21}$$

By arranging the inequality (21), we obtain

$$(1 - Ck)\|\theta^n\|^2_{L^2(\Omega)} \le Ck\|\theta^0\|^2_{L^2(\Omega)} + 2Ck \sum_{j=1}^{n-1} \|\theta^j\|^2_{L^2(\Omega)} + C \sum_{j=1}^n k\|u^j - u_h^j\|^2_{L^2(\Omega_U)}. \tag{22}$$

For $1 - Ck > 0$, we apply the discrete Gronwall's Lemma to obtain

$$\|y_h - y_h(u)\|_{L^\infty(I;L^2(\Omega))} \le C\|u - u_h\|_{L^2(I;L^2(\Omega_U))}. \tag{23}$$

Similarly, we derive the following inequality subtractin (8b) from (18b)

$$\|\zeta\|_{L^\infty(I;L^2(\Omega))} \leq C\|y_h - y_h(u)\|_{L^2(I;L^2(\Omega))}. \tag{24}$$

Therefore, Lemma 4.4 is proved through (23)-(24). □

In order to find an upper bound to the difference between the optimal u and the fully-discrete control u_h^n, we divide the domain Ω_U as the active and inactive regions of the control u:

$$\begin{aligned}
\Omega_U^*(t) &= \{\cup_{K_U} : K_U \subset \Omega_U, u_a < u(\cdot,t)|_{K_U} < u_b\},\\
\Omega_U^c(t) &= \{\cup_{K_U} : K_U \subset \Omega_U, u(\cdot,t)|_{K_U} = u_a \text{ or } u(\cdot,t)|_{K_U} = u_b\},\\
\Omega_U^b(t) &= \Omega_U \backslash (\Omega_U^*(t) \cup \Omega_U^c(t)).
\end{aligned}$$

It is assumed that the intersection of the three sets is empty, i.e., $\Omega_U^i \cap \Omega_U^j = \emptyset$ for $i \neq j$ and $\Omega_U = \Omega_U^*(t) \cup \Omega_U^c(t) \cup \Omega_U^b(t)$. $\Omega_U^b(t)$ consists of elements which lie close to the free boundary between the active and the inactive sets for each time interval. We also assume

$$\text{meas}(\Omega_U^b(t)) \leq Ch_U \ \forall \ t \in [0,T] \tag{25}$$

for the regularity of u and $\mathfrak{T}_h^U$. This assumption is valid if the boundary of the level set $\Omega_U^c(t)$ consists of a finite number of rectifiable curves [14]. In addition, we set $\Omega^*(t) = \{x \in \Omega_U : u_a < u(x,t) < u_b\}$, which includes $\Omega_U^*(t) \subset \Omega^*(t)$ [10].

Lemma 4.5. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (4) and discrete OCP, respectively. We assume that $u \in L^2(I;W^{1,\infty}(\Omega_U)), u|_{\Omega^*} \in L^2(I;H^2(\Omega^*)), p \in L^2(I;W^{1,\infty}(\Omega))$, we have*

$$\begin{aligned}
\|u-u_h\|_{L^2(I;L^2(\Omega_U))} \leq Ck\left\|\frac{\partial p}{\partial t}\right\|_{L^2(I;L^2(\Omega))} &+ C(1+\tau h^{-1})\|p_h(u)-p\|_{L^2(I;L^2(\Omega))}\\
&+ Ch_U^{3/2}\|u\|_{L^2(I;L^2(\Omega_U))} + \tau h_U^{3/2}h^{-1}\|p\|_{L^2(I;L^2(\Omega))}.
\end{aligned} \tag{26}$$

Proof. Following [17] and using the variational inequality (8c), we define

$$(J_h'(u), v-u)_U = \sum_{n=1}^{N+1} k(\alpha u^n - B^*p_h^{n-1}(u) - \tau\beta\cdot\nabla B^*p_h^{n-1}(u), v^n - u^n)_U,$$

where $p_h^{n-1}(u)$ is the solution of (18b). With $\underline{P}^{n-1} = p_h^{n-1}(v) - p_h^{n-1}(u)$, we consider

$$\begin{aligned}
&(J_h'(v) - J_h'(u), v-u)_U\\
&= \alpha\|v-u\|^2_{L^2(I;L^2(\Omega_U))} - \sum_{n=1}^{N+1} k(B^*\underline{P}^{n-1} + \tau\beta\cdot\nabla B^*\underline{P}^{n-1}, v^n-u^n)_U.
\end{aligned} \tag{27}$$

