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SOME RESULTS ON GENERALIZED TOEPLITZ OPERATOR ON GENERALIZED HARDY SPACE

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Abstract. In this paper, we define and study some properties of the generalized Hardy space $H_{F,2}$, where F is an injective linear transform from $L^p(\Pi)$ into $L^p(\Pi)$ and Π is the unit circle in the complex plane $\mathbb{C}$. Also we introduce the concept of a generalized Toeplitz operator on $H_{F,2}$ and prove some of its properties. Further results are presented.

Keywords: Hardy space, Toeplitz operator.

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1. Introduction

Let $\Pi = \{z \in \mathbb{C} : |z| = 1\}$ represent the unit circle in the complex plane $\mathbb{C}$ and μ be the Lebesgue measure on Π. Then $L^p(\Pi)$ shall denote the Banach space of Lebesgue measurable functions on Π with

$$||f||_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^p d\mu(\theta) \right\}^{\frac{1}{p}} < \infty, 1 \le p < \infty,$$

and $L^\infty(\Pi)$ denotes the Banach space of bounded measurable functions f on Π with $||f||_\infty = \text{ess sup}\{|f(\theta)|, \theta \in [0, 2\pi]\} < \infty$, see [2, 3]. If $z \in \Pi$, we can write z in the form $z = e^{i\theta}$ for some $\theta \in [0, 2\pi]$. For all $n \in \mathbb{Z}$, the complex valued function χ_n is defined on the set Π by $\chi_n(z) = z^n$ or we write $\chi_n(e^{i\theta}) = e^{in\theta}$. The set $\wp = \left\{ \sum_{n=-N}^{N} \alpha_n \chi_n : \alpha_n \in \mathbb{C} \right\}$ is called the set of trigonometric polynomials, while the set of all polynomials, $\wp_+ = \left\{ \sum_{n=0}^{N} \alpha_n \chi_n : \alpha_n \in \mathbb{C} \right\}$ is called the set of analytic trigonometric polynomials.

The Hardy space H^p is the space of all functions $f \in L^p(\Pi)$ such that

$$\int_0^{2\pi} f\left(e^{i\theta}\right) \chi_n\left(e^{i\theta}\right) d\mu(\theta) = 0 \text{ for all } n > 0,\ p = 1, 2, \infty.$$

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It is known [3] that χ_n are orthonormal Schauder basis for $L^p(\Pi)$ and H^p is a closed subspace of $L^p(\Pi)$.

For the case $p = 2$, $L^2(\Pi)$ is a Hilbert space and H^2 is a complemented subspace of $L^2(\Pi)$ see, [3]. That is there exists a bounded projection $P : L^2(\Pi) \to H^2$. If $\varphi \in L^\infty(\Pi)$, then $\varphi\left(H^2\right) \subseteq L^2(\Pi)$. So we can define the operator $T_\varphi : H^2 \to H^2$ by $T_\varphi(f) = P(\varphi f)$. T_φ is called the Toeplitz operator with symbol φ. For more on Toeplitz operator and Hardy spaces we refer the reader to [3]-[10] and references therein.

In this paper, we define, study and prove some properties of the generalized Toeplitz operator $T_{\varphi,F}$ on the generalized Hardy space $H_{F,p}$, where F is an injective linear transform from $L^p(\Pi)$ into $L^p(\Pi)$ and Π is the unit circle in the complex plane $\mathbb{C}$.

2. Generalized Hardy space

Let $F : L^p(\Pi) \to L^p(\Pi)$ be a linear operator such that rang$(F) \cap H^p \neq \{0\}$, and $F(f) = 0$ if and only if $f = 0$, that is, F is one to one. For $p = 1, 2, \infty$, the generalized Hardy space $H_{F,p}(\Pi) = H_{F,p}$ is defined to be the collection of all functions $f \in L^p(\Pi)$ for which

$$\int_0^{2\pi} F(f)\left(e^{i\theta}\right) \cdot \chi_n\left(e^{i\theta}\right)\, d\mu(\theta) = 0, \text{ for } n > 0.$$

The condition that rang$(F) \cap H^p$ is to avoid that $H_{F,p} = \{0\}$. It is clear that if F is the identity operator, then $H_{F,p} = H^p$.

