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NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH DEFORMED MATSUMOTO METRIC

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Abstract. The purpose of present paper is to find the nonholonomic frames for the deformed Matsumoto type metric which are given in the forms

$I.\ (\frac{\alpha^2}{\alpha-\beta})\alpha = \frac{\alpha^3}{\alpha-\beta}$

$II.\ (\frac{\alpha^2}{\alpha-\beta})\beta = \frac{\alpha^2\beta}{\alpha-\beta}$

where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. The first metric of the above deformation is obtained by the product of Matsumoto and Riemannian metric and second one is the product of Matsumoto and 1-form metric.

Keywords: Riemannian metric, one form metric, Matsumoto metric, GL-metric, nonholonomic Finsler frame.

AMS Subject Classification: 53C60

1. Introduction

P.R. Holland [1] [2] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field in the year 1982. In the year 1987, R.S. Ingarden [3] was the first person, to point out that the Lorentz force law can be written in above case as geodesic equation on a Finsler space called Randers space. Further in 1995, R.G. Beil [5] [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper, we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered the fundamental tensor field might be the deformation of two different special Finsler spaces from the (α, β)- metrics. First we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as first product of Matsumoto metric[11] and Riemannian metric and second is the product of Matsumoto metric[11] and 1-form metric. Further, we obtain a corresponding frame for each of these

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two Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [10], Tripathi [14, 16] and Narasimhamurthy [15].

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)-metrics [11].

Definition 2.1. *A Finsler space $F^n = \{M, F(x,y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\} \tag{1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

The first Finsler spaces with (α,β)–metrics were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case by considering, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a\,(x,y) > 0$ and $b\,(x,y) \geq 0$ Two functions on TM and $B\,(x,y) = B_i\,(x,y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y)$$

Now a days the above generalized Lagrange metric is known as the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = \exp(2\sigma(x,y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6]. Further Bucataru [12] considered the class of Lagrange spaces with (α,β)-metric and obtained some new and interesting results in the year 2002.

A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)-metric. Next we defined some different Finsler space with (α,β)-metrics.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of a and F we have the following formulae:

$$\begin{gathered} p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i}; \\ l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i \\ l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha}; \\ b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L} \end{gathered} \tag{2}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j \tag{3}$$

Theorem 2.1. [10]*: For a Finsler space (M,F) consider the metric with the entries:*

$$Y_j^i = \sqrt{\frac{\alpha}{L}}(\delta_j^i - l^i l_j + \sqrt{\frac{\alpha}{L}} p^i p_j) \tag{4}$$

defined on TM. Then $Y_j = Y_j^i(\frac{\partial}{\partial y^i})$, $\quad j \in 1,2,3,\ldots,n$ *is a non holonomic frame.*

Theorem 2.2. [7]*: With respect to frame the holonomic components of the Finsler metric tensor* $a_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}. \tag{5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L\{\alpha(x,y),\beta(x,y)\}$ we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}) \tag{6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.
For a Finsler space with (α, β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13].

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j \tag{8}$$

From (8) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$

$$II. \qquad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho_{-1}^2)b_i b_j \tag{9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (9) is according to the theorem (7.9.1) in [10], given by,

$$X_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) \tag{10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl} \tag{11}$$

Again the frame that corresponds to the II_{nd} deformation (9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^i b_j \tag{12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij} \tag{13}$$

Theorem 2.3. [10]*: Let $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (10) and (12) respectively.

3. Nonholonomic frames for Finsler space with deformed Matsumoto Metric

In this section we consider two cases of nonholonomic Finlser frames with special (α, β)-metrics, such a I^{st} Finsler frame product of Matusmoto metric and Riemannian metric and II^{nd} Finsler frame product of Matsumoto metric and 1-form metric.

