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FIXED POINT THEOREMS FOR GENERALIZED (ψ,φ)-WEAK CONTRACTIONS

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Abstract. In this paper, we prove some fixed point theorems for generalized (ψ,φ)-weak contractive mappings in a metric space. Our result generalized and extend recent results of Singh et al.[16, Theorem 2.1], Dorić [7, Theorem 2.1], Rhoades [15, Theorem 1] and Dutta and Choudhary [9, Theorem 2.1]. Also, we provid an example to support the useability of our results.

Keywords: fixed point, metric space, weak contractions, generalized (ψ,φ)-weak contractions.

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1. Introduction and preliminaries

Let (X,d) be a metric space. A mapping $T : X \to X$ is called contraction if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \le kd(x,y),\ \forall x,y \in X. \tag{1}$$

The best known and important fixed point theorem is the Banach contraction principle [1], which assures that every contraction T from a complete metric space X into itself has a unique fixed point. The simplicity of its proof and the possibilities of attaining the fixed point by using successive approximations let this theorem become a very useful tool in analysis and its applications. Due to importance and simplicity of Banach contraction principle, several authors have obtained many interesting extensions and generalizations of Banach contraction principle (see [5], [17]- [19] and references therein). The following important generalization is due to Ćirić [6].

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to X$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$*

$$d(Tx,Ty) \le rM_g(x,y), \tag{2}$$

where

$$M_g(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}. \tag{3}$$

Then T has a unique fixed point.

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A map T satisfying (2) is called a generalized contraction. The following is the quasi-contraction theorem, given by Ćirić [5], and is considered the most general contraction theorem in metric fixed point theory (cf. [14, 15]).

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$*

$$d(Tx, Ty) \leq rM(x, y), \tag{4}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{5}$$

Then T has a unique fixed point.

The Banach contraction theorem and its several extensions have been generalized using recently developed notion of weakly contractive maps (see [4, 10, 2, 11, 3, 12, 13]). The following basic result is due to Rhoades [15]

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{6}$$

where $\varphi\colon [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

If one takes $\varphi(t) = kt$ where $0 < k < 1$, then (6) reduces to (1). Introducing a new generalization of Banach contraction principle Dutta and Choudhary [9] proved the following generalization of Theorem 1.3.

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{7}$$

where

(*i*) *$\psi\colon [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,*
(*ii*) *$\varphi\colon [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.*

Then T has a unique fixed point.

Dorić [7] obtained the following generalization of Theorem 1.1 and Theorem 1.4

Theorem 1.5. *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$\psi(d(Tx, Ty)) \leq \psi(M_g(Tx, Ty)) - \varphi(M_g(Tx, Ty)), \quad \forall x, y \in X \tag{8}$$

where $M_g(Tx, Ty)$ is defined by (3), ψ and φ are defined as in Theorem 1.4. Then T has a unique fixed point.

Theorem 1.1, Theorem 1.3, Theorem 1.4 and Theorem 1.5 have been generalized by Singh et al. [16] in the following manner.

Theorem 1.6. *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \psi(d(Tx, Ty)) \leq \psi(M_g(Tx, Ty)) - \varphi(M_g(Tx, Ty)), \tag{9}$$

for all $x, y \in X$, where $M_g(Tx, Ty)$ is defined by (3), ψ and φ are defined as in Theorem 1.4. Then T has a unique fixed point.

2. MAIN RESULTS

In this section, we established a fixed point theorem which generalized Theorem 1.4, Theorem 1.5 and Theorem 1.6. Also, we give an illustrative example.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ satisfy*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M_T(x,y)) - \varphi(M_T(x,y)), \quad \forall x,y \in X \tag{10}$$

where

$$M_T(x,y) = \max\left\{ \begin{array}{c} d(x,y), d(T^2x,Tx), d(T^2x,y), \\ \frac{d(x,Ty)+d(Tx,y)}{2}, \frac{d(T^2x,Tx)+d(T^2x,Ty)}{2}, \\ d(T^2x,Ty)+d(Tx,x), d(Tx,y)+d(y,Ty) \end{array} \right\}, \tag{11}$$

and

(*i*) $\psi\colon [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(*ii*) $\varphi\colon [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

Then T has a unique fixed point.

