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# GENERALIZED $\alpha - \psi$-GERAGHTY MULTIVALUED MAPPINGS ON $b$-METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. In this paper, we provide some conditions for the existence of a coincidence point of single-valued and multivalued mappings involving generalized $\alpha - \psi$-Geraghty contractions endowed with a graph. Our main results improve the existing results in the corresponding literature. We also present examples to support the obtained results.

Keywords: $b$-metric space, generalized $\alpha-\psi$-Geraghty multivalued mappings, coincidence point.

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## 1. Introduction

The study of $b$-metric spaces was initiated in the works of Bakhtin, Heinonen, and Czerwik [6, 8]. Afterwards, several articles which deal with fixed point theorems for single-valued and multivalued mappings in the class of $b$-metric spaces appeared in [2, 3, 4, 5, 8, 10] and related references therein.

**Definition 1.1.** [9] *Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to [0, \infty)$ is said to be a b-metric and the pair $(X, d)$ is called a b- metric space if, for all $x, y, z \in X$, the following conditions are satisfied:*
$(bM_1)$ $d(x, y) = 0$ *if and only if* $x = y$;
$(bM_2)$ $d(x, y) = d(y, x)$;
$(bM_3)$ $d(x, z) \leq s[d(x, y) + d(y, z)]$.

**Remark 1.1.** *Since a metric space turns into a b-metric space by taking the constant $s = 1$, the class of b-metric spaces is larger than the class of metric spaces.*

The following example shows that there exists a $b$-metric which is not a metric.

**Example 1.1.** *Let $X = \{a, b, c\}$ with $0 < a < 2b < c$ and $d : X \times X \to [0, \infty)$ be defined by*

$$d(a, b) = b, \quad d(a, c) = \frac{b}{2} \quad and \quad d(b, c) = c,$$

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*with $d(x,x)=0$ and $d(x,y)=d(y,x)$ for all $x,y\in X$. Notice that $d$ is not a metric since $d(b,c)>d(a,b)+d(a,c)$. However, it is easy to see that d is a b-metric space with coefficient $s\geq 2$.*

Let $\mathbb{N}$ be the set of positive integers. A sequence $\{x_n\}$ in a $b$-metric space $X$ is said to be convergent if and only if there exists $x\in X$ such that $d(x_n,x)\to 0$ as $n\to\infty$. In this case, we write $\to \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in a $b$-metric space $X$ is said to be Cauchy if and only if $d(x_n,x_m)\to 0$ as $m,n\to\infty$. A $b$-metric space $(X,d)$ is complete if every Cauchy sequence in $X$ converges. In general, a $b$-metric is not continuous. The famous Banach contraction principle [7] infers that every contraction on a complete metric space has a unique fixed point. Jachymski [11] introduced the notion of a Banach $G$-contraction to generalize the Banach contraction principle as follows. Let $(M,d)$ be a metric space. Consider $\Delta$ the diagonal of the Cartesian product $M\times M$ and $G$ a directed graph such that the set $V(G)$ of its vertices coincides with $M$ and the set $E(G)$ of its edges contains all loops; that is, $E(G)\supseteq\Delta$. Assume that $G$ has no parallel edges. A mapping $f:M\to M$ is called a Banach $G$-contraction if:
$(i)$ for every $x,y\in X$,

$$(x,y)\in E(G)\Rightarrow (f(x),f(y))\in E(G)$$

$(ii)$ there exists $0<\alpha<1$ such that for all $x,y\in X$,

$$(x,y)\in E(G)\Rightarrow d(f(x),f(y))\leq \alpha d(x,y)$$

Now, let $(X,d)$ be a $b$-metric space. Take $P_{b,cl}(X)$ the set of bounded and closed sets in $X$. For $x\in X$ and $A,B\in P_{b,cl}(X)$, as in [8], we define

$$D(x,A)=\inf_{a\in A} d(x,a),$$
$$D(A,B)=\sup_{a\in A} D(a,B).$$

Define a mapping $H: P_{b,cl}(X)\times P_{b,cl}(X)\to[0,\infty)$ such that

$$H(A,B)=\max\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,B)\},$$

for every $A,B\in CB(X)$. Then the mapping $H$ forms a $b$-metric. $H$ is called as the Hausdorff $b$-metric induced by the $b$-metric $d$. The proof of the following lemmas can be found in [8].