We find an upper bound for the last term of (27) with $\underline{Y}^{n-1} = y_h^{n-1}(v) - y_h^{n-1}(u)$ using (8a-8b)

$$\begin{aligned}
&\sum_{n=1}^{N+1} k(\underline{P}^{n-1} + \tau\beta\cdot\nabla\underline{P}^{n-1}, Bv^n - Bu^n) \\
&= \sum_{n=1}^{N+1} k\left((\underline{Y}^n - \underline{Y}^{n-1}, \underline{P}^{n-1}) + ka_h^s(\underline{Y}^n, \underline{P}^{n-1}) + \sum_{K\in\mathfrak{T}_h} \tau(\underline{Y}^n - \underline{Y}^{n-1}, \beta\cdot\nabla\underline{P}^{n-1})_K\right) \\
&= \sum_{n=1}^{N+1} k\left((\underline{P}^{n-1} - \underline{P}^n, \underline{Y}^n) + ka_h^s(\underline{Y}^n, \underline{P}^{n-1}) + \sum_{K\in\mathfrak{T}_h} \tau(\underline{Y}^n - \underline{Y}^{n-1}, \beta\cdot\nabla\underline{P}^{n-1})_K\right) \\
&= \sum_{n=1}^{N+1} k\left(-(\underline{Y}^n, \underline{Y}^n) - \tau(\underline{P}^{n-1} - \underline{P}^n, \beta\cdot\nabla\underline{Y}^n) + \sum_{K\in\mathfrak{T}_h} \tau(\underline{Y}^n - \underline{Y}^{n-1}, \beta\cdot\nabla\underline{P}^{n-1})_K\right) \\
&= -\sum_{n=1}^{N+1} k(\underline{Y}^n, \underline{Y}^n) - \|\underline{Y}\|^2_{L^2(I;L^2(\Omega))} \leq 0. \qquad (28)
\end{aligned}$$

Let $\Pi_h u^n \in U_h$ be the standard Lagrange interpolation of u at time t_n such that $\Pi_h u_n(x) = u_n(x)$ for all vertices x. Then, $\Pi_h u_n$ belongs to U_h^{ad} at time t_n. Then, by (27-28), we have

$$\begin{aligned}
\alpha\|u - u_h\|^2_{L^2(I;L^2(\Omega_U))} &\leq (J_h'(u) - J_h'(u_h), u - u_h)_U \\
&= \sum_{n=1}^{N+1} k(\alpha u^n - B^* p_h^{n-1}(u) - \tau\beta\cdot\nabla B^* p_h^{n-1}(u), u^n - u_h^n)_U \\
&- \sum_{n=1}^{N+1} k(\alpha u_h^n - B^* p_h^{n-1} - \tau\beta\cdot\nabla B^* p_h^{n-1}, u^n - u_h^n)_U.
\end{aligned}$$

We add and substract the term $\sum_{n=1}^{N+1} k(B^* p^n, u^n - u_h^n)_U$ to the first term. For the second term, we rewrite $u^n - u_h^n$ as $u^n - \Pi_h u^n + \Pi_h u^n - u_h^n$. Then, we obtain

$$\begin{aligned}
\alpha\|u - u_h\|^2_{L^2(I;L^2(\Omega_U))} &= \sum_{n=1}^{N+1} k(\alpha u^n - B^* p^n, u^n - u_h^n)_U + \sum_{n=1}^{N+1} k(B^* p^n - B^* p_h^{n-1}(u), u^n - u_h^n)_U \\
&+ \sum_{n=1}^{N+1} k(\alpha u_h^n - B^* p_h^{n-1} - \tau\beta\cdot\nabla B^* p_h^{n-1}, \Pi_h u^n - u^n)_U \\
&+ \sum_{n=1}^{N+1} k(\alpha u_h^n - B^* p_h^{n-1} - \tau\beta\cdot\nabla B^* p_h^{n-1}, u_h^n - \Pi_h u^n)_U \\
&- \sum_{n=1}^{N+1} k(\tau\beta\cdot\nabla B^* p_h^{n-1}(u), u^n - u_h^n)_U.
\end{aligned}$$