Proof. Take $x \in X$. We construct a sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$x_n = Tx_{n-1}, \ \ \forall n \in \mathbb{N}, \tag{12}$$

where $x_0 = x$. If there exists $m \in \mathbb{N}$ such that $d(x_m, Tx_m) = 0$ then $x_* = x_m$ becomes a fixed point of T, which completes the proof. Consequently, in the rest of the proof, we assume that

$$0 < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \tag{13}$$

Hence, we have

$$\frac{1}{2}d(x_n,Tx_n) < d(x_n,Tx_n) = d(x_n,x_{n+1}), \quad \forall n \in \mathbb{N}. \tag{14}$$

Therefore by (10), we have

$$\begin{aligned} &\psi(d(Tx_n,Tx_{n+1})) \\ &\quad \leq \psi(M_T(x_n,x_{n+1})) - \varphi(M_T(x_n,x_{n+1})) \\ &\quad = \psi\left(\max\left\{ \begin{array}{c} d(x_n,x_{n+1}), d(x_{n+2},x_{n+1}), d(x_{n+2},x_{n+1}), \\ \frac{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}{2}, \frac{d(x_{n+2},x_{n+1})+d(x_{n+2},x_{n+2})}{2}, \\ d(x_{n+2},x_{n+2})+d(x_{n+1},x_n), d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2}) \end{array} \right\} \right) \\ &\qquad - \varphi\left(\max\left\{ \begin{array}{c} d(x_n,x_{n+1}), d(x_{n+2},x_{n+1}), d(x_{n+2},x_{n+1}), \\ \frac{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}{2}, \frac{d(x_{n+2},x_{n+1})+d(x_{n+2},x_{n+2})}{2}, \\ d(x_{n+2},x_{n+2})+d(x_{n+1},x_n), d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2}) \end{array} \right\} \right). \end{aligned}$$

So

$$\begin{aligned} \psi(d(x_{n+1},x_{n+2})) \leq \psi\left(\max\left\{ d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}) \right\} \right) \\ - \varphi\left(\max\left\{ d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}) \right\} \right). \end{aligned} \tag{15}$$

If $d(x_{n+1},x_{n+2}) > d(x_n,x_{n+1})$, then

$$\max\left\{ d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}) \right\} = d(x_{n+1},x_{n+2}),$$

so (15) becomes

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_{n+1}, x_{n+2})) - \varphi(d(x_{n+1}, x_{n+2})),$$

which is a contradiction (from (13) and property of φ, we have $\varphi(d(x_{n+1}, x_{n+2})) > 0$). Thus, we conclude that

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})). \tag{16}$$

Consequently,

$$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1})),$$

and by the property of ψ, we can get

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \quad \forall n \in \mathbb{N}.$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is monotonic nonincreasing and bounded below. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = \lim_{n\to\infty} d(x_{n+1}, x_{n+2}). \tag{17}$$

Therefore, by the lower semi-continuity of φ, we have

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(d(x_n, x_{n+1})). \tag{18}$$

We claim that $r = 0$. In fact taking upper limits as $n \to \infty$ to each side of the (16) and using (17) and (18), we get

$$\psi(r) \leq \psi(r) - \varphi(r).$$

Consequently $\varphi(r) \leq 0$. Hence by the property of the function φ, $\varphi(r) = 0$. But $\varphi(r) = 0$ implies $r = 0$. So, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{19}$$

Also from (19) and the inequality

$$0 \leq d(x_n, x_{n+m}) \leq \Sigma_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}),$$

for all $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{20}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \;\; d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \;\; d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{21}$$