3.1. **Nonholonomic frame for $L = (\frac{\alpha^2}{\alpha-\beta})\alpha = \frac{\alpha^3}{\alpha-\beta}$.** In the first case, for a Finsler space with the fundamental function $L = (\frac{\alpha^2}{\alpha-\beta})\alpha = \frac{\alpha^3}{\alpha-\beta}$ the Finsler invariants (6) are given by

$$\rho_1 = \frac{2\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^2}, \quad \rho_0 = \frac{\alpha^3}{(\alpha-\beta)^3}, \tag{14}$$
$$\rho_{-1} = \frac{\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^3}, \quad \rho_{-2} = \frac{3\beta^2 - \alpha\beta}{2\alpha(\alpha-\beta)^3},$$
$$B^2 = \frac{(\alpha - 3\beta)^2(\alpha^2 b^2 - \beta^2)}{4(\alpha-\beta)^6}$$

Using (14) in (10) we have,

$$X_j^i = \sqrt{\frac{2\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^2}}\delta_j^i - \frac{(\alpha^2 - 3\alpha\beta)^2}{4\beta^2(\alpha-\beta)^6}[\sqrt{\frac{2\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^2}} + \tag{15}$$
$$\sqrt{\frac{2\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^2} + \frac{2\alpha\beta(\alpha-\beta)^3}{(3\beta-\alpha)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j)$$

Again using (14) in (12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\beta(\alpha-\beta)^2 C^2}{\alpha^3}}\}b^i b_j \tag{16}$$

where $C^2 = \frac{2\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^2}b^2 + \frac{(3\beta-\alpha)}{2\alpha\beta(\alpha-\beta)^3}(\alpha^2 b^2 - \beta^2)^2$

Theorem 3.1. *Let $L = (\frac{\alpha^2}{\alpha-\beta})\alpha = \frac{\alpha^3}{\alpha-\beta}$ be the metric function of a Finsler space with (α, β) metric for which the condition (7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (15) and (16) respectively.

3.2. **Nonholonomic frame for $L = (\frac{\alpha^2}{\alpha-\beta})\beta = \frac{\alpha^2\beta}{\alpha-\beta}$.** In the second case, for a Finsler space with the fundamental function $L = (\frac{\alpha^2}{\alpha-\beta})\beta = \frac{\alpha^2\beta}{\alpha-\beta}$ the Finsler invariants (6) are given by

$$\rho_1 = \frac{\alpha\beta - 2\beta^2}{2(\alpha-\beta)^2}, \quad \rho_0 = \frac{\alpha^3}{(\alpha-\beta)^3}, \tag{17}$$
$$\rho_{-1} = \frac{\alpha^2 - 3\alpha\beta}{2(\alpha-\beta)^3}, \quad \rho_{-2} = \frac{3\beta^2 - \alpha\beta}{2\alpha(\alpha-\beta)^3},$$
$$B^2 = \frac{(\alpha-3\beta)^2(\alpha^2 b^2 - \beta^2)}{4(\alpha-\beta)^6}$$

Using (17) in (10) we have,

$$X^i_j = \sqrt{\frac{\alpha\beta - 2\beta^2}{2(\alpha-\beta)^2}}\delta^i_j - \frac{(\alpha^2 - 3\alpha\beta)^2}{4\beta^2(\alpha-\beta)^6}[\sqrt{\frac{\alpha\beta - 2\beta^2}{2(\alpha-\beta)^2}} + \tag{18}$$
$$\sqrt{\frac{\alpha\beta - 2\beta^2}{2(\alpha-\beta)^2} + \frac{2\alpha\beta(\alpha-\beta)^3}{(3\beta-\alpha)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j)$$

Again using (17) in (12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\beta(\alpha-\beta)^2 C^2}{\alpha^3}}\}b^i b_j \tag{19}$$

where $C^2 = \frac{\alpha\beta-2\beta^2}{2(\alpha-\beta)^2}b^2 + \frac{(3\beta-\alpha)}{2\alpha\beta(\alpha-\beta)^3}(\alpha^2 b^2 - \beta^2)^2$

Theorem 3.2. *Let $L = (\frac{\alpha^2}{\alpha-\beta})\beta = \frac{\alpha^2\beta}{\alpha-\beta}$ be the metric function of a Finsler space with (α,β) metric for which the condition (7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is nonholonomic Finsler Frame with X^i_k and Y^k_j are given by (18) and (19) respectively.

4. Conclusions

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru ([7][8]), has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α,β)-metric, it appears a natural question: Does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α, β)-metrics.

In this work, we consider the Matsumoto Finsler metrics, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many((α,β)-metrics, in future work we can determine the frames for them also.

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