We observe that the first term and the fourth term of above equation ≤ 0 due to (4c) and (8c), respectively. Then, we obtain

$$\begin{aligned}
\alpha\|u - u_h\|^2_{L^2(I;L^2(\Omega_U))} &\leq \sum_{n=1}^{N+1} k(B^* p^n - B^* p_h^{n-1}(u), u^n - u_h^n)_U \\
+ \sum_{n=1}^{N+1} k(\alpha u_h^n - B^* p_h^{n-1} &- \tau\beta\cdot\nabla B^* p_h^{n-1}, \Pi_h u^n - u^n)_U - \sum_{n=1}^{N+1} k(\tau\beta\cdot\nabla B^* p_h^{n-1}(u), u^n - u_h^n)_U.
\end{aligned}$$

Now, for the first term of the equation, we add and substract the term $\sum_{n=1}^{N+1} k(B^*p^{n-1}, u^n - u_h^n)_U$. Then, we add and substract the term $\sum_{n=1}^{N+1} k(\tau\beta\cdot\nabla B^*p_h^{n-1}, u^n - u_h^n)_U$ and arrange the resulting sum to derive the following inequality

$$\alpha\|u-u_h\|^2_{L^2(I;L^2(\Omega_U))} \leq \sum_{n=1}^{N+1} k(B^*p^{n-1} - B^*p_h^{n-1}(u), u^n - u_h^n)_U + \sum_{n=1}^{N+1} k(B^*p^n - B^*p^{n-1}, u^n - u_h^n)_U$$
$$+ \sum_{n=1}^{N+1} k(\alpha u_h^n - B^*p_h^{n-1}, \Pi_h u^n - u^n)_U - \sum_{n=1}^{N+1} k(\tau\beta\cdot\nabla B^*p_h^{n-1}(u) - \tau\beta\cdot\nabla B^*p_h^{n-1}, u^n - u_h^n)_U$$
$$+ \sum_{n=1}^{N+1} k(\tau\beta\cdot\nabla B^*p_h^{n-1}, u_h^n - \Pi_h u^n)_U$$

The following estimates are derived by using Young's inequality as in [17].

$$\begin{aligned} T_1 &\leq C_1 \sum_{n=1}^{N+1} k\|p^{n-1} - p_h^{n-1}(u)\|^2_{L^2(\Omega)} + C_2 \sum_{n=1}^{N+1} k\|u^n - u_h^n\|^2_{L^2(\Omega_U)} \\ &\leq C_1\|p - p_h(u)\|^2_{L^2(0,T;L^2(\Omega))} + C_2\|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))}, \end{aligned} \tag{29}$$

$$\begin{aligned} T_2 &\leq C_1 \sum_{n=1}^{N+1} k\|p^n - p^{n-1}\|^2_{L^2(\Omega)} + C_2 \sum_{n=1}^{N+1} k\|u^n - u_h^n\|^2_{L^2(\Omega_U)} \\ &\leq C_1 k^2 \left\|\frac{\partial p}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))} + C_2\|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))}, \end{aligned} \tag{30}$$

$$\begin{aligned} T_4 &\leq \tau^2\|\beta\|^2 C(\delta) \sum_{n=1}^{N+1} k\|\nabla(p^{n-1} - p_h^{n-1}(u))\|^2_{L^2(\Omega)} + C\delta \sum_{n=1}^{N+1} k\|u^n - u_h^n\|^2_{L^2(\Omega_U)} \\ &\leq \tau^2 h^{-2}\|\beta\|^2 C(\delta)\|p - p_h(u)\|^2_{L^2(I;L^2(\Omega))} + C\delta\|u - u_h\|^2_{L^2(I;L^2(\Omega_U))}. \end{aligned} \tag{31}$$

