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A NOTE ON LINE GRAPHS

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ABSTRACT. The line graph and 1-quasitotal graph are well-known concepts in graph theory. In Satyanarayana, Srinivasulu, and Syam Prasad [13], it is proved that if a graph G consists of exactly m connected components G_i $(1 \leq i \leq m)$ then $L(G) = L(G_1) = L(G_2) \oplus ... \oplus L(G_m)$ where $L(G)$ denotes the line graph of G, and $\oplus$ denotes the ring sum operation on graphs. In [13], the authors also introduced the concept 1-quasitotal graph and obtained that $Q_1(G) = G \oplus L(G)$ where $Q_1(G)$ denotes 1-quasitotal graph of a given graph G. In this note, we consider zero divisor graph of a finite associate ring R and we will prove that the line graph of K_{n-1} contains the complete graph on n vertices where n is the number of elements in the ring R.

Keywords: line graph, quasi-total graph, zero-divisor graph, associate ring, complete graph.

AMS Subject Classification: 05C25, 05C76, 05C99, 13E15, 68R10.

1. Introduction

Let $G = (V, E)$ be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge is identified as an unordered pair of vertices v_i, v_j. An edge associated with a vertex pair v_i, v_i is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and $\delta(v)$ denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loop or parallel edges is called a simple graph. We consider simple graphs only. For a commutative ring R, the notion of zero divisor graph is given in Beck [1]. In this paper, we consider the associative rings (need not be commutative) and we will provide some examples on the zero divisor graphs of $\mathbb{Z}_n$ where n is a positive integer.

A graph $G = (V, E)$ is said to be a star graph, if there exists a fixed vertex v such that $E = \{vu : u \in V, u \neq v\}$. A star graph is said to be an n-star graph, if the number of vertices of the graph is n. A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. The complete graph on n vertices is denoted by K_n. In a graph G, a subset S of $V(G)$ is said to be a dominating set, if every vertex not in S has a neighbor in S. The domination number denoted by $\gamma(G)$ is defined as min $\{|S| : S$

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is a dominating set in G }. A subgraph of $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that $V^1 \subseteq V, E^1 \subseteq E$ and each edge of G^1 has the same end vertices in G^1 as in G. For any graph G with vertex set $V(G)$ and edge set $E(G)$, the 1quasitotal graph, (denoted by $Q_1(G)$) of G consists of the vertex set of $Q_1(G)$, that is $V(Q_1(G)) = V(G) \cup E(G)$ and two vertices x, y in $V(Q_1(G))$ are adjacent if they satisfy conditions: (i) x, y are in $V(G)$ and $\overline{xy} \in G$ and (ii) x, y are in $E(G)$ and x, y are incident in G.

For other preliminary results and notations, we use Satyanarayana and Syam Prasad [9].

2. Zero divisor graph of a ring

Let R denotes a finite associative ring.

Definition 2.1. *A graph is said to be the zero divisor graph of R if and* $E = \{\overline{xy} : x, y \in R, x \neq 0 \neq y, xy = 0\} \cup \{\overline{x0} : 0 \neq x \in R\}$ *where* $\overline{ab}$ *denotes an edge between* $a, b \in V$.

This definition zero divisor graph is same as that of Beck [1] in case of commutative rings.

Notation 2.1. (i) *The zero divisor graph of ring R is denoted by* $ZDG(R)$,

(ii) *In the graph* $ZDG(R)$, *we have that* $V(ZDG(R)) = R$ *and* $E(ZDG(R)) = \{\overline{xy} : x, y \in R, x \neq 0 \neq y, xy = 0\} \cup \{\overline{x0} : 0 \neq x \in R\}$.

