TWMS J. App. Eng. Math. V.7, N.2, 2017, pp. 332-336

SOME RESULTS ON TOTAL CHROMATIC NUMBER OF A GRAPH

S.K. VAIDYA[1], RAKHIMOL V.ISAAC[2], §

Abstract. A total coloring of a graph is a proper coloring in which no two adjacent or incident graph elements receive the same color. The total chromatic number of a graph is the smallest positive integer for which the graph admits a total coloring. In this paper, we derive some results on total chromatic number of a graph.

Keywords: total coloring, total chromatic number, splitting graph.

AMS Subject Classification: 05C15; 05C76.

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The elements of $V(G)$ and $E(G)$ are commonly called the graph elements. A coloring of a graph G is to assign colors (numbers) to the vertices or edges or both. A vertex coloring (edge coloring) is called proper if no two vertices (edges) receive the same color. Many variants of proper colorings are available in the literature such as a- coloring, b- coloring, list coloring, total coloring etc. The present work is focused on total coloring of graphs.

A function $\pi : V(G) \cup E(G) \rightarrow \mathbb{N}$ is called a *total coloring* if no two adjacent or incident graph elements are assigned the same color. The total chromatic number of G, denoted by $\chi_T(G)$, is the smallest positive integer k for which there exists a total coloring $\pi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$.

The concept of total coloring was introduced independently by Behzad [1] and Vizing [10] and they have also posed the following conjecture termed as Total Coloring Conjecture (TCC)

Conjecture 1.1. $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$ *where* $\Delta(G)$ *denotes the maximum degree of* G.

This conjecture was confirmed for $\Delta(G) = 3$ by Rosenfeld [5] and Vijayaditya [9] and for $\Delta(G) \leq 3$ by Kostochka [4]. The total chromatic number for complete graph K_n is determined by Behzad *et al* [2] while Yap [11] have determined the total chromatic number for cycle C_n. Vaidya and Rakhimol [8] have verified TCC for some cycle related graphs.

[1] Saurashtra University, Rajkot - 360005, Gujarat, India.
e-mail: samirkvaidya@yahoo.co.in; ORCID: http://orcid.org/ 0000-0002-5394-1424

[2] Shree M. and N. Virani Science College, Rajkot - 360005, Gujarat, India.
e-mail: rakhiisaac@yahoo.co.in; ORCID: http://orcid.org/0000-0001-8520-9863

§ Manuscript received: August 27, 2015; accepted: August 24, 2017.
TWMS Journal of Applied and Engineering Mathematics Vol.7, No.2; © Işık University, Department of Mathematics, 2017; all rights reserved.

Definition 1.1. [7] *Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(k)}$. Then $G =< W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} >$ is the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.*

Definition 1.2. [6] *Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(k)}$. Then $G=< W_n^{(1)} \blacktriangle W_n^{(2)} \blacktriangle \ldots \blacktriangle W_n^{(k)} >$ is the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ by an edge as well as to a new vertex x_{p-1} where $2 \leq p \leq k$.*

2. Some General Results

Theorem 2.1. *Let G be a graph with $\Delta(G) = k$ and if there are exactly two vertices with degree k which are adjacent and all other vertices are of degree less than or equal to $k-2$, then $\chi_T(G) = k + 1$.*

Proof. : Let G be the graph with $\Delta(G) = k$. Let v_1 and v_2 be the only two vertices in G such that $d(v_1) = d(v_2) = k$ and the remaining vertices have degree less than k.