Observe that

$$\begin{aligned} \epsilon \leq d(x_{p(n)}, x_{q(n)}) &\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\leq d(x_{p(n)-1}, Tx_{p(n)-1}) + \epsilon. \end{aligned}$$

It follows from (19) and the above inequality that

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{22}$$

Also from (19), (22) and the inequality

$$\begin{aligned} d(x_{p(n)}, x_{q(n)}) &\leq d(x_{p(n)}, x_{p(n)+m}) + d(x_{p(n)+m}, x_{q(n)}) \\ &\leq 2d(x_{p(n)}, x_{p(n)+m}) + d(x_{p(n)}, x_{q(n)}) \\ &\leq 2\Sigma_{i=0}^{m-1} d(x_{p(n)+i}, x_{p(n)+i+1}) + d(x_{p(n)}, x_{q(n)}), \end{aligned}$$

for all $m \in \mathbb{N}$, we can get

$$\lim_{n\to\infty}[d(x_{p(n)}, x_{p(n)+m}) + d(x_{p(n)+m}, x_{q(n)})] = \epsilon. \tag{23}$$

From (20) and the inequality

$$0 \le d(x_{p(n)}, x_{p(n)+m}) \le \Sigma_{i=0}^{m-1} d(x_{p(n)+i}, x_{p(n)+i+1}),$$

we have

$$\lim_{n\to\infty} d(x_{p(n)}, x_{p(n)+m}) = 0, \quad \forall m \in \mathbb{N}. \tag{24}$$

So, from (23) and (24), we get

$$\lim_{n\to\infty} d(x_{p(n)+m}, x_{q(n)}) = \epsilon, \quad \forall m \in \mathbb{N}. \tag{25}$$

Then from (19) and (22), we can choose a positive integer $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_{p(n)}, Tx_{p(n)}) < \frac{1}{2}\epsilon < d(x_{p(n)}, x_{q(n)}), \quad \forall n \ge n_1.$$

Therefore by (10), we obtain

$$\begin{aligned}
&\psi(d(Tx_{p(n)}, Tx_{q(n)})) \\
&\quad \le \psi(M_T(x_{p(n)}, x_{q(n)})) - \varphi(M_T(x_{p(n)}, x_{q(n)})) \\
&\quad = \psi\left(\max\left\{ \begin{array}{c} d(x_{p(n)}, x_{q(n)}), d(x_{p(n)+2}, x_{p(n)+1}), d(x_{p(n)+2}, x_{q(n)}), \\ \frac{d(x_{p(n)}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)})}{2}, \\ \frac{d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+2}, x_{q(n)+1})}{2}, \\ d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{p(n)}), \\ d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) \end{array} \right\} \right) \\
&\quad\quad - \varphi\left(\max\left\{ \begin{array}{c} d(x_{p(n)}, x_{q(n)}), d(x_{p(n)+2}, x_{p(n)+1}), d(x_{p(n)+2}, x_{q(n)}), \\ \frac{d(x_{p(n)}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)})}{2}, \\ \frac{d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+2}, x_{q(n)+1})}{2}, \\ d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{p(n)}), \\ d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) \end{array} \right\} \right).
\end{aligned}$$

Taking limits as $n \to \infty$ on each side of the above inequality and using (22) and (25), we have $\psi(\epsilon) \le \psi(\epsilon) - \varphi(\epsilon)$, which is a contradiction with $\epsilon > 0$, so it follows that $\{x_n\}$ is a Cauchy sequence in X. By completenes of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore,

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{26}$$

Now, we claim that,

$$(I)\ \frac{1}{2}d(x_n, Tx_n) < d(x_n, x^*) \ or\ (II)\ \frac{1}{2}d(Tx_n, T^2x_n) < d(Tx_n, x^*),\ \forall n \in \mathbb{N}. \tag{27}$$