**Lemma 1.1.** *Let $(X,d)$ be a b-metric space. For any $A,B\in P_{b,cl}(X)$ and any $x,y\in X$, we have the following:*
(1) *$D(x,B)\leq d(x,b)$ for any $b\in B$,*
(2) *$D(x,B)\leq H(A,B)$,*
(3) *$D(x,A)\leq s(d(x,y)+D(y,B))$.*

**Lemma 1.2.** *Let $A$ and $B$ be nonempty closed and bounded subsets of a b-metric space $(X,d)$. Choose $q>1$. Then for all $a\in A$, there exists $b\in B$ such that $d(a,b)\leq qH(A,B)$.*

**Definition 1.2.** [16] *Let $X$ be a nonempty set and $G=(V(G),E(G))$ be a graph such that $V(G)=X$. $T:X\to P_{b,cl}(X)$ is said to be graph preserving if it satisfies the following:*

$$if\quad (x,y)\in E(G),\quad then\quad (u,v)\in E(G)\quad for\ all\quad u\in Tx\quad and\quad v\in Ty.$$

**Definition 1.3.** [16] *Let $X$ be a nonempty set and $G=(V(G),E(G))$ be a graph such that $V(G)=X$. $T:X\to P_{b,cl}(X)$ is said to be g-graph preserving if it satisfies the following: for $x,y\in X$,*

$$if\quad (g(x),g(y))\in E(G),\quad then\quad (u,v)\in E(G)\quad for\ all\quad u\in Tx\quad and\quad v\in Ty.$$

Let $\Phi$ be set of all increasing and continuous functions $\phi : [0,\infty) \to [0,\infty)$ satisfying

$$\phi(ct) \leq c\phi(t) \quad \text{for all} \;\; c > 1.$$

Let $s \geq 1$. We denote by $\mathcal{F}_s$ the family of all functions $\beta : [0,\infty) \to [0, \frac{1}{s^2})$.

The notation of an $\alpha-\psi$-Geraghty contraction-type multivalued mapping in the setting of metric spaces was introduced by Karapinar and Samet [12, 13, 14]. Newly, Afshari et al. [1] proved some results on generalized $\alpha-\psi$-Geraghty contraction-type multivalued mappings. Precisely, they have proved the following theorem.

**Theorem 1.1.** *Let $(X, d)$ be a complete b-metric space with a coefficient $s \geq 1$. Let $T : X \to P_{b,cl}(X)$ be a multivalued mapping. Suppose that there exists $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)\psi(s^3 H(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)),$$

*for all $x, y \in X$, where $\beta \in \mathcal{F}_s$ and $\psi, \phi \in \Phi$ with*

$$M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2s}\}$$

$$\text{and} \quad N(x,y) = \min\{D(x,Tx), D(y,Tx)\}.$$

*Suppose also that*
*(i) $T$ is $\alpha$-admissible;*
*(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;*
*(iii) $T$ is continuous or $X$ is $\alpha$-regular.*
*Then $T$ has a fixed point.*

Mention that the concept of $\alpha$-regularity is stated as follows.

**Definition 1.4.** [15] *Let $(X, d)$ be a b-metric space and $\alpha : X \times X \to [0,\infty)$. $X$ is said $\alpha$-regular, if for every sequence $\{x_n\}$ in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$.*

In this paper, we improve Theorem 1.1 by proving the existence of a coincidence point of single-valued and multivalued mappings in the class of $b$-metric spaces endowed with a graph, but without the function $\alpha$. We do not need the assumption that $T$ is continuous to establish our main results.

## 2. Auxiliary results: the case $s = 1$

Here, we treat the case $s = 1$. First, let $\Psi$ be the set of all increasing and continuous functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:
$(i)$ $\psi(r + t) \leq \psi(r) + \psi(t)$ for all $r, t > 0$;
$(ii)$ $\psi(ct) \leq c\psi(t)$ for all $c > 1$;
$(iii)$ $\psi(0) = 0$.

**Definition 2.1.** *Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. For $g : X \to X$ and $T : X \to P_{b,cl}(X)$, $T$ is said to be a generalized $g$-Geraghty-type $G$-multivalued mapping provided that*
*(i) $T$ is $g$-graph preserving;*

*(ii) for every $x, y \in X$ such that $(g(x), g(y)) \in E(G)$, whenever there exists some $L \geq 0$ such that for*

$$M(g(x), g(y)) = \max\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), \frac{D(g(x), Ty) + D(g(y), Tx)}{2}\} \quad (1)$$

$$\text{and} \quad N(g(x), g(y)) = \min\{D(g(x), Tx), D(g(y), Tx)\}, \quad (2)$$

*we have*

$$\psi(H(Tx, Ty)) \leq \theta(\psi(M(g(x), g(y))))\psi(M(g(x), g(y))) + L\phi(N(g(x), g(y))), \quad (3)$$

*where $\theta \in \mathcal{F}_1$ and $\psi, \phi \in \Psi$.*

**Lemma 2.1.** *Let $(X, d)$ be a metric space with a directed graph $G$. Assume that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is a generalized $g$-Geraghty-type $G$-multivalued mapping in $(X, d)$. Suppose also that*
*(i) there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*(ii) if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx$ and $w \in Ty$.*
*Then there exists a sequence $\{x_k\}_{k \in \mathbb{N} \cup \{0\}}$ in $X$ such that for each $k \in \mathbb{N}$, we have*

$$\begin{cases} g(x_k) \in Tx_{k-1} \\ (g(x_{k-1}), g(x_k)) \in E(G) \\ \{g(x_k)\} \quad \text{is a Cauchy sequence in} \quad X. \end{cases}$$