In order to bound T_3, T_5, let us mention the following interpolation error estimate given in [17, 10]. Assuming $\Pi_h u^n$ is the standard Lagrangian interpolation satisfying $\Pi_h u^n(x) = u(x, t^n)$ for any vertex x. With $\Pi_h u^n$ belonging to U_h^{ad}, we obtain

$$\|u^n - \Pi_h u^n\|_{L^2(\Omega_U^*(t_n))} \leq Ch_U^2\|u^n\|_{H^2(\Omega_U^*(t_n))}, \quad \|u^n - \Pi_h u^n\|_{W^{0,\infty}(\Omega_U^b(t_n))} \leq Ch_U\|u^n\|_{W^{1,\infty}(\Omega_U^b(t_n))},$$

for $u \in L^2(I; W^{1,\infty}(\Omega_U))$ and $u(t)_{|\Omega^*} \in H^2(\Omega^*(t))$. Hence,

$$\begin{aligned} &\|u - \Pi_h u\|^2_{L^2(I;L^2(\Omega_U))} \\ &= \sum_{n=1}^{N+1} k \left(\int_{\Omega_U^*(t_n)} (u^n - \Pi_h u^n)^2 + \int_{\Omega_U^c(t_n)} (u^n - \Pi_h u^n)^2 + \int_{\Omega_U^b(t_n)} (u^n - \Pi_h u^n)^2 \right) \\ &\leq Ch_U^4 \sum_{n=1}^{N+1} k\|u\|^2_{H^2(\Omega_U^*(t_n))} + 0 + Ch_U^2 \sum_{n=1}^{N+1} k\|u\|^2_{W^{1,\infty}(\Omega_U^b(t_n))} \text{ meas } (\Omega_U^b(t^n)) \\ &\leq Ch_U^3(\|u\|^2_{L^2(I;H^2(\Omega^*(t)))} + \|u\|^2_{L^2(I;W^{1,\infty}(\Omega_U))}) \leq Ch_U^3. \end{aligned} \tag{32}$$

The term T_5 is bounded as in [10, Lemma 4.5] where an integral average operator $\tilde{\Pi}_h$ is used. In addition, Young's inequality and inverse inequality are used to eliminate the gradient operator.

$$
\begin{aligned}
T_5 &= \sum_{n=1}^{N+1} k(\tau\beta\cdot(\nabla B^* p^{n-1} - \tilde{\Pi}_h(\nabla B^* p^{n-1})), u_h^n - \Pi_h u^n)_U \\
&\leq C\sum_{n=1}^{N+1} k\tau^2\|\beta\|^2\|\nabla p^{n-1} - \tilde{\Pi}_h\nabla p^{n-1}\|^2_{L^2(I,L^2(\Omega))} + C\sum_{n=1}^{N+1} k\|u_h^n - \Pi_h u^n\|^2_{L^2(I,L^2(\Omega_u))} \\
&\leq Ch_U^3(\tau^2 h^{-2}\|\beta\|^2\|p\|^2_{L^2(I;L^2(\Omega))} + \|u\|^2_{L^2(I;L^2(\Omega_U))}).
\end{aligned} \tag{33}
$$

We proceed with T_3 by adding and subtracting the appropriate terms

$$
\begin{aligned}
T_3 = &\sum_{n=1}^{N+1} k(\alpha u^n - B^* p^n, \Pi_h u^n - u^n)_U + \sum_{n=1}^{N+1} k(\alpha(u_h^n - u^n), \Pi_h u^n - u^n)_U \\
&+ \sum_{n=1}^{N+1} k(B^* p^{n-1} - B^* p_h^{n-1}(u), \Pi_h u^n - u^n)_U + \sum_{n=1}^{Nv} k(B^* p_h^{n-1}(u) - B^* p_h^{n-1}, \Pi_h u^n - u^n)_U \\
&+ \sum_{n=1}^{N+1} k(B^* p^n - B^* p^{n-1}, \Pi_h u^n - u^n)_U = \sum_{i=1}^{5} S_i.
\end{aligned} \tag{34}
$$