Again, assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Tx_m) \ge d(x_m, x^*) \ and\ \frac{1}{2}d(Tx_m, T^2x_m) \ge d(Tx_m, x^*). \tag{28}$$

Therefore,

$$2d(x_m, x^*) \le d(x_m, Tx_m) \le d(x_m, x^*) + d(x^*, Tx_m),$$

which implies that

$$d(x_m, x^*) \le d(x^*, Tx_m), \tag{29}$$

Since $\frac{1}{2}d(x_m, Tx_m) < d(x_m, Tx_m)$, by the assumption of theorem, we get

$$\begin{aligned}
&\psi(d(Tx_m, T^2x_m)))\\
&\quad \leq \psi(M_T(x_m, Tx_m)) - \varphi(M_T((x_m, Tx_m))\\
&\quad = \psi\left(\max\left\{\begin{array}{c} d(x_m, Tx_m), d(T^2x_m, Tx_m),\\ d(T^2x_m, Tx_m), \frac{d(x_m,T^2x_m)+d(Tx_m,Tx_m)}{2},\\ \frac{d(T^2x_m,Tx_m)+d(T^2x_m,T^2x_m)}{2},\\ d(T^2x_m, T^2x_m) + d(Tx_m, x_m),\\ d(Tx_m, Tx_m) + d(Tx_m, T^2x_m)\end{array}\right\}\right)\\
&\quad\quad - \varphi\left(\max\left\{\begin{array}{c} d(x_m, Tx_m), d(T^2x_m, Tx_m),\\ d(T^2x_m, Tx_m), \frac{d(x_m,T^2x_m)+d(Tx_m,Tx_m)}{2},\\ \frac{d(T^2x_m,Tx_m)+d(T^2x_m,T^2x_m)}{2},\\ d(T^2x_m, T^2x_m) + d(Tx_m, x_m),\\ d(Tx_m, Tx_m) + d(Tx_m, T^2x_m)\end{array}\right\}\right)\\
&\quad = \psi\left(\max\left\{d(x_m, Tx_m), d(T^2x_m, Tx_m)\right\}\right)\\
&\quad\quad - \varphi\left(\max\left\{d(x_m, Tx_m), d(T^2x_m, Tx_m)\right\}\right). \qquad (30)
\end{aligned}$$

If $d(T^2x_m, Tx_m) > d(x_m, Tx_m)$, then (30) becomes

$$\psi(d(Tx_m, T^2x_m)) \leq \psi(d(Tx_m, T^2x_m)) - \varphi(d(Tx_m, T^2x_m)),$$

which is a contradiction (from (13) and property of φ, we have $\varphi(d(T^2x_m, Tx_m)) > 0$). Thus, we conclude that

$$\psi(d(Tx_m, T^2x_m)) \leq \psi(d(x_m, Tx_m)) - \varphi(d(x_m, Tx_m)). \qquad (31)$$

Consequently,

$$\psi(d(Tx_m, T^2x_m)) < \psi(d(x_m, Tx_m)),$$

and by the property of ψ, we have

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m). \qquad (32)$$

It follows from (28), (29) and (32) that

$$\begin{aligned}
d(Tx_m, T^2x_m) < d(x_m, Tx_m) &\leq d(x_m, x^*) + d(x^*, Tx_m)\\
&\leq 2d(x^*, Tx_m) \leq d(Tx_m, T^2x_m).
\end{aligned}$$