*Proof.* Since $g$ is surjective, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$ and $(g(x_0), g(x_1)) \in E(G)$. Let $q = \frac{1}{\sqrt{\theta(\psi(d(g(x_0), g(x_1))))}}$. We have $q > 1$. Then

$$0 < D(g(x_1), Tx_1) \leq H(Tx_0, Tx_1) < qH(Tx_0, Tx_1).$$

By Lemma 1.2, again $g$ is surjective, so there exists $x_2 \in X$ such that $g(x_2) \in Tx_1$ and

$$\begin{aligned} \psi(d(g(x_1), g(x_2))) &< \psi(qH(Tx_0, Tx_1)) \leq q\psi(H(Tx_0, Tx_1)) \\ &\leq q\theta(\psi(M(g(x_0), g(x_1))))\psi(M(g(x_0), g(x_1))) + qL\phi(N(g(x_0), g(x_1))), \end{aligned} \quad (4)$$

where

$$\begin{aligned} M(g(x_0), g(x_1)) &= \max\{d(g(x_0), g(x_1)), D(g(x_0), Tx_0), D(g(x_1), Tx_1), \\ &\quad \frac{D(g(x_0), Tg(x_1)) + D(g(x_1), Tx_0)}{2}\} \\ &\leq \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1), \frac{D(g(x_0), Tx_1)}{2}\} \\ &\leq \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1), \frac{D(g(x_0), Tx_1)}{2}\} \end{aligned} \quad (5)$$

and

$$\begin{aligned} N(g(x_0), g(x_1)) &= \min\{D(g(x_0), Tx_0), D(g(x_1), Tx_0)\} \\ &\leq \min\{d(g(x_0), g(x_1)), d(g(x_1), g(x_1))\} = 0. \end{aligned} \quad (6)$$

In view of

$$\begin{aligned} \frac{D(g(x_0), Tx_1)}{2} &\leq \frac{[d(g(x_0), g(x_1)) + D(g(x_1), Tx_1)]}{2} \\ &\leq \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1)\}, \end{aligned}$$

we get

$$M(x_0,,x_1) \leq \max\{d(g(x_0),g(x_1)),D(g(x_1),Tx_1)\}.$$

If $\max\{d(g(x_0),g(x_1)),D(g(x_1),Tx_1)\} = D(g(x_1),Tx_1)$, then by (4), we have

$$\begin{aligned}\psi(D(g(x_1),Tx_1)) &\leq \psi(d(g(x_1),g(x_2))) \\ &< \sqrt{\theta(\psi(D(g(x_1),Tx_1)))}\psi(D(g(x_1),Tx_1)) < \psi(D(g(x_1),Tx_1)),\end{aligned}$$

which is a contradiction. Hence, we obtain $\max\{d(g(x_0),g(x_1)),D(g(x_1),Tx_1)\} = d(g(x_0),g(x_1))$ and so by (4),

$$\psi(d(g(x_1),g(x_2))) \leq \sqrt{\theta(\psi(d(g(x_0),g(x_1))))}\psi(d(g(x_0),g(x_1))). \tag{7}$$

Having in mind that $\psi \in \Psi$ and $\sqrt{\theta(\psi(d(g(x_0),g(x_1)))} < 1$, so we get

$$\begin{aligned}&\psi(\frac{1}{\sqrt{\theta(\psi(d(g(x_0),g(x_1))))}}d(g(x_1),g(x_2))) \qquad (8)\\ &\leq \frac{1}{\sqrt{\theta(\psi(d(g(x_0),g(x_1))))}}\psi(d(g(x_1),g(x_2))) < \psi(d(g(x_0),g(x_1))).\end{aligned}$$

Since $\psi$ is increasing, we have

$$d(g(x_1),g(x_2)) \leq \sqrt{\theta(\psi(d(g(x_0),g(x_1))))}d(g(x_0),g(x_1)).$$

Recall that $g(x_2) \in Tx_1$ and $g(x_1) \notin Tx_1$, so it is clear that $g(x_2) \neq g(x_1)$. Choose

$$q_1 = \frac{\sqrt{\theta(\psi(d(g(x_0),g(x_1))))}\psi(d(g(x_0),g(x_1)))}{\psi(d(g(x_1),g(x_2)))}.$$

By (5) and (7), we have $q_1 > 1$. If $g(x_2) \in Tx_2$, then $x_2$ is a coincidence point of $g$ and $T$. Assume that $g(x_2) \notin Tx_2$. We get

$$0 < \psi(d(g(x_2),Tx_2)) \leq \psi(H(Tx_1,Tx_2)) < q_1\psi(H(Tx_1,Tx_2)).$$

Hence, there exists $g(x_3) \in Tg(x_2)$ such that

$$\begin{aligned}&\psi(d(g(x_2),g(x_3))) < q_1\psi(H(Tx_1,Tx_2)) \\ &\leq q_1\theta(\psi(M(g(x_1),g(x_2))))\psi(M(g(x_1),g(x_2))) + q_1L\phi(N(g(x_1),g(x_2))).\end{aligned}$$