If $e = v_1v_2$ then in the graph $G - \{e\}$, the vertex v_1 and its incident edges need only k colors. Similarly, the vertex v_2 and its incident edges also need only k colors for the total coloring. The vertices which are not adjacent to v_1 and v_2 and the edges which are not incident to v_1 and v_2 can be properly colored using any of these k colors as such vertices have degree less than k. Thus the total chromatic number of $G - \{e\}$ is k. Finally, to color the edge $e = v_1v_2$, we need a new color which is not assigned earlier. Hence $\chi_T(G) = k + 1$. □

Example 2.1. *We illustrate the Theorem 2.1 by means of following example.*

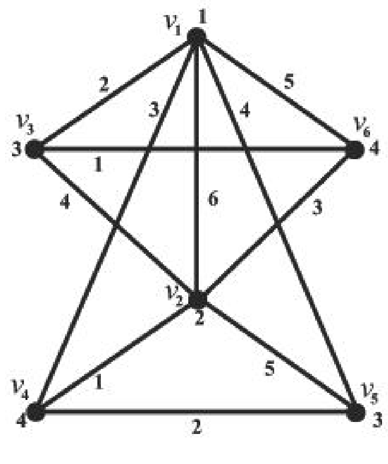

Figure 1. The graph illustrating Theorem 2.1.

In the graph of Figure 1, $d(v_1) = d(v_2) = 5 = \Delta(G)$ and the remaining vertices v_3, v_4, v_5 and v_6 have degree less than 3. Now assign the proper coloring as follows: $\pi(v_1) = 1$, $\pi(v_1v_3) = 2$, $\pi(v_1v_4) = 3$, $\pi(v_1v_5) = 4$, $\pi(v_1v_6) = 5$, $\pi(v_2) = 2$, $\pi(v_2v_4) = 1$, $\pi(v_2v_5) = 5$, $\pi(v_2v_3) = 4$, $\pi(v_2v_6) = 3$, $\pi(v_3) = 3$, $\pi(v_4) = 4$, $\pi(v_5) = 3$, $\pi(v_6) = 4$, $\pi(v_3v_6) = 1$, $\pi(v_3v_2) = 4$. For the remaining edges assign the colors as $\pi(v_3v_6) = 3$ and $\pi(v_4v_5) = 2$. Now we used five colors for the vertices v_1 and v_2 and their incident edges. Now for the edge $e = v_1v_2$, $\pi(e) = 6$ as the colors from 1 to 5 have been assigned already.

Corollary 2.1. $\chi_T(G) = \Delta(G) + 1$, *if G is $B_{n,n}$ or $< W_n^{(1)} : W_n^{(2)} >$ or $< W_n^{(1)} \blacktriangle W_n^{(2)} >$.*

Proof. : The proof is obvious from the Theorem 2.1. □

Theorem 2.2. *If $\chi_b(G) = \chi_T(G)$, then $\chi_b(G) = \Delta(G) + 1$.*

Proof. : It is given that $\chi_b(G) = \chi_T(G)$. As stated by Behzad [1], $\chi_T(G)$ is either $\Delta(G)+1$ or $\Delta(G)+2$. Hence $\chi_b(G)$ is either $\Delta(G)+1$ or $\Delta(G)+2$. But $\chi_b(G)$ can never takes the value $\Delta(G)+2$ as $\chi_b(G) \leq \Delta(G)+1$ as proved in [3]. Thus, $\chi_b(G) = \Delta(G)+1$. □

Remark 2.1. *The converse of above theorem is not true. To illustrate this we consider the graph $W_3 : C_3 + K_1$ as shown in Figure 2. Here, $\Delta(W_3) = 3$ and $\chi_b(W_3) = 4$ but $\chi_b(W_3) \neq \chi_T(W_3)$ as $\chi_T(W_3) = 5$.*