This is a contradiction. Hence, (27) holds. Suppose part (I) of (27) is true, then from assumption of theorem, we have

$$\begin{aligned}
&\psi(d(x_{n+1}, Tx^*))\\
&\quad \leq \psi(M_T(x_n, x^*)) - \varphi(M_T(x_n, x^*))\\
&\quad = \psi\left(\max\left\{\begin{array}{c} d(x_n, x^*), d(Tx_{n+1}, x_{n+1}), d(x_{n+2}, x^*), \frac{d(x_n,Tx^*)+d(x_{n+1},x^*)}{2},\\ \frac{d(Tx_{n+1},x_{n+1})+d(x_{n+2},Tx^*)}{2},\\ d(x_{n+2}, Tx^*) + d(Tx_n, x_n), d(x_{n+1}, x^*) + d(x^*, Tx^*)\end{array}\right\}\right)\\
&\quad\quad - \varphi\left(\max\left\{\begin{array}{c} d(x_n, x^*), d(Tx_{n+1}, x_{n+1}), d(x_{n+2}, x^*), \frac{d(x_n,Tx^*)+d(x_{n+1},x^*)}{2},\\ \frac{d(Tx_{n+1},x_{n+1})+d(x_{n+2},Tx^*)}{2},\\ d(x_{n+2}, Tx^*) + d(Tx_n, x_n), d(x_{n+1}, x^*) + d(x^*, Tx^*)\end{array}\right\}\right).
\end{aligned}$$

Taking $n \to \infty$ and using (26), we get

$$\psi(d(x^*, Tx^*)) \leq \psi(d(x^*, Tx^*)) - \varphi(d(x^*, Tx^*)).$$

This yields $x^* = Tx^*$. Now suppose part (II) of (27) is true, then from assumption of theorem, we have

$$\begin{aligned}
&\psi(d(x_{n+2}, Tx^*)) \\
&\quad = \psi(d(Tx_{n+1}, Tx^*)) \\
&\quad \leq \psi(M_T(x_{n+1}, x^*)) - \varphi(M_T(x_{n+1}, x^*)) \\
&\quad = \psi\left(\max\left\{ \begin{array}{c} d(x_{n+1}, x^*), d(T^2x_{n+1}, Tx_{n+1}), d(T^2x_{n+1}, x^*), \frac{d(x_{n+1},Tx^*)+d(Tx_{n+1},x^*)}{2}, \\ \frac{d(T^2x_{n+1},Tx_{n+1})+d(T^2x_{n+1},Tx^*)}{2}, \\ d(T^2x_{n+1}, Tx^*) + d(Tx_{n+1}, x_{n+1}), d(Tx_{n+1}, x^*) + d(x^*, Tx^*) \end{array} \right\} \right) \\
&\qquad - \varphi\left(\max\left\{ \begin{array}{c} d(x_{n+1}, x^*), d(T^2x_{n+1}, Tx_{n+1}), d(T^2x_{n+1}, x^*), \frac{d(x_{n+1},Tx^*)+d(Tx_{n+1},x^*)}{2}, \\ \frac{d(T^2x_{n+1},Tx_{n+1})+d(T^2x_{n+1},Tx^*)}{2}, \\ d(T^2x_{n+1}, Tx^*) + d(Tx_{n+1}, x_{n+1}), d(Tx_{n+1}, x^*) + d(x^*, Tx^*) \end{array} \right\} \right) \\
&\quad = \psi\left(\max\left\{ \begin{array}{c} d(x_{n+1}, x^*), d(Tx_{n+2}, x_{n+2}), d(x_{n+3}, x^*), \frac{d(x_{n+1},Tx^*)+d(x_{n+2},x^*)}{2}, \\ \frac{d(Tx_{n+2},x_{n+2})+d(x_{n+3},Tx^*)}{2}, \\ d(x_{n+3}, Tx^*) + d(Tx_{n+1}, x_{n+1}), d(x_{n+2}, x^*) + d(x^*, Tx^*) \end{array} \right\} \right) \\
&\qquad - \varphi\left(\max\left\{ \begin{array}{c} d(x_{n+1}, x^*), d(Tx_{n+2}, x_{n+2}), d(x_{n+3}, x^*), \frac{d(x_{n+1},Tx^*)+d(x_{n+2},x^*)}{2}, \\ \frac{d(Tx_{n+2},x_{n+2})+d(x_{n+3},Tx^*)}{2}, \\ d(x_{n+3}, Tx^*) + d(Tx_{n+1}, x_{n+1}), d(x_{n+2}, x^*) + d(x^*, Tx^*) \end{array} \right\} \right).
\end{aligned}$$