Example 2.1. *Consider* Z_n, *the ring of integers modulo n. Let us construct the* $ZDG(R)$ *where* $R = Z_{10} = \{0, 1, 2, ..., 9\}$. *So* $V(ZDG(R)) = \{0, 1, 2, ..., 9\}$. *Since* $5 \cdot 8 = 5 \cdot 4 = 5 \cdot 6 = 0$, *there exist edges between the vertices 5 and 8, 5 and 4 also 5 and 6. Since 0 is adjacent to all the elements in R, we get* $\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{09} \in E(ZDG(R))$. *Therefore* $E(ZDG(R)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{09}\}$. *Now* $ZDG(R)$ *is given in figure 1.3 (i). We observe that* $ZDG(Z_{10})$ *contains 10-star graph as its subgraph. The domination number is 1.*

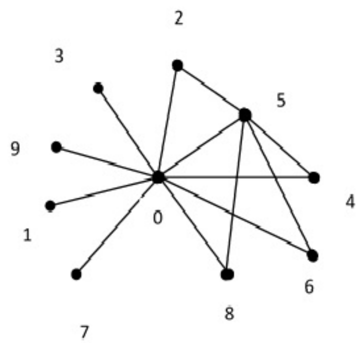

Figure-1.3(i)

Now we construct line graph of $ZDG(Z_{10})$.

$V(ZDG(Z_{10})) = \{e_1 = \overline{01}, e_2 = \overline{02}, e_3 = \overline{03}, e_4 = \overline{04}, e_5 = \overline{05}, e_6 = \overline{06}, e_7 = \overline{07}, e_8 = \overline{08}, e_9 = \overline{09}, e_{10} = \overline{25}, e_{11} = \overline{54}, e_{12} = \overline{56}, e_{13} = \overline{58}\}$.

and

*$E(ZDG(Z_{10})) = \{\overline{e_1e_2}, \overline{e_1e_3}, \overline{e_1e_4}, \overline{e_1e_5}, \overline{e_1e_6}, \overline{e_1e_7}, \overline{e_1e_8}, \overline{e_2e_3},$
$\overline{e_2e_4}, \overline{e_2e_5}, \overline{e_2e_6}, \overline{e_2e_7}, \overline{e_2e_8}, \overline{e_2e_9}, \overline{e_3e_4}, \overline{e_3e_5},$
$\overline{e_3e_6}, \overline{e_3e_7}, \overline{e_3e_8}, \overline{e_3e_9}, \overline{e_4e_5}, \overline{e_4e_6},$
$\overline{e_4e_7}, \overline{e_4e_8}, \overline{e_4e_9}, \overline{e_5e_6}, \overline{e_5e_7}, \overline{e_5e_8},$
$\overline{e_5e_9}, \overline{e_6e_7}, \overline{e_6e_8}, \overline{e_6e_9}, \overline{e_7e_8}, \overline{e_7e_9},$
$\overline{e_8e_9}, \overline{e_{10}e_2}, \overline{e_{10}e_5}, \overline{e_{10}e_{11}}, \overline{e_{10}e_{12}}, \overline{e_{10}e_{13}},$
$\overline{e_{11}e_5}, \overline{e_{11}e_{13}}, \overline{e_{11}e_{12}}, \overline{e_{11}e_4}, \overline{e_{12}e_6}, \overline{e_{12}e_{13}},$
$\overline{e_{12}e_5}, \overline{e_{13}e_8}, \overline{e_{13}e_5}\}$.*

The graph $E(ZDG(Z_{10}))$ is given in figure 1.3 (ii).

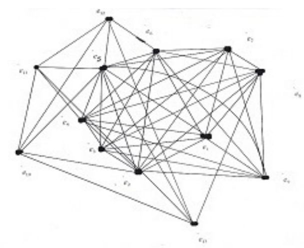

Figure-1.3(ii)

In Satyanarayana, Srinivasulu, and Syam Prasad [13], it is proved that $Q_1 = G \oplus L(G)$ where $Q_1(G)$ denotes 1-quasitotal graph of a given graph G.

The 1-quasitotel graph of $(ZDG(Z_{10}))$ is the ring sum of both the graphs given in the figure 1.3 (i) and 1.3 (ii).