Proposition 2.1. *$f \in H_{F,p}$ if and only if $F(f) \in H^p$.*

Proof. For all $f \in L^p(\Pi)$, $F(f) \in L^p(\Pi)$ and so, $f \in H_{F,p}$ if and only if

$$\int_0^{2\pi} F(f)\left(e^{i\theta}\right) \cdot \chi_n\left(e^{i\theta}\right) d\mu(\theta) = 0,$$

for $n > 0$ if and only if $F(f) \in H^p$. □

Lemma 2.1. *$H_{F,p}$ is a normed space under the norm $\|f\|_{F,p} = \|F(f)\|_p$, for all $f \in H_{F,p}$, $p = 1, 2, \infty$.*

Proof. For $p = 1, 2, \infty, f \in H_{F,p}, \|f\|_{F,p} \geq 0$, follows from the definition. Suppose that $\|f\|_{F,p} = \|F(f)\|_p = 0$. Since F is one to one and $\|.\|_p$ is a norm on H^p, it follows that $f = 0$. The other properties of the norm follows from linearity of F. □

In the following we give conditions under which $H_{F,p}(\Pi)$ is a Banach space.

Theorem 2.1. *For $P = 1, 2$, if F is continuous, then $H_{F,p}(\Pi)$ is closed subspace of $L^p(\Pi)$ and hence $H_{F,p}$ is a Banach space.*

Proof. Let f_n be a sequence in $H_{F,p}$ which converges to f. To show that $H_{F,p}$ is a *closed subspace of* $L^p(\Pi)$ it is sufficient to show that $f \in H_{F,p}$. Since

$$\begin{aligned} \underset{n\to\infty}{Lim} \|f_n - f\|_{F,p} &= \underset{n\to\infty}{Lim} \|F(f_n - f)\|_p \\ &= \left\|F\left(\underset{n\to\infty}{Lim}(f_n - f)\right)\right\|_p \\ &= \|F(0)\|_p = \|0\|_p = 0, \end{aligned}$$

we have $f \in H_{F,p}$. Since a closed subspace of a Banach space is Banach space using Lemma 2.1, $H_{F,p}$ *is a Banach space.* □

Example 2.1. *Let $\varphi \in L^{\infty}(\Pi)$ and $F : L^2(\Pi) \to L^2(\Pi)$ be the multiplication operator $F(f) = \varphi.f$. Then F is bounded. Hence continuous and $H_{F,p}$ is a Banach space.*

The following example shows that if F is not continuous, $H_{F,p}$ needs not to be a Banach space.

Example 2.2. *Let $F : L^2(\Pi) \to L^2(\Pi)$ be such that $F\left(\sum\limits_{n=1}^{\infty} a_n z^n\right) = \sum\limits_{n=1}^{\infty} \frac{a_n}{4^n} z^{n+1}$. Let $f \in L^2(\Pi)$ be such that $f(z) = \sum\limits_{n=2}^{\infty} \frac{1}{3^n} z^n \in H^2$. If $f \in rang(F)$, then $f = F(g)$ for some $g = \sum\limits_{n=1}^{\infty} a_n z^n \in L^2(\Pi)$. Hence*

$$f(z) = \sum_{n=2}^{\infty} \frac{1}{3^n} z^n = \sum_{n=1}^{\infty} \frac{a_n}{4^n} z^{n+1}.$$

Therefore $a_n = \frac{4^n}{3^{n+1}}$ and $f \notin rang(F)$. By Theorem 2.2 in [5], *$H_{F,2}$ is not a Banach space. Hence F is not continuous.*

Proposition 2.2. *If F is continuous, then $H_{F,2}$ is a Hilbert space.*

Proof. Since H^2 is a Hilbert space, there exists an inner product on H^2 denoted it by $\langle x, y \rangle_{H^2}$. Define an inner product on $H_{F,2}$ by

$$\langle f, g \rangle_{F,2} = \langle F(f), F(g) \rangle_{H^2}$$

for all f, $g \in H_{F,2}$. Using Theorem 2.1 and the properties of the inner product on H^2 it follows easily that $H_{F,2}$ is a Hilbert space. □

Let F be an injective linear transform from $L^2(\Pi)$ into $L^2(\Pi)$ such that rang$(F) \cap H^p \neq \{0\}$. For $\varphi \in L^{\infty}(\Pi)$, the multiplication operator $M_{\varphi,F} : L^2(\Pi) \to L^2(\Pi)$ is defined by $M_{\varphi,F}(f) = \varphi F(f)$.