Taking $n \to \infty$ and using (26), we get

$$\psi(d(x^*, Tx^*)) \leq \psi(d(x^*, Tx^*)) - \varphi(d(x^*, Tx^*)).$$

This yields $x^* = Tx^*$. Hence, x^* is a fixed point of T. Now let us to show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ be two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$, then $d(x^*, y^*) > 0$. So, we have $0 = \frac{1}{2}d(x^*, Tx^*) < d(x^*, y^*)$ and from the assumption of theorem, we obtain

$$\begin{aligned}
&\psi(d(y^*, x^*)) \\
&\quad \leq \psi(M_T(y^*, x^*)) - \varphi(M_T(y^*, x^*)) \\
&\quad = \psi\left(\max\left\{ \begin{array}{c} d(y^*, x^*), d(T^2y^*, Ty^*), d(T^2y^*, x^*), \frac{d(y^*,Tx^*)+d(Ty^*,x^*)}{2}, \\ \frac{d(T^2y^*,Ty^*)+d(T^2y^*,Tx^*)}{2}, \\ d(T^2y^*, Tx^*) + d(Ty^*, y^*), d(Ty^*, x^*) + d(x^*, Tx^*) \end{array} \right\} \right) \\
&\qquad - \varphi\left(\max\left\{ \begin{array}{c} d(y^*, x^*), d(T^2y^*, Ty^*), d(T^2y^*, x^*), \frac{d(y^*,Tx^*)+d(Ty^*,x^*)}{2}, \\ \frac{d(T^2y^*,Ty^*)+d(T^2y^*,Tx^*)}{2}, \\ d(T^2y^*, Tx^*) + d(Ty^*, y^*), d(Ty^*, x^*) + d(x^*, Tx^*) \end{array} \right\} \right) \\
&\quad = \psi(d(y^*, x^*)) - \varphi(d(y^*, x^*)).
\end{aligned}$$

This gives $\varphi(d(y^*, x^*)) \leq 0$. Hence $y^* = x^*$. This completes the proof. □

The following two theorems can be obtained easily by repeating the steps in the proof of Theorem 2.1.

Theorem 2.2. *Let (X,d) be a complete metric space and $T : X \to X$ satisfy*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M_T(x,y)) - \varphi(M_g(Tx,Ty)), \quad \forall x,y \in X,$$

where $M_T(x,y)$ is defined by (11), $M_g(Tx,Ty)$ is defined by (3), ψ and φ are defined as in Theorem 2.1. Then T has a unique fixed point.

Theorem 2.3. *Let (X,d) be a complete metric space and $T : X \to X$ satisfy*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M_T(x,y)) - \varphi(d(x,y)), \quad \forall x,y \in X,$$

where $M_T(x,y)$ is defined by (11), ψ and φ are defined as in Theorem 2.1. Then T has a unique fixed point.

Remark 2.1. *Since $d(x,y) \leq M_g(Tx,Ty) \leq M_T(x,y)$ and ψ is nondecreasing function, the result of Singh et al.* [16, Theorem 2.1] *and Dorić* [7, Theorem 2.1] *are obtained from Theorem 2.2. Also the result of Rhoades* [15, Theorem 1] *and Dutta and Choudhary* [9, Theorem 2.1] *are obtained from Theorem 2.3.*

Example 2.1 shows the generality of Theorem 2.1 over Theorem 1.2, Theorem1.3, Theorem 1.4, Theorem 1.5 and Theorem 1.6. Further, it is interesting to note that the map T of Example 2.1 does not satisfy the hypotheses of Theorem 1.2, Theorem1.3, Theorem 1.4, Theorem 1.5 and Theorem 1.6.