Example 2.2. *Let us construct the $(ZDG(R))$, where $R = Z_8$. We know that $R = Z_8 = \{0, 1, 2, ..., 7\}$, so $V = (ZDG(R)) = \{0, 1, 2, ..., 7\}$. Since $2 \cdot 4 = 4 \cdot 6 = 0$, there exists an edge between the vertices 2 and 4, 4 and 6. Also, since 0 is adjacent to all the elements in R, we get*

$E(ZDG(R)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{24}, \overline{46}\}$.

Now $(ZDG(R))$ is given in figure 1.4(i). We observe that $(ZDG(Z_8))$ contains 8-star graph as its subgraph. The domination number is 1.

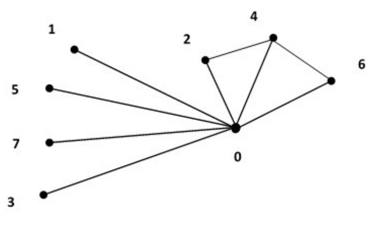

Figure-1.4(i)

Now we construct line graph of $(ZDG(Z_8))$.

$V(ZDG(Z_8)) = \{e_1 = \overline{01}, e_2 = \overline{02}, e_3 = \overline{03}, e_4 = \overline{04}, e_5 = \overline{05}, e_6 = \overline{06}, e_7 = \overline{07}, e_8 = \overline{24}, e_9 = \overline{46}\}$ and

$E(ZDG(Z_{10})) = \{\overline{e_1e_2}, \overline{e_1e_3}, \overline{e_1e_4}, \overline{e_1e_5}, \overline{e_1e_6}, \overline{e_1e_7}, \overline{e_2e_3}, \overline{e_2e_4}, \overline{e_2e_5}, \overline{e_2e_6}, \overline{e_2e_7}, \overline{e_3e_4}, \overline{e_3e_5}, \overline{e_3e_6}, \overline{e_3e_7}, \overline{e_4e_5}, \overline{e_4e_6}, \overline{e_4e_7}, \overline{e_4e_8}, \overline{e_4e_9}, \overline{e_5e_6}, \overline{e_5e_7}, \overline{e_2e_8}, \overline{e_8e_9}, \overline{e_4e_8}, \overline{e_4e_9}, \overline{e_6e_9}\}$.

The graph $L(ZDG(Z_8))$ is given by figure 1.4(ii).

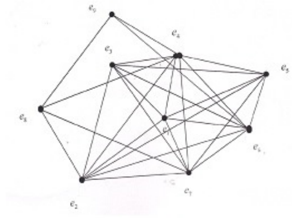

Figure-1.4(ii)

The 1-quasitotal graph of $(ZDG(Z_8))$ is the ring sum of both the graphs given in figures 1.4(i) and 1.4(ii).

Note 2.1. *(i) $(ZDG(Z_p))$ is a $p-$star graph for any prime number p. (ii) The domination number of $(ZDG(Z_n))$ is equal to 1 for any positive integer n.*

3. Line graphs

In this section, we construct line graphs for two zero divisor graphs and observe some properties. Finally, we prove that $L(ZDG(R))$ contains the complete graph $K_{(n-1)}$ as a subgraph where n denotes the number of elements in the ring R.

Example 3.1. *Take $R = Z_4$, the ring of integers modulo 4. We know that $R = Z_4 = \{0, 1, 2, 3\}$, so $V(ZDG(R)) = \{0, 1, 2, 3\}$. Since 0 is adjacent to all the elements in R, we get $\overline{01}, \overline{02}, \overline{03} \in E(ZDG(R))$. Now $ZDG(R))$ is given in figure 2.1.*

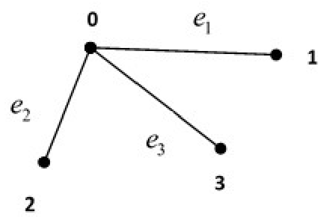

Figure-2.1

Now we construct line graph of $ZDG(Z_4)$. Now $V(ZDG(Z_4)) = \{e_1 = \overline{01}, e_2 = \overline{02}, e_3 = \overline{03}\}$ and $E(ZDG(Z_4)) = \{\overline{e_1e_2}, \overline{e_2e_3}, \overline{e_3e_1}\}$. The graph $L(ZDG(Z_4))$ is given by figure 2.1 (ii).