Similarly, $M(g(x_1),g(x_2)) \leq d(g(x_1),g(x_2))$ and $N(g(x_1),g(x_2)) = 0$. By (7) and a property of $(\theta)$, we have

$$\begin{aligned}\psi(d(g(x_2),g(x_3))) &\leq \sqrt{\theta(\psi(d(g(x_1),g(x_2))))}\psi(d(g(x_1),g(x_2))) \qquad (9)\\ &\leq \sqrt{\theta(\psi(d(g(x_1),g(x_2))))}\sqrt{\theta(\psi(d(g(x_0),g(x_1))))}\psi(d(g(x_0),g(x_1))). \qquad (10)\end{aligned}$$

By (7) and that assumption that $\sqrt{\theta(\psi(d(g(x_0),g(x_1))))} < 1$, we have

$$\psi(d(g(x_1),g(x_2))) \leq \psi(d(g(x_0),g(x_1))).$$

The function $\theta$ is increasing, by (9), we obtain

$$\psi(d(g(x_2),g(x_3))) \leq (\sqrt{\theta(\psi(d(g(x_0),g(x_1))))})^2\psi(d(g(x_0),g(x_1))). \tag{11}$$

Again, by (8),

$$d(g(x_2),g(x_3)) \leq (\sqrt{\theta(\psi(d(g(x_0),g(x_1))))})^2 d(g(x_0),g(x_1))$$

It is clear that $g(x_2) \neq g(x_1)$. Take

$$q_2 = \frac{(\sqrt{\theta(\psi(d(x_0, x_0)))})^2 \psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_2), g(x_3)))}.$$

Then $q_2 > 1$. If $g(x_3) \in Tx_3$, then $x_3$ is a coincidence point of $g$ and $T$. Assume that $g(x_3) \notin Tx_3$. Then

$$0 < \psi(d(g(x_3), Tx_3)) \leq \psi(H(Tx_2, Tx_3)) < q_2 \psi(H(Tx_2, Tx_3)).$$

Thus there exists $g(x_4) \in Tx_3$ such that

$$\begin{aligned} \psi(d(g(x_3), g(x_4))) &< q_2 \psi(H(Tx_2, Tx_3)) \\ &\leq q_2 \theta(\psi(M(g(x_2), g(x_3)))) \psi(M(g(x_2), g(x_3))) + q_2 L \phi(N(g(x_2), g(x_3))) \end{aligned} \tag{12}$$

Similarly, $M(g(x_2), g(x_3)) \leq d(g(x_2), g(x_3).)$ and $N(g(x_2), g(x_3)) = 0$. So, by (12),

$$\begin{aligned} \psi(d(g(x_3), g(x_4))) &\leq \sqrt{\theta(\psi(d(g(x_2), g(x_3))))} \psi(d(g(x_2), g(x_3))) \\ &\leq \sqrt{\theta(\psi(d(g(x_2), g(x_3))))} (\sqrt{\theta(\psi(d(g(x_0), g(x_1))))})^2 \psi(d(g(x_0), g(x_1))). \end{aligned} \tag{13}$$

By (11) and the assumption $\sqrt{\theta(\psi(d(g(x_0), g(x_1))))}^2 < 1$, we have

$$\psi(d(g(x_2), g(x_3))) \leq \psi(d(g(x_0), g(x_1))).$$

Again, $\theta$ is increasing, so using (13),

$$d(g(x_3), g(x_4)) \leq (\sqrt{\theta(\psi(d(g(x_0), g(x_1))))})^3 d(g(x_0), g(x_1)).$$

It is clear that $g(x_3) \neq g(x_2)$. Put

$$q_3 = \frac{(\sqrt{\theta(\psi(d(g(x_0), g(x_1))))})^3 \psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_2), g(x_3)))}.$$

Then $q_3 > 1$. By continuing this process, we are arrived to construct a sequence $\{x_n\}$ in $X$ such that $g(x_n) \in Tx_{n-1}$, $g(x_n) \neq g(x_{n-1})$ and

$$d(g(x_n), g(x_{n+1})) < (\sqrt{\theta(\psi(d(g(x_0), g(x_1))))})^n d(g(x_0), g(x_1))$$

for all $n$. Let $t = \sqrt{\theta(\psi(d(g(x_0), g(x_1))))}$, then $0 < t < 1$. For $n < m$, by the triangle inequality

$$\begin{aligned} d(g(x_n), g(x_m)) &\leq d(g(x_n), g(x_{n+1})) + d(g(x_{n+1}), g(x_{n+2})) + \ldots \\ &\quad + d(g(x_{m-2}), g(x_{m-1})) + d(g(x_{m-1}), g(x_m)) \\ &\leq t^n (1 + t + t^2 + \ldots) d(g(x_0), g(x_1)) \\ &= (\frac{t^n}{1-t}) d(g(x_0), g(x_1)) \to 0 \ as \ n \to \infty. \end{aligned}$$

Therefore, for $n < m$, we obtain

$$d(g(x_n), g(x_m)) \to 0 \ as \ n \to \infty.$$

We deduce

$$\lim_{m,n \to \infty} d(g(x_n), g(x_m)) = 0.$$

Thus $\{g(x_n)\}$ is a Cauchy sequence in $(X, d)$. The proof is completed. □

The following hypothesis is required for the rest.