Example 2.1. *Let $X = \{-2,-1,0,1,2\}$ and define a metric d on X by*

$$d(x,y) = \begin{cases} 0, & if\ x = y, \\ 2, & if\ (x,y) \in \{(1,-1),(-1,1)\}, \\ 1, & otherwise. \end{cases}$$

Then, (X,d) is a complete metric space. Let $T\colon X \to X$ be defined by

$$T(-2) = T(0) = T(2) = 2, \quad T(-1) = 0, \ \ T(1) = -2.$$

First observe that

$$d(Tx,Ty) > 0 \Leftrightarrow [(x \in \{-2,0,2\} \wedge y = 1) \vee (x \in \{-2,0,2\} \wedge y = -1) \vee (x = 1 \wedge y = -1)],$$

and

$$d(Tx,Ty) = 0 \Leftrightarrow [(x = -2 \wedge y = 0) \vee (x = -2 \wedge y = 2) \vee (x = 0 \wedge y = 2)],$$

Now we consider the following cases:
***Case1**. Let $x \in \{-2,0,2\} \wedge y = 1$, then*

$$d(Tx,Ty) = d(2,-2) = 1, d(x,y) = d(x,1) = 1, d(x,Tx) = d(x,2) = 0 \vee 1,$$
$$d(y,Ty) = d(1,-2) = 1, \frac{d(x,Ty)+d(Tx,y)}{2} = \frac{d(x,-2)+d(2,1)}{2} = \frac{1}{2} \vee 1,$$
$$\frac{d(T^2x,x)+d(T^2x,Ty)}{2} = \frac{d(2,x)+d(2,-2)}{2} = \frac{1}{2} \vee 1, d(x,Ty) = d(x,-2) = 0 \vee 1,$$
$$d(Tx,y) = d(2,-2) = 1, d(T^2x,Tx) = d(2,2) = 0, d(T^2x,y) = d(2,1) = 1,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,-2) + d(x,2) = 1 \vee 2,$$
$$d(Tx,y) + d(y,Ty) = d(2,1) + d(1,-2) = 2.$$

Case2. *Let* $x \in \{-2, 0, 2\} \wedge y = -1$, *then*

$$d(Tx,Ty) = d(2,0) = 1, d(x,y) = d(x,-1) = 1, d(x,Tx) = d(x,2) = 0 \vee 1,$$
$$d(y,Ty) = d(-1,0) = 1, \frac{d(x,Ty) + d(Tx,y)}{2} = \frac{d(x,0) + d(2,-1)}{2} = \frac{1}{2} \vee 1,$$
$$\frac{d(T^2x,x) + d(T^2x,Ty)}{2} = \frac{d(2,x) + d(2,0)}{2} = \frac{1}{2} \vee 1, d(x,Ty) = d(x,0) = 0 \vee 1,$$
$$d(Tx,y) = d(2,-1) = 1, d(T^2x,Tx) = d(2,2) = 0, d(T^2x,y) = d(2,-1) = 1,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,0) + d(x,2) = 1 \vee 2$$
$$d(Tx,y) + d(y,Ty) = d(2,-1) + d(-1,0) = 2.$$

Case3. *Let* $x = 1 \wedge y = -1$, *then*

$$d(Tx,Ty) = d(-2,0) = 1, d(x,y) = d(1,-1) = 2, d(x,Tx) = d(1,-2) = 1,$$
$$d(y,Ty) = d(-1,0) = 1, \frac{d(x,Ty) + d(Tx,y)}{2} = \frac{d(1,0) + d(-2,-1)}{2} = 1,$$
$$\frac{d(T^2x,x) + d(T^2x,Ty)}{2} = \frac{d(2,1) + d(2,0)}{2} = 1, d(x,Ty) = d(1,0) = 1,$$
$$d(Tx,y) = d(-2,-1) = 1, d(T^2x,Tx) = d(2,-2) = 1, d(T^2x,y) = d(2,-1) = 1,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,0) + d(1,-2) = 1,$$
$$d(Tx,y) + d(y,Ty) = d(-2,-1) + d(-1,0) = 2.$$