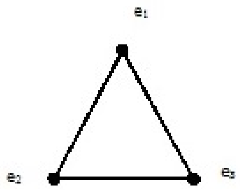

Figure 2.1(ii)

We can observe that $L(ZDG(Z_4))$ is a triangle (since $ZDG(Z_4)$) is a 3-stargraph). The domination number of both of these graphs is equal to 1. $L(ZDG(Z_4))$ is eqaul to K_3, the complete graph on three vertices.

Example 3.2. *(i) Let us construct $ZDG(Z_6)$, so $V(ZDG(Z_6)) = \{1, 2, 3, 4, 5\}$. Since $2 \cdot 3 = 3 \cdot 4 = 0$, there exist an edge between the vertices 2 adn 3, 3 and 4. Also since 0 is adjacent to all the elements in R, we get $\overline{e_1e_2}, \overline{e_2e_3}, \overline{e_3e_1} \in E(ZDG(Z_6))$. The graph $ZDG(Z_6)$ is given in figure 2.2(i).*

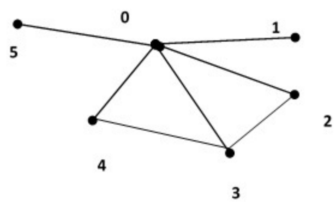

Figure-2.2(i)

Now we construct line graph of $ZDG(Z_6)$. Now $V(ZDG(Z_6)) = \{e_1 = \overline{01}, e_2 = \overline{02}, e_3 = \overline{03}, e_4 = \overline{04}, e_5 = \overline{05}, e_6 = \overline{23}, e_7 = \overline{34}\}$. The graph $L(ZDG(Z_6))$ is given by figure 2.2 (ii).

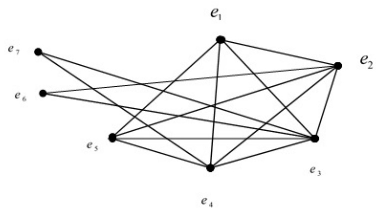

Fig 2.2(ii)

We can observe that subgraph of $L(ZDG(Z_6))$ is generated by the vertices e_1, e_2, e_3, e_4, e_5 forms a complete graph K_5 on five vertices which is a subgraph of $L(ZDG(Z_6))$.

The proof of the following Lemma 2.3 is parallel to the first part of Lemma 1.3 of [13]. For completeness we present the proof here.

Lemma 3.1. *Let G be a graph and H be a subgraph of G. Then $L(H) \subseteq L(G)$.*

Proof: Since H is a subgraph of G, we have that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Now $V(L(H)) = E(H) \subseteq E(G) = V(L(G))$. Let $\overline{e_1e_2} \in E(L(H))$. This implies $e_1, e_2 \in E(H)$ such that e_1 and e_2 are adjacent. Since H is a subgraph of G, we have that $e_1, e_2 \in E(H) \subseteq E(G)$ and e_1, e_2 are adjacent in G. Therefore $\overline{e_1e_2} \in E(L(G))$. Now we verified that $V(L(H)) \subseteq V(L(G))$ and $E(L(H)) \subseteq E(L(G))$. This shows that $L(H)$ is a subgraph of $L(G)$.

Lemma 3.2. *If $G = (V, E)$ is a stargraph of degree n, then $L(G) = K_{(n-1)}$, the complete graph on $(n-1)$ vertices.*

Proof: Let $G = (V, E)$ be a star graph with center 0 and $V = \{0, x_1, x_2, ..., x_{(n-1)}\}$. It is clear that $E = \{\overline{0x_i} : 1 \leq i \leq (n-1)\}$. Write $e_i = \overline{0x_i}$ for $1 \leq i \leq (n-1)$.