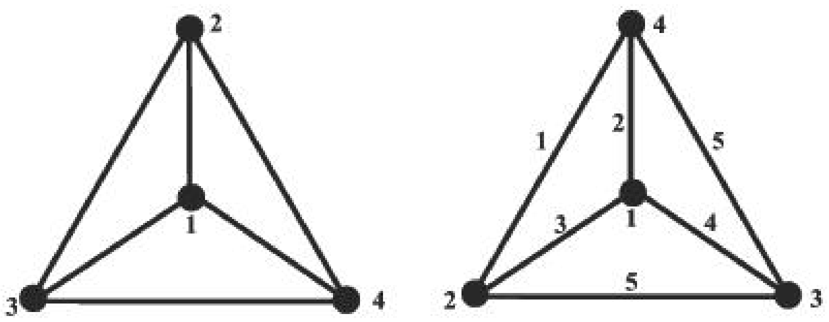

FIGURE 2. b-coloring of W_3 total coloring of W_3

Definition 2.1. *The splitting graph $S'(G)$ of a graph G is obtained by adding new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively.*

Theorem 2.3. $\chi_T(S'(G)) = 2.\Delta(G) + 1$.

Proof. : Let G be a graph with $\Delta(G) = k$. Let v be a vertex in G with $d(v) = k$. By the definition of splitting graph it is clear that $d(v) = k$; $v \in S'(G)$ and hence $\Delta(S'(G)) = 2k$. Therefore,

$$\chi_T(S'(G)) \geq \Delta(S'(G)) + 1 = 2k + 1. \tag{1}$$

Again, by the definition of splitting graph, $\cap N(v'_i) = \phi$, where v'_i are the newly added vertices. Also, v_i and v'_i are non adjacent in $S'(G)$. Then for the total coloring of $S'(G)$, by assigning the color 1 to the vertex v, all the remaining vertices will receive the colors $2, 3, \ldots, 2k+1$. Thus,

$$\chi_T(S'(G)) \leq 2k + 1. \tag{2}$$

From equations (1) and (2), we get $\chi_T(S'(G))$=$2k$+1=$2.\Delta(G)$+1. Hence the theorem. □

3. Total chromatic number of some wheel related graphs

Definition 3.1. *A subdivision of a graph G is a graph obtained from G by inserting vertices of degree 2 into the edges of G.*

Definition 3.2. *The gear graph, G_n, is obtained from wheel $W_n = C_n + K_1$ by subdividing each of its rim edges exactly once. Then obviously, $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.*

Theorem 3.1. *For the gear graph G_n, $\chi_T(G_n) = \Delta(G_n) + 1$.*

Proof. : We know $V(G_n)= \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ with apex vertex v. Then $\Delta(G_n) = d(v) = n$ and v is the vertex of maximum degree. Obviously, $\chi_T(G_n) \geq \Delta(G_n)+1$.
Now define the coloring $\pi : V(G_n) \cup E(G_n) \to \mathbb{N}$ such that,
$\pi(v) = \pi(u_n v_1) = \pi(u_k v_{k+1}) = 1$, $\pi(v_k) = k + 2$; $k = 1, 2, \ldots, n-1$,
$\pi(v_n) = 2$, $\pi(vv_i) = \pi(u_i) = i + 1$; $i = 1, 2, \ldots, n$,
$\pi(v_1u_1) = n + 1$, $\pi(v_j u_i) = j$; $j = 2, 3, \ldots, n$. This coloring gives the total coloring with $n + 1$ colors only. Thus $\chi_T(G_n) = \Delta(G_n) + 1$. □

Theorem 3.2. *If G be a graph with W_n as a sub graph and $d(v) = \Delta(G)$ where v is the apex vertex then $\chi_T(G) = \Delta(G) + 1$.*

Proof. : Let G be a graph which contains W_n as a sub graph. Consider the sub graph W_n with vertex set $\{v, v_1, v_2, \ldots, v_n\}$ where v is the apex vertex and $d(v) = n = \Delta(G)$. All other vertices which are adjacent to v in G have degree less than n. So we need at most $n + 1$ colors for the total coloring of G.
Now assign the color as $\pi(v) = 1$, $\pi(vv_k) = k+1$, $\pi(v_k) = k+2$, $\pi(v_n) = 2$, $\pi(v_k v_{k+1}) = 1$, $\pi(v_k v_1) = 1$, gives the total chromatic number $n + 1$. Thus $\chi_T(G) = n + 1 = \Delta(G) + 1$. □

Definition 3.3. *The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 3.4. *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

Definition 3.5. *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.*

Definition 3.6. *The web graph is the graph obtained by joining the pendant vertices of a Helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.*

The graph $W(t, n)$ is the generalized web graph with t number of n- cycles.