Theorem 2.2. *Let $H^2 \subset rang(F)$. If $H_{F,2}$ is an invariant subspace for $M_{\varphi.F}$, then φ is in $H_{F,\infty}$.*

Proof. Since $1 \in H^2$ and $H^2 \subset \text{rang}(F)$, there exists $w \in L^2(\Pi)$ such that $F(w) = 1$. But $M_{\varphi,F}(H_{F,2})$ is contained in $H_{F,2}$. Therefore

$$M_{\varphi,F}(w) = \varphi F(w) = \varphi.1 = \varphi \in H_{F,2},$$

which implies that

$$\int_0^{2\pi} F(\varphi)\left(e^{i\theta}\right) \cdot \chi_n\left(e^{i\theta}\right) \, d\mu(\theta) = 0, \text{ for } n > 0.$$

But $\varphi \in L^{\infty}(\Pi)$. Hence $\varphi \in H_{F,\infty}$. □

Theorem 2.3. *If $\varphi \in H^{\infty}$ and $H^2 \subset range(F)$, then $H_{F,2}$ is an invariant subspace for $M_{\varphi.F}$.*

Proof. Let $\varphi \in H^{\infty}$, $f \in H_{F,2}$. Since

$$\begin{aligned} M_{\varphi,F}(H_{F,2}) &= \{\varphi F(f) : f \in H_{F,2}\} \\ &= \left\{\varphi F(f) : F(f) \in H^2\right\} \\ &\subset M_{\varphi}\left(H^2\right) \subset H^2 \subset \text{range}(F), \end{aligned}$$

it follows that

$$M_{\varphi,F}(H_{F,2}) \subset H_{F,2}.$$

□

3. Generalized Toeplitz operator $T_{\varphi,F}$ on $H_{F,2}$

Let $F : L^p(\Pi) \to L^p(\Pi)$ be a linear operator such that $\text{rang}(F) \cap H^p \neq \{0\}$. If $H_{F,2}$ is closed subspace of $L^2(\Pi)$, there exists a bounded projection P of $L^2(\Pi)$ onto $H_{F,2}$. For φ in $L^\infty(\Pi)$, the generalized Toeplitz operator $T_{\varphi,F}$ on $H_{F,2}$ is defined by

$$T_{\varphi,F}(f) = P(\varphi.F(f)).$$

Since for $f \in H_{F,2}$, $F(f) \in H^2$, $T_{\varphi,F}(f) = T_\varphi(F(f))$, it is easy to define a map ζ from $L^\infty(\Pi)$ into $\pounds(H_{F,2})$ by $\zeta(f) = T_{\varphi,F}(f)$, where $\pounds(H_{F,2})$ is the space of all bounded linear operators on $H_{F,2}$.

In the following, we prove some properties of the generalized Toeplitz operator $T_{\varphi,F}$.

Theorem 3.1. *The mapping ζ is a contractive $*-$linear from $L^\infty(\Pi)$ into $\pounds(H_{F,2})$.*

Proof. 1) ζ is contractive: For $f \in H_{F,2}$, φ, $\psi \in L^\infty(\Pi)$

$$\begin{aligned}
\|(\zeta(\varphi) - \zeta(\psi)) f\|_{F,2} &= \|(T_{\varphi,F} - T_{\psi,F}) f\|_{F,2} \\
&= \|T_{\varphi,F}(f) - T_{\psi,F}(f)\|_{F,2} \\
&= \|P(\varphi.F(f)) - P(\psi.F(f))\|_{F,2} \\
&= \|P(\varphi.F(f) - \psi.F(f))\|_{F,2} \\
&= \|P((\varphi - \psi).F(f))\|_{F,2} \\
&\leq \|P\| \|(\varphi - \psi).F(f)\|_{F,2} \\
&\leq \|(\varphi - \psi).F(f)\|_{F,2} \\
&\leq \|(\varphi - \psi)\|_{F,2} \|F(f)\|_{F,2} \\
&\leq \|(\varphi - \psi)\|_{F,2}.
\end{aligned}$$