By the inequality in (4c), we have $\alpha u^n - B^* p^n = 0$ on $\Omega_U^*(t)$. In addition, there exists $x_0 \in K_U \in \Omega_U^b$ with $u_a < u(x_0,t) < u_b$ satisfying $(\alpha u^n - B^* p^n)(x_0) = 0$. Then, we adapt the following estimate motivated by [17]

$$
\begin{aligned}
\|\alpha u^n - B^* p^n\|_{W^{0,\infty}(\Omega_U^b(t))} &= \|\alpha u^n - B^* p^n - (\alpha u^n - B^* p^n)(x_0)\|_{W^{0,\infty}(\Omega_U^b(t))} \\
&\leq Ch_U\|\alpha u^n - B^* p^n\|_{W^{1,\infty}(\Omega_U^b(t))}.
\end{aligned}
$$

Then, the first term S_1 in the sum (34) is rewritten as

$$
\begin{aligned}
&\sum_{n=1}^{N+1} k(\alpha u^n - B^* p^n, \Pi_h u^n - u^n)_U \\
&= \sum_{n=1}^{N+1} k\int_{\Omega_U^*(t_n)} (\alpha u^n - B^* p^n, \Pi_h u^n - u^n) + \sum_{n=1}^{N+1} k\int_{\Omega_U^c(t_n)} (\alpha u^n - B^* p^n, \Pi_h u^n - u^n) \\
&+ \sum_{n=1}^{N+1} k\int_{\Omega_U^b(t_n)} (\alpha u^n - B^* p^n, \Pi_h u^n - u^n) \\
&\leq \sum_{n=1}^{N+1} k\|\alpha u^n - B^* p^n\|_{W^{0,\infty}(\Omega_U^b(t_n))}\|\Pi_h u^n - u^n\|_{W^{0,\infty}(\Omega_U^b(t_n))}\mathrm{meas}(\Omega_U^b(t_n)) \\
&\leq Ch_U^3\left(\|u\|^2_{L^2(I;W^{1,\infty}(\Omega_U))} + \|p\|^2_{L^2(I;W^{1,\infty}(\Omega))}\right) \leq Ch_U^3.
\end{aligned} \tag{35}
$$

The remaining terms S_i are bounded using similarly in [17] to find the estimate (26). □

Lemma 4.6. *Let (y,p) and $(y_h(u), p_h(u))$ be the solutions of (4a-4b) and (18a-18b), respectively. Assume that $y,\ p \in L^\infty(I; H_0^1(\Omega)\cap H^2(\Omega)) \cap H^1(I; H^2(\Omega)) \cap H^2(I; L^2(\Omega))$, $y_d \in L^2(I; L^2(\Omega))$ and $\tau = \mathcal{O}(k)$. Then,*

$$
\|y - y_h(u)\|_{L^\infty(I;L^2(\Omega))} + \|p - p_h(u)\|_{L^\infty(I;L^2(\Omega))} \leq C\left(h^2 + k + \tau^{1/2}(h^2 + h + \epsilon) + h^2\tau^{-1/2}\right) \tag{36}
$$

where C depends on some spatial and temporal derivatives of y, p, and y_d.

Proof. Let us start by subtracting (4a) from (18a) to obtain an error equation

$$
\begin{aligned}
&(\frac{y^n - y^{n-1}}{k}, \varphi) + a(y^n, \varphi) - \left(\frac{y_h^n(u) - y_h^{n-1}(u)}{k}, \varphi \right) - a_h^s(y_h^n(u), \varphi) \\
&+ \left[\sum_{K \in \mathfrak{T}_h} \tau \left(f^n + u^n, \beta \cdot \nabla \varphi\right)_K \right] - \sum_{K \in \mathfrak{T}_h} \tau \left(\frac{y_h^n(u) - y_h^{n-1}(u)}{k}, \nabla \varphi \right)_K = 0. \qquad (37)
\end{aligned}
$$

As in [12], we decompose the error $y^n - y_h^n(u)$ as

$$
y_h^n(u) - y^n = (y_h^n(u) - \pi_h^n y) + (\pi_h^n y - y^n) = e_h^n + \eta_h^n. \qquad (38)
$$