Corollary 3.1. $\chi_T(G) = \Delta(G) + 1$, *if G is H_n or CH_n or $W(t, n)$ or $< W_n^{(1)} : W_n^{(2)} >$ or $< W_n^{(1)} \blacktriangle W_n^{(2)} >$ then $\chi_T(G) = \Delta(G) + 1$.*

Proof. : The proof is obvious as the graphs Helm H_n, Closed Helm CH_n, Generalized Web graph $W(t, n)$, $< W_n^{(1)} : W_n^{(2)} >$ and $< W_n^{(1)} \blacktriangle W_n^{(2)} >$ contain W_n as a subgraph. Thus by Theorem 3.2 their total chromatic number is $\Delta + 1$. □

4. Conclusion

The total coloring is a variant of proper coloring. We derive several general results on this concept and investigate total chromatic number of some wheel related graphs.

Acknowledgement

The present work is a part of the research work done under the Minor Research Project No. GUJCOST/MRP/2015-16/2620 of Gujarat Council on Science and Technology(GUJCOST).

References

[1] Behzad,M., (1965), Graphs and their chromatic numbers, Ph.D Thesis, Michigan University.
[2] Behzad,M., Chartrand,G., and Cooper,Jr.J.K., (1967), The colour numbers of complete graphs, J. London Math. Soc., 42, pp.226-228.
[3] Irving,R.W. and Manlove,D.F., (1999), The *b*-chromatic number of a graph, Discrete Applied Mathematics, 91,(1-3), pp.127-141.
[4] Kostochka,A.V., (1996), The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Math, 162, pp.199-214.
[5] Rosenfeld,M., (1971), On the total coloring of certain graphs, Israel J. Math., 9(3), pp.396-402.
[6] Vaidya,S.K. and Barasara,C.M., (2011), Product Cordial Graphs in the Context of Some Graph Operations, International Journal of Mathematics and Scientific Computing, 1(3), pp.1-6.
[7] Vaidya,S.K., Dani,N.A., Kanani,K.K., and Vihol,P.L., (2011), Cordial and 3-equitable Labeling for Some Wheel Related Graphs, IAENG International Journal of Applied Mathematics , 41(2), pp.99-105.
[8] Vaidya,S.K. and Isaac,Rakhimol V., (2015), Total coloring of some cycle related graphs, IOSR Journal of Mathematics, 11(3)(V), pp.51-53.
[9] Vijayaditya,N., (1971), On total chromatic number of a graph, J. London Math Soc., 2(3), pp.405-408.
[10] Vizing,V.G., (1968), Some unsolved problems in graph theory, Uspekhi Mat. Nauk (in Russian), 23(6), pp.117-134 and in Russian Mathematical Surveys, 23(6), pp.125-141.
[11] Yap,H.P., (1996), Total colourings of graphs,in: Lecture Notes in Mathematics, 1623, pp.6.

S. K. Vaidya for the photography and short autobiography, see TWMS J. App. Eng. Math., V.4, N.1, 2014.

Rakhimol V. Isaac was born in 1980. She got her B. Sc. Degree and M. Sc. Degree in mathematics from Mahatma Gandhi University, Kottayam, Kerala in 2001 and 2003 respectively. She obtained her M. Phil. Degree in 2010 from Saurashtra University, Rajkot-360 005, Gujarat (India). At present, she is working as an assistant professor in mathematics in Shree M. and N. Virani Science College, Rajkot (Gujarat) and also pursuing her research under the supervision of S. K. Vaidya.