2) ζ is linear: For $f \in H_{F,2}$, $\lambda \in \mathbb{C}$

$$\begin{aligned}
(\lambda\zeta(\varphi) + \zeta(\psi)) f &= ((\lambda T_{\varphi,F} + T_{\psi,F}) f) \\
&= P(\lambda\varphi.F(f)) + P(\psi.F(f)) \\
&= P(\lambda\varphi.F(f) + \psi.F(f)) \\
&= P((\lambda\varphi + \psi).F(f)) \\
&= T_{\lambda\varphi+\psi,F}(f) = \zeta(\lambda\varphi + \psi)
\end{aligned}$$

3) To prove that $\zeta(\varphi)^* = \zeta(\overline{\varphi})$, let $f,\, g \in H_{F,2}$. Then

$$\begin{aligned}
\langle T_{\overline{\varphi},F}(f), F(g)\rangle_{F,2} &= \langle P(\overline{\varphi}F(f)), F(g)\rangle_{F,2} \\
&= \langle \overline{\varphi}F(f), P(F(g))\rangle_{F,2} \\
&= \langle \overline{\varphi}F(f), F(g)\rangle_{F,2} \\
&= \langle F(f), \varphi F(g)\rangle_{F,2} \\
&= \langle P(F(f)), \varphi F(g)\rangle_{F,2} \\
&= \langle F(f), P(\varphi F(g))\rangle_{F,2} \\
&= \langle F(f), T_{\varphi}(F(g))\rangle_{F,2} \\
&= \left\langle T_{\varphi}^{*}(F(f)), F(g)\right\rangle_{F,2} \\
&= \left\langle T_{\varphi,F}^{*}(f), F(g)\right\rangle_{F,2},
\end{aligned}$$

which implies that

$$\zeta(\varphi)^* = T_{\varphi,F}^{*} = T_{\overline{\varphi},F} = \zeta(\overline{\varphi}).$$

□

Theorem 3.2. *If φ is in $L^{\infty}(\Pi)$ and $\psi \in H^{\infty}$, then $T_{\varphi}T_{\psi,F} = T_{\varphi\omega,F}$.*

Proof. Let $f \in H_{F,2}$. Since $\psi \in H^{\infty}$ and $F(f) \in H^2$, $\psi F(f) \in H^2$. Hence $P(\psi F(f)) = \psi F(f)$ and

$$\begin{aligned}
T_{\varphi}T_{\psi,F}(f) &= T_{\varphi}(P(\psi F(f))) \\
&= T_{\varphi}(\psi F(f)) \\
&= P(\varphi\psi F(f)) \\
&= T_{\varphi\omega,F}(f),
\end{aligned}$$

that is, $T_{\varphi}T_{\psi,F} = T_{\varphi\psi,F}$. □

Theorem 3.3. *If $\varphi \in L^{\infty}(\Pi)$, $\overline{\theta} \in H^{\infty}$, then $T_{\theta,F}T_{\varphi} = T_{\theta\varphi,F}$.*

Proof.

$$\begin{aligned}
(T_{\theta,F}T_{\varphi})^* &= T_{\varphi}^{*}T_{\theta,F}^{*} \\
&= T_{\overline{\varphi}}T_{\overline{\theta},F} \\
&= T_{\overline{\varphi}\overline{\theta},F} \\
&= T_{\overline{\theta}\overline{\varphi},F} \\
&= T_{\overline{\theta\varphi},F} \\
&= T_{\theta\varphi,F}^{*},
\end{aligned}$$

which implies that $(T_{\theta,F}T_{\varphi})^* = T_{\theta\varphi,F}^{*}$. By taking adjoints to both sides, we get $T_{\theta,F}T_{\varphi} = T_{\theta\varphi,F}$. □

Theorem 3.4. *Let $\varphi \in L^{\infty}(\Pi)$ and $H^2 \subset rang(F)$. Then φ is invertible in $L^{\infty}(\Pi)$, if $T_{\varphi,F}$ is invertible.*