The term η_h^n can be estimated with (11) by taking the degree of local polynomials $r = 1$. We need then only to derive an estimate for e_h^n. We rewrite (37) by using (38) as follows

$$
\begin{aligned}
&(e_h^n - e_h^{n-1}, \varphi) + k a_h^s(e_h^n, \varphi) \\
&= k \left(\underbrace{y_t^n - \pi_h^n y_t + \left(\pi_h^n y_t - \frac{\pi_h^n y - \pi_h^{n-1} y}{k} \right)}_{Y_1}, \varphi \right) + k \left(\underbrace{\sigma \left(y^n - \pi_h^n y\right) + \beta \cdot \nabla (y^n - \pi_h^n y)}_{Y_2}, \varphi \right) \\
&+ k \sum_{K \in \mathfrak{T}_h} \tau (\underbrace{Y_1 + Y_2 + \epsilon \Delta (\pi_h^n y - y^n)}_{Y_3}, \beta \cdot \nabla \varphi)_K - \sum_{K \in \mathfrak{T}_h} \tau (e_h^n - e_h^{n-1}, \beta \cdot \nabla \varphi)_K. \qquad (39)
\end{aligned}
$$

The error equation (39) is similar to discrete OCP problem . Thus, we can apply the techniques used in the proof of Lemma 4.2 to (39) by choosing $\varphi = e_h^n$ with $e_h^0 = 0$ to arrive at

$$
\|e_h^n\|_{L^2(\Omega)}^2 + \frac{3k}{40} \sum_{j=1}^n \|e_h^j\|_s^2 \leq Ck \left(\sum_{j=1}^n (\|Y_1^j\|_{L^2(\Omega)}^2 + \|Y_2^j\|_{L^2(\Omega)}^2 + \|Y_3^j\|_{L^2(\Omega)}^2) \right). \qquad (40)
$$

By inserting the bounds for the right-hand side given in [12] to (40) and combining with the well-known estimate (11), we finish the proof for the state equation. For the adjoint equation, we subtract (4b) from (18b) and proceed as in the proof of (36) and use the stability estimate of adjoint equation Lemma 4.3 to obtain

$$
\|p - p_h(u)\|_{L^\infty(I; L^2(\Omega))} \leq C \|y - y_h(u)\|_{L^2(I; L^2(\Omega))}. \qquad (41)
$$

By combining the estimates for state and adjoint, we derive (36). □

We present the main result of this study by combining Lemma 4.4, 4.5, 4.6.

Theorem 4.1. *Let $(y,\ p,\ u)$ and $(y_h,\ p_h,\ u_h)$ be the solutions of (4) and (8), respectively. With $\tau = \mathcal{O}(k)$, we have*

$$
\begin{aligned}
&\|y - y_h\|_{L^\infty(I; L^2(\Omega))} + \|p - p_h\|_{L^\infty(I; L^2(\Omega))} + \|u - u_h\|_{L^2(I; L^2(\Omega_U))} \\
&\leq C \left((1 + \tau h^{-1})(h^2 + k + \tau^{1/2}(h^2 + h + \epsilon) + h^2 \tau^{-1/2}) + k + \tau h_U^{3/2} h^{-1} + h_U^{3/2} \right), \qquad (42)
\end{aligned}
$$

where C depends on some spatial and temporal derivatives of y, p, y_d and u.

By Theorem 4.1, we derive a priori error estimates for the SUPG method applied to the distributed OCPs. There are different approaches in the a priori error analysis according to the stabilization parameter proportional to the length of the time step or mesh size. We choose the optimal scaling of the mesh size h and time step size k so that the error bounds of these estimates are balanced to obtain L_∞ error estimates for the state and adjoint, L^2 error estimates for the control. Numerical results in the next section confirm the predicted a priori error estimates.

5. Numerical Results

In this section, we present numerical results for the unsteady control constrained optimal control problems governed by the convection diffusion equation. We consider two examples from [9, 10] which are solved by using the characteristic finite element method in space with backward Euler method in time. The control constraints are given as $u \geq 0$.

We have only one asymptotic order of convergence by setting the stabilization parameter proportional to time step k, i.e., $\tau = \mathcal{O}(k)$ and scaling $k \cong h^{4/3}$. Thus, we balance the terms $\mathcal{O}(k)$ and $\mathcal{O}(h^2\tau^{-1/2}) = \mathcal{O}(h^2k^{-1/2})$ to obtain the optimal L^2 error due to (42). Hence, the backward Euler leads the expected order to be $\mathcal{O}(h^{4/3})$.