Hypothesis $(A)$: For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $n_k \in \mathbb{N}$.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space with a directed graph $G$. Assume that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is $g$-graph preserving. Suppose that $T$ is a generalized $g$-Geraghty-type $G$-multivalued mapping in $(X, d)$. Assume also that*
*$(i)$ there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*$(ii)$ if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx, w \in Ty$;*
*$(iii)$ the hypothesis $(A)$ holds.*
*Then there exists $u \in X$ such that $g(u) \in Tu$, that is, $u$ is a coincidence point of $g$ and $T$.*

*Proof.* By $(i)$, let $x_0 \in X$ be such that $(g(x_0), g(x_1)) \in E(G)$ for some $g(x_1) \in Tx_0$. From Lemma 2.1, there exists a sequence $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$ in $X$ such that for each $k \in \mathbb{N}$,

$$g(x_k) \in Tx_{k-1} \quad \text{and} \quad (g(x_{k-1}), g(x_k)) \in E(G).$$

$\{g(x_k)\}$ is also a Cauchy sequence in $X$. Since $X$ is complete, the sequence $\{g(x_k)\}$ converges to a point $w$ for some $w \in X$. Let $u \in X$ be such that $g(u) = w$. In view of $(iii)$, there is a subsequence $\{g(x_{k_n})\}$ such that $(g(x_k), g(u)) \in E(G)$ for any $n \in \mathbb{N}$. We claim that $g(u) \in Tu$. We have

$$\begin{aligned}\psi(D(g(u), Tu)) &\leq \psi(d(g(u), g(x_{k_n})) + D(g(x_{k_n}), Tu))\\ &\leq \psi(d(g(u), g(x_{k_n}))) + \psi(D(g(x_{k_n}), Tu))\\ &\leq \psi(d(g(u), g(x_{k_n}))) + \psi(H(Tx_{k_n}, Tu))\\ &\leq \psi(d(g(u), g(x_{k_n}))) + \theta(\psi(M(g(x_{k_n}), g(u))))\psi(M(g(x_{k_n}), g(u)))\\ &+ L\phi(N(g(x_{k_n}), g(u))).\end{aligned}$$

Referring to (5) and (6),

$$M(g(x_{k_n}), g(u)) \leq d(g(x_{k_n}), g(u)) \quad \text{and} \quad N(g(x_{k_n}), g(u)) = 0.$$

Since $\{g(x_{k_n})\}$ is subsequence of $\{g(x_k)\}$, it converges to $g(u)$ as $n \to \infty$, so $D(g(u), Tu) = 0$. Since $Tu$ is closed, we conclude that $g(u) \in Tu$, that is, $u$ is a coincidence point of $g$ and $T$. □

**Example 2.1.** *Let $X = [0, 1]$ be endowed with the usual metric $d$. Consider the directed graph $G$ defined by $V(G) = X$ and*

$$E(G) = \{(x, x), (0, \frac{1}{2}), (\frac{1}{2}, 0), (0, \frac{1}{4}), (\frac{1}{4}, 0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}) : x \in X\}.$$

*Let $T : X \to P_{b,cl}(X)$ be defined by*

$$Tx = \begin{cases} \{\frac{1}{4}\} \text{ if } x = 1,\\ \{0, \frac{1}{2}\} \text{ if } x \in (0, 1) - \{\frac{1}{2}, \frac{1}{\sqrt{2}}\},\\ \{\frac{1}{2}\} \text{ if } x \in \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}\}.\end{cases}$$

*Let $g : X \to X$ be defined by $g(x) = x^2$. Consider $\psi(t) = t$ and $\theta(t) = \frac{t+1}{t+2}$. Then it is easy to check that $T$ is a $g$-Geraghty-type $G$-multivalued mapping. It is straightforward to check that the conditions $(i)$, $(ii)$, and $(iii)$ of Theorem 2.1 are satisfied. On the other hand, if $(g(x), g(y)) \in E(G)$, then $H(Tg(x), Tg(y)) = 0$. Hence, if for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$, then*

$$\psi(H(Tx, Ty)) \leq \theta(\psi(M(g(x), g(y))))\psi(M(g(x), g(y))) + L\phi(N(g(x), g(y))).$$

*By Theorem 2.1, there exists $u \in X$ such that $g(u) \in Tu$. In this example, $u = \frac{1}{\sqrt{2}}$.*

## 3. Main results: the case $s > 1$

Here, we consider the case $s > 1$. First, we introduce the notion of a $g$-Geraghty-type $G$-contraction multivalued mapping in the setting of $b$-metric spaces.

**Definition 3.1.** *Let $(X, d)$ be a b-metric space with a directed graph $G$ and with a coefficient $s > 1$. Let $T : X \to P_{b,cl}(X)$ be a multivalued mapping. We say that $T$ is a generalized g-Geraghty-type G-contraction multivalued mapping in the b-metric space $(X, d)$ provided that*
*(i) $T$ is g-graph preserving;*
*(ii) for every $x, y \in X$ such that $(g(x), g(y)) \in E(G)$, whenever there exists some $L \geq 0$ such that for*

$$M(x,y) = \max\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), \frac{D(g(x), Ty) + D(g(y), Tx)}{2s}\} \tag{14}$$

$$\text{and} \quad N(g(x), g(y)) = \min\{D(g(x), Tx), D(g(y), Tx)\}, \tag{15}$$

*we have*

$$\psi(s^3 H(Tx, Ty)) \leq \beta(\psi(M(g(x), g(y))))\psi(M(g(x), g(y))) + L\phi(N(g(x), g(y))), \tag{16}$$

*for all $x, y \in X$, where $\beta \in \mathcal{F}_s$ and $\psi, \phi \in \Psi$.*

**Remark 3.1.** *The functions belonging to $\mathcal{F}$ are strictly smaller than $\frac{1}{s^2}$. Then, the expression $\beta(\psi(M(g(x), g(y))))$ in (16) satisfies*

$$\beta(\psi(M(g(x), g(y)))) < \frac{1}{s^2} \text{ for all } \quad x, y \in X \text{ with } x \neq y.$$

**Lemma 3.1.** *Let $(X, d)$ be a b-metric space with a directed graph $G$ and with a coefficient $s > 1$. Assume that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is g-graph preserving. Suppose also that $T$ is a generalized g-Geraghty-type G-contraction multivalued mapping in $(X, d)$. Assume that*
*(i) there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*(ii) if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx$ and $w \in Ty$.*
*Then there exists a sequence $\{x_k\}_{k \in \mathbb{N} \cup \{0\}}$ in $X$ such that for each $k \in \mathbb{N}$, we have*

$$\begin{cases} g(x_k) \in Tx_{k-1} \\ (g(x_{k-1}), g(x_k)) \in E(G) \\ \{g(x_k)\} \quad \text{is a Cauchy sequence in} \quad X. \end{cases}$$

*Proof.* Since $g$ is surjective, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$ and $(g(x_0), g(x_1)) \in E(G)$. Let us take a real $q$ such that $1 < q < s$. Then

$$0 < D(g(x_1), Tx_1) \leq H(Tx_0, Tx_1) < qH(Tx_0, Tx_1).$$

Hence, By Lemma 1.2 and regarding again as $g$ is surjective, there exists $x_2 \in X$ such that $g(x_2) \in Tx_1$ and

$$\begin{aligned} &\psi(d(g(x_1), g(x_2))) < \psi(qH(Tx_0, Tx_1)) \leq q\psi(s^3 H(Tx_0, Tx_1)) \qquad (17) \\ &\leq q\beta(\psi(M(g(x_0), g(x_1))))\psi(M(g(x_0), g(x_1))) + qL\phi(N(g(x_0), g(x_1))) \\ &< \frac{q}{s^2}\psi(M(g(x_0), g(x_1))) + qL\phi(N(g(x_0), g(x_1))), \end{aligned}$$

where

$$M(g(x_0), g(x_1)) = \max\{d(g(x_0), g(x_1)), D(g(x_0), Tx_0), D(g(x_1), Tx_1), \frac{D(g(x_0), Tx_1) + D(g(x_1), Tx_0)}{2s}\} \le \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1), \frac{D(g(x_0), Tx_1)}{2s}\} \le \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1), \frac{D(g(x_0), Tx_1)}{2s}\} \tag{18}$$

and

$$N(g(x_0), g(x_1)) = \min\{D(g(x_0), Tx_0), D(g(x_1), Tx_0)\} \le \min\{d(g(x_0), g(x_1)), d(g(x_1), g(x_1))\} = 0. \tag{19}$$

Since

$$\frac{D(g(x_0), Tx_1)}{2s} \le \frac{[d(g(x_0), g(x_1)) + D(g(x_1), Tx_1)]}{2s} \le \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1)\},$$

we get

$$M(x_0, , x_1) \le \max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1)\}.$$

If $\max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1)\} = D(g(x_1), Tx_1)$, then by (17), we have

$$\psi(D(g(x_1), Tg(x_1))) \le \psi(d(g(x_1), g(x_2))) < \frac{q}{s^2}\psi(D(g(x_1), Tx_1)) < \psi(D(g(x_1), Tx_1)),$$

which is a contradiction. Hence, $\max\{d(g(x_0), g(x_1)), D(g(x_1), Tx_1)\} = d(g(x_0), g(x_1))$, and so by (17),

$$\psi(d(g(x_1), g(x_2))) \le \frac{q}{s^2}\psi(d(g(x_0), g(x_1))). \tag{20}$$

Since $\psi \in \Psi$ and $\frac{q}{s^2} < 1$, we have

$$\psi(\frac{s^2}{q}d(g(x_1), g(x_2))) \le \frac{s^2}{q}\psi(d(g(x_1), g(x_2))) \le \psi(d(g(x_0), g(x_1))). \tag{21}$$

The function $\psi$ is increasing, so

$$d(g(x_1), g(x_2)) \le \frac{q}{s^2}d(g(x_0), g(x_1)).$$

Recall that $g(x_2) \in Tx_1$ and $g(x_1) \notin Tx_1$, so it is clear that $g(x_2) \neq g(x_1)$. Put

$$q_1 = \frac{q}{s^2}\frac{\psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_1), g(x_2)))}.$$

By (18) and (20), we have $q_1 > 1$. If $g(x_2) \in Tx_2$, then $x_2$ is a coincidence point of $g$ and $T$. Assume that $g(x_2) \notin Tx_2$. Then,

$$0 < \psi(d(g(x_2), Tx_2)) \le \psi(H(Tx_1, Tx_2)) < q_1\psi(H(Tx_1, Tx_2)).$$

Hence, there exists $g(x_3) \in Tx_2$ such that

$$\begin{aligned}&\psi(d(g(x_2), g(x_3))) < q_1\psi(s^3H(Tx_1, Tx_2))\\&\leq q_1\beta(\psi(M(g(x_1), g(x_2))))\psi(M(g(x_1), g(x_2))) + q_1L\phi(N(g(x_1), g(x_2))).\end{aligned}$$

Similarly, $M(g(x_1), g(x_2)) \leq d(g(x_1), g(x_2))$ and $N(g(x_1), g(x_2)) = 0$. So, in addition to (20), by a property of $(\beta)$, we have

$$\begin{aligned}\psi(d(g(x_2), g(x_3))) &\leq \frac{q}{s^2}\frac{\psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_1), g(x_2)))}\psi(d(g(x_1), g(x_2))) \\ &= (\frac{q}{s^2})^2\psi(d(g(x_0), g(x_1))).\end{aligned} \tag{22}$$

Again, by (21), we obtain

$$d(g(x_2), g(x_3)) \leq (\frac{q}{s^2})^2 d(g(x_0), g(x_1))$$

It is clear that $g(x_2) \neq g(x_3)$. Let

$$q_2 = \frac{(\frac{q}{s^2})^2\psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_2), g(x_3)))}.$$

Then $q_2 > 1$. If $g(x_3) \in Tx_3$, then $x_3$ is a coincidence point of $g$ and $T$. Assume that $g(x_3) \notin Tx_3$. Then,

$$0 < \psi(d(g(x_3), Tx_3)) \leq \psi(H(Tx_2, Tx_3)) < q_2\psi(s^3H(Tx_2, Tx_3)).$$

Thus, there exists $g(x_4) \in Tx_3$ such that

$$\begin{aligned}&\psi(d(g(x_3), g(x_4))) < q_2\psi(s^3H(Tx_2, Tx_3))\\&\leq q_2\beta(\psi(M(g(x_2), g(x_3))))\psi(M(g(x_2), g(x_3))) + q_2L\phi(N(g(x_2), g(x_3)))\end{aligned} \tag{23}$$

Similarly $M(g(x_2), g(x_3)) \leq d(g(x_2), g(x_3))$ and $N(g(x_2), g(x_3)) = 0$. So, by (12),

$$\begin{aligned}\psi(d(g(x_3), g(x_4))) &\leq \frac{q_2}{s^2}\psi(d(g(x_2), g(x_3))) \leq \frac{(\frac{q}{s^2})^3\psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_2), g(x_3)))}\psi(d(g(x_2), g(x_3))) \\ &= (\frac{q}{s^2})^3\psi(d(g(x_0), g(x_1))).\end{aligned} \tag{24}$$

Again, by (21),

$$d(g(x_3), g(x_4)) \leq (\frac{q}{s^2})^3 d(g(x_0), g(x_1)).$$

Put

$$q_3 = \frac{(\frac{q}{s^2})^3\psi(d(g(x_0), g(x_1)))}{\psi(d(g(x_3), g(x_4)))}.$$

Then $q_3 > 1$. By continuing this process, we are arrived to construct a sequence $\{g(x_n)\}$ in $X$ such that $g(x_n) \in Tx_{n-1}$ and $g(x_n) \neq g(x_{n-1})$. Also,

$$d(g(x_n), g(x_{n+1})) < (\frac{q}{s^2})^n\psi(d(g(x_0), g(x_1)))$$

for all $n$. Now, using the triangle inequality, we write for $n < m$

$$\begin{aligned} d(g(x_n), g(x_m)) &\leq sd(g(x_n), g(x_{n+1})) + s^2 d(g(x_{n+1}), g(x_{n+2})) + \dots \\ &+ s^{m-n-2}[d(g(x_{m-2}), g(x_{m-1})) + d(g(x_{m-1}), g(x_m))]]) \\ &\leq s(\frac{q}{s^2})^n (1 + s(\frac{q}{s^2}) + s^2(\frac{q}{s^2})^2 + \dots) d(g(x_0), g(x_1)) \\ &= [\frac{s(\frac{q}{s^2})^n}{1 - s(\frac{q}{s^2})}] d(g(x_0), g(x_1)) \to 0 \ as \ n \to \infty. \end{aligned}$$

Therefore, by symmetry

$$\lim_{m,n\to\infty} d(g(x_n), g(x_m)) = 0.$$

We deduce that $\{g(x_n)\}$ is a Cauchy sequence in $(X, d)$. □

Our main result is stated as follows.

**Theorem 3.1.** *Let $(X, d)$ be a complete b-metric space with a directed graph $G$ and with a coefficient $s > 1$. Suppose that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is g-graph preserving. Assume also that $T$ is a generalized g-Geraghty-type G-contraction multivalued mapping in $(X, d)$. Suppose that*
*(i) there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*(ii) if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx$ and $w \in Ty$;*
*(iii) (A) holds.*
*Then there exists $u \in X$ such that $g(u) \in Tu$, that is, $u$ is a coincidence point of $g$ and $T$.*

*Proof.* By $(i)$, choose $x_0 \in X$ such that $(g(x_0), g(x_1)) \in E(G)$ for some $g(x_1) \in Tx_0$. By Lemma 3.1, there exists a sequence $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$ in $X$ such that for each $k \in \mathbb{N}$

$$g(x_k) \in Tx_{k-1}, \quad (g(x_{k-1}), g(x_k)) \in E(G),$$

and $\{g(x_k)\}$ is a Cauchy sequence in $X$. The $b$-metric space $(X, d)$ is complete, so the sequence $\{g(x_k)\}$ converges to a point $w$ for some $w \in X$. $g$ is surjective, then there exists $u \in X$ such that $g(u) = w$. In view that $(A)$ holds, there is a subsequence $\{g(x_{k_n})\}$ such that that $(g(x_k), g(u)) \in E(G)$ for any $n \in \mathbb{N}$. We claim that $g(u) \in Tu$. We have

$$\begin{aligned} \psi(D(g(u), Tu)) &\leq \psi(sd(g(u), g(x_{k_n})) + s^3 D(g(x_{k_n}), Tu)) \\ &\leq \psi(sd(g(u), g(x_{k_n}))) + \psi(s^3 H(Tx_{k_n}, Tu)) \\ &\leq s(\psi(d(g(u), g(x_{k_n}))) + \beta(\psi(M(g(x_{k_n}), g(u)))))\psi(M(g(x_{k_n}), g(u))) \\ &+ L\phi(N(g(x_{k_n}), g(u))). \end{aligned}$$

By (18) and (19), we obtain

$$M(g(x_{k_n}), g(u)) \leq d(g(x_{k_n}), g(u)) \quad \text{and} \quad N(g(x_{k_n}), g(u)) = 0.$$

Because $\{g(x_{k_n})\}$ is a subsequence of $\{g(x_k)\}$, so it converges to $g(u)$ as $n \to \infty$. Thus $D(g(u), Tu) = 0$. Having in mind that $Tu$ is closed, we conclude that $g(u) \in Tu$. □

## 4. CONSEQUENCES

Taking $L = 1$ and $\psi(t) = t$ in (16), we obtain the following result.

**Corollary 4.1.** *Let $(X, d)$ be a complete b-metric space with a directed graph $G$ and with a coefficient $s > 1$. Assume that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is*

*$g$-graph preserving satisfying the following:*
*if for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$, then*

$$s^3 H(Tx, Ty) \leq \beta(M(g(x), g(y)))M(g(x), g(y)).$$

*Suppose also that*
*$(i)$ there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*$(ii)$ if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx, w \in Ty$;*
*$(iii)$ $(A)$ holds.*
*Then there exists $u \in X$ such that $g(u) \in Tu$.*

**Corollary 4.2.** *Let $(X, d)$ be a complete $b$-metric space with a directed graph $G$ and with a coefficient $s > 1$. Assume that $g : X \to X$ is a surjective map and $T : X \to P_{b,cl}(X)$ is $g$-graph preserving satisfying the following:*
*for all $x, y \in X$, if $(g(x), g(y)) \in E(G)$, then*

$$\psi(s^3 H(Tx, Ty)) \leq \beta(\psi((d(g(x), g(y)))))\psi(d(g(x), g(y))) + L\phi(N(g(x), g(y))),$$

*for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$ and*

$$\text{and } N(x, y) = \min\{d(x, Tx), d(y, Tx)\}. \tag{25}$$

*Suppose also that*
*$(i)$ there exists $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Tx_0$;*
*$(ii)$ if $(g(x), g(y)) \in E(G)$, then $(z, w) \in E(G)$ for all $z \in Tx, w \in Ty$;*
*$(iii)$ $(A)$ holds.*
*Then there exists $u \in X$ such that $g(u) \in Tu$.*

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