Proof. It is sufficient to show that M_{φ} is an invertible operator if $T_{\varphi,F}$ is. If $T_{\varphi,F}$ is invertible, then there exists $\epsilon > 0$ such that

$$\|T_{\varphi,F}(f)\| = \|T_{\varphi}F(f)\| \geq \epsilon \|F(f)\|$$

for all $f \in H_{F,2}$. which implies that for each $n \in \mathbb{Z}$, $f \in H_{F,2}$, we have

$$\begin{aligned} \|M_{\varphi}(\chi_n F(f))\| &= \|\varphi\chi_n F(f)\| \\ &= \|\varphi F(f)\| \\ &\geq \|P(\varphi F(f))\| \\ &= \|T_{\varphi,F}(f)\| \\ &\geq \epsilon\|F(f)\| \\ &= \epsilon\|\chi_n F(f)\|. \end{aligned}$$

Since the set $\left\{\chi_n F(f) : F(f) \in H^2,\ n \in \mathbb{Z}\right\}$ is dense in $L^2(\Pi)$, we get that:

$$\|M_{\varphi}(g)\| \geq \epsilon\|g\|.$$

for all $g \in L^2(\Pi)$. Similarly $\|M_{\overline{\varphi}}(f)\| \geq \epsilon\|f\|$ using that $T_{\overline{\varphi},F}$ is invertible and then M_{φ} its self is invertible. □

4. **Further Results**

Let $Y = \left\{f \in L^1[0,2\pi] : f(t) = 0 \text{ for all } 0 \leq t < \pi\right\}$. Since for all $\varphi \in L^{\infty}[0,2\pi]$, $f \in L^1[0,2\pi]$,

$$\begin{aligned} \int_0^{2\pi} |\varphi(\theta) . f(\theta)|\, d\mu(\theta) &= \int_0^{2\pi} |\varphi(\theta)|\,|f(\theta)|\, d\mu(\theta) \\ &\leq \|\varphi(\theta)\|_{\infty} \int_0^{2\pi} |f(\theta)|\, d\mu(\theta) < \infty, \end{aligned}$$

the multipliers of $L^1[0,2\pi]$ is the space $L^{\infty}[0,2\pi]$.

Theorem 4.1. *Y is a complemented subspace of $L^1[0,2\pi]$.*

Proof. For $f \in L^1[0,2\pi]$, define f_1, f_2 as

$$f_1(t) = \begin{cases} 0 & , 0 \leq t < \pi \\ f(t) & , \pi \leq t \leq 2\pi \end{cases},$$

and

$$f_2(t) = \begin{cases} f(t) & , 0 \leq t < \pi \\ 0 & , \pi \leq t \leq 2\pi \end{cases}.$$

Then $f = f_1 + f_2$, $f_1 \in Y$, that is, $L^1[0,2\pi] = Y + K$, where

$$K = \{f \in X : f(t) = 0 \text{ for all } \pi \leq t \leq 2\pi\}.$$

It is easy to that $\|f\|_1 = \|f_1\|_1 + \|f_2\|_1$ □

Since Y is a complemented subspace of $L^1(0,2\pi)$, there exists a bounded projection $P : L^1(0,2\pi) \to Y$. For $\varphi \in L^{\infty}[0,2\pi]$, $\varphi g \in L^1(0,2\pi)$ for all $g \in Y$. Define $T_{\varphi}(g) = P(\varphi g)$. Then T_{φ} is a linear mapping from Y into Y. T_{φ} is called a Toeplitz Type operator.

Theorem 4.2. *The mapping $\zeta : \pounds\left(L^1(0,2\pi)\right) \longrightarrow \pounds(Y)$ defined by $\zeta(\varphi) = T_{\varphi}$ is a bounded linear operator such that $\zeta(\varphi^*) = (\zeta(\varphi))^*$, where $\pounds(X)$ is the space of all bounded linear operators on X.*

Proof. 1) ζ **is linear:** Let $f,\, g \in L^{\infty}[0, 2\pi]$, $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}\zeta(f+g)\varphi &= T_{f+g}(\varphi)\\ &= P((f+g)\varphi)\\ &= P(f\varphi + g\varphi)\\ &= P(f\varphi) + P(g\varphi)\\ &= T_f(\varphi) + T_g(\varphi)\\ &= \zeta(f)\