We use a numerical test problem is taken from [10], of a highly convection dominated OCP problem

$$Q = (0,1] \times \Omega, \quad \Omega = (0,1)^2, \quad \epsilon = 10^{-5}, \quad \beta = (0.5, 0.5)^T, \quad \sigma = 0, \quad \alpha = 1.$$

We take $\Omega = \Omega_U$ and $B = I$. The source function f and the initial condition y_0 are computed using the following exact solutions of the state, adjoint and control, respectively,

$$\begin{aligned}
y(x,t) &= p(\frac{1}{2\sqrt{\epsilon}}\sin(t_x) - 8\epsilon\pi^2 - \frac{\sqrt{\epsilon}}{2} + \frac{1}{2}\sin(t_x)^2) \\
&- \pi\cos(\pi t)\sin(2\pi x_1)\sin(2\pi x_2)\exp(\frac{-1+\cos(t_x)}{\sqrt{\epsilon}}), \\
p(x,t) &= \sin(\pi t)\sin(2\pi x_1)\sin(2\pi x_2)\exp(\frac{-1+\cos(t_x)}{\sqrt{\epsilon}}), \\
u(x,t) &= \max(-p, 0) \\
y_d(x,t) &= \pi(1 + 2\sqrt{\epsilon}\sin(t_x))\sin(\pi t)\sin(2\pi(x_1+x_2))\exp(\frac{-1+\cos(t_x)}{\sqrt{\epsilon}}), \\
t_x &= t - 0.5(x_1 + x_2).
\end{aligned}$$

We choose the stabilization parameter as $\tau = 4k/5$. In Table 1, we present the error and convergence rates. Theoretical convergence rate $\mathcal{O}(h^{4/3})$ is achieved for the numerical results of the state, adjoint and control. Indeed, the rate of convergence for the state increases up to 1.5. The Errors are in the same range as in [10].

We observe that the state, adjoint and control variables are are approximated well in Figure 1 as in [10].

$h/\sqrt{2}$	$\|y-y_h\|_\infty$	order	$\|p-p_h\|_\infty$	order	$\|u-u_h\|_2$	order
2^{-2}	3.3791e-1	-	3.3772e-2	-	1.8472e-2	-
2^{-3}	1.2205e-1	1.47	1.5106e-2	1.16	7.0489e-3	1.39
2^{-4}	6.3586e-2	0.94	7.3456e-3	1.04	3.3531e-3	1.07
2^{-5}	2.6872e-2	1.24	3.1713e-3	1.21	1.4818e-3	1.18
2^{-6}	9.2758e-3	1.53	1.4241e-3	1.16	6.8664e-4	1.11

Table 1. Error and convergence rates for the stabilization parameter $\tau = 4k/5$

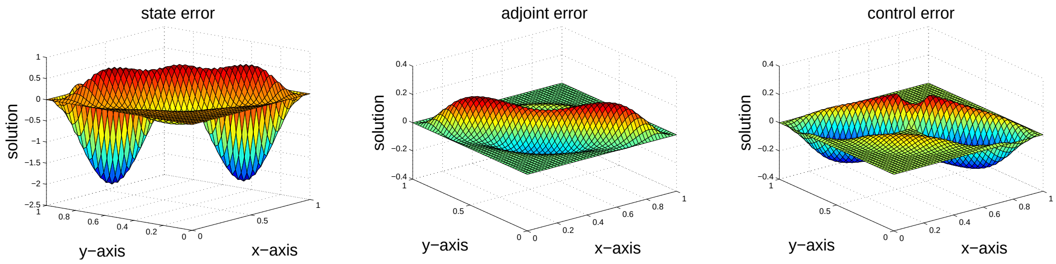

Figure 1. $h = 2^{-6}\sqrt{2},\ \tau = 4k/5,\ k \cong h^{4/3},\ t = 0.5$: Errors for the state, adjoint and control errors.

6. Conclusion

We have shown that by balancing the errors, the convergence rates of the optimal solutions can be improved under SUPG discretization in space, which is common for OCPs, when the state, control and adjoint are discretized by linear finite elements. In case of higher order finite elements, for SUPG discretized diffusion-convection-reaction equations, the difference between the *DO* and *OD* is more significant. But this does not imply more accurate controls. This will be investigated in a future work for time-dependent diffusion-convection-reaction equations.

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