Case4. *Let* $x = -2 \wedge y = 0$, *then*

$$d(Tx,Ty) = d(2,2) = 0, d(x,y) = d(-2,0) = 1, d(x,Tx) = d(-2,2) = 1,$$
$$d(y,Ty) = d(0,2) = 1, \frac{d(x,Ty) + d(Tx,y)}{2} = \frac{d(2,2) + d(2,0)}{2} = 1,$$
$$\frac{d(T^2x,x) + d(T^2x,Ty)}{2} = \frac{d(2,-2) + d(2,2)}{2} = \frac{1}{2}, d(x,Ty) = d(-2,2) = 1,$$
$$d(Tx,y) = d(2,2) = 0, d(T^2x,Tx) = d(2,2) = 0, d(T^2x,y) = d(2,0) = 1,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,2) + d(-2,2) = 1,$$
$$d(Tx,y) + d(y,Ty) = d(2,0) + d(0,2) = 2.$$

Case5. *Let* $x = -2 \wedge y = 2$, *then*

$$d(Tx,Ty) = d(2,2) = 0, d(x,y) = d(-2,2) = 1, d(x,Tx) = d(-2,2) = 1,$$
$$d(y,Ty) = d(2,2) = 0, \frac{d(x,Ty) + d(Tx,y)}{2} = \frac{d(2,2) + d(2,2)}{2} = 0,$$
$$\frac{d(T^2x,x) + d(T^2x,Ty)}{2} = \frac{d(2,-2) + d(2,2)}{2} = \frac{1}{2}, d(x,Ty) = d(-2,2) = 1,$$
$$d(Tx,y) = d(2,2) = 0, d(T^2x,Tx) = d(2,2) = 0, d(T^2x,y) = d(2,2) = 0,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,2) + d(-2,2) = 1,$$
$$d(Tx,y) + d(y,Ty) = d(2,2) + d(2,2) = 0.$$

Case6. *Let* $x = 0 \wedge y = 2$, *then*

$$d(Tx,Ty) = d(2,2) = 0, d(x,y) = d(0,2) = 1, d(x,Tx) = d(0,2) = 1,$$
$$d(y,Ty) = d(2,2) = 0, \frac{d(x,Ty) + d(Tx,y)}{2} = \frac{d(0,2) + d(2,2)}{2} = \frac{1}{2},$$
$$\frac{d(T^2x,x) + d(T^2x,Ty)}{2} = \frac{d(2,0) + d(2,2)}{2} = \frac{1}{2}, d(x,Ty) = d(0,2) = 1,$$
$$d(Tx,y) = d(2,2) = 0, d(T^2x,Tx) = d(2,2) = 0, d(T^2x,y) = d(2,2) = 0,$$
$$d(T^2x,Ty) + d(x,Tx) = d(2,2) + d(0,2) = 1,$$
$$d(Tx,y) + d(y,Ty) = d(2,2) + d(2,2) = 0.$$

In Case1 and Case2, we have

$$d(Tx,Ty) = d(x,y) = M(Tx,Ty) = M_g(Tx,Ty) = 1.$$

This proves that for all function ψ *and* φ, T *does not satisfy the hypotheses of Theorem 1.6, Theorem 1.5, Theorem 1.4, Theorem 1.3 and Theorem 1.2. However, we see that for all* $x, y \in X$

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ and } M_T(x,y) = 2.$$

We set $\psi(t) = \frac{3}{4}t$ *and* $\varphi(t) = \frac{1}{4}t$, *then we have*

$$\psi(d(Tx,T2)) \leq \psi(M_T(x,y)) - \varphi(M_T(x,y)).$$

Hence , T satisfies in assumption of Theorem 2.1.

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