Consider the line graph L(G). Now $V(L(G)) = \{e_1, e_2, ..., e_{(n-1)}\}$, for each e_i, e_j, $i \neq j$, we have that $e_i = \overline{x_i0}$, $e_j = \overline{0x_j}$ are adjacent and so $\overline{e_ie_j} \in E(L(G))$. Hence $L(G)$ is a complete graph on vertices $(n-1)$ vertices $e_1, e_2, ..., e_{(n-1)}$. Hence $L(G) = K_{(n-1)}$.

Theorem 3.1. *If R is a finite associative ring with $|R| = n$, then $L(ZDG(R))$ contains the complete graph $K_{(n-1)}$ as a subgraph.*

Proof: Let $R = \{0, x_1, x_2, ..., x_{(n-1)}\}$. By definition of $L(ZDG(R))$ we have that $\overline{0x_i} \in E(ZDG(R))$ for $1 \leq i \leq (n-1)$. Now $V = \{0, x_1, x_2, ..., x_{(n-1)}\}$ together with the edge set $\{\overline{0x_i} : 1 \leq i \leq (n-1)$ forms a star graph S which is also a subgraph of $ZDG(R)$. By Lemma 3.1, $L(S)$ is a subgrah of $L(ZDG(R))$. Since S is a stargraph with $V(S) = n$, by Lemma 3.2, we have that $L(S) = K_{(n-1)}$. Thus $L(S) = K_{(n-1)}$ is subgraph of $L(ZDG(R))$.

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References

[1] Beck, (1956), Coloring of Commutative Ring, J. Algebra, 116 (1988), 208-226.
[2] Bondy,J.A. and Murty,U.S.R., (1976), Graph Theory with Applications, The Macmillan Press Ltd.
[3] Harary,F., (1972), Graph Theory, Addison-Wesley Publishing Company, USA.
[4] Narsing Deo, (1997), Graph Theory with Applications to Engineering and Computer Science, Prentice Hall of India Pvt. Ltd, New Delhi.
[5] Armugam,S. and Ramachandran,S., (2001), Invitation to Graph Theory , Scitech Publications (India) Pvt. Ltd, Chennai.
[6] Kulli,V.R., (1989), Minimally Non outerplanar Graphs: A survey, (in the Book: Recent Studies in Graph theory ed: V. R. Kulli) , Vishwa International Publication, pp.177-189.
[7] Satyanarayana,Bh. and Syam Prasad,K., (2003), An Isomorphism Theorem on Directed Hypercubes of Dimension n, Indian J. Pure and Appl. Math, 34(10), pp.1453-1457.
[8] Satyanarayana,Bh. and Syam Prasad,K., (2009), Discrete Mathematics and Graph Theory, Prentice Hall India Pvt. Ltd., New Delhi, ISBN 978-81-203-3842-5.
[9] Satyanarayana,Bh. and Syam Prasad,K., (2014), Discrete Mathematics and Graph Theory, Prentice Hall India Pvt. Ltd., New Delhi, (Second Edition), ISBN 978-81-203-4948-3.
[10] Satyanarayana,Bh. and Syam Prasad,K., (2013), Nearrings, Fuzzy Ideals and Graph Theory CRC Press (Taylor and Francis Group, London, New York), ISBN 13: 9781439873106.

[11] Satyanarayana Bh., Syam Prasad,K., and Nagaraju,D., (2010), Prime Graph of a Ring, J. Combinatorics, Informations and System Sciences, (35), pp.27-42.
[12] Satyanarayana,Bh., Srinivasulu,D., and Syam Prasad,K., (2012), Some Results on Degree of Vertices in Semitotal Block Graph and Total Block Graph, International Journal of Computer Applications (0975 - 8887), Vol.50, No.9, pp.19-22.
[13] Satyanarayana,Bh., Srinivasulu,D., and Syam Prasad,K., (2014), Line Graphs and Quasi-Total Block Graphs, International Journal of Computer Applications (0975 - 8887) Vol.105, No.3, pp.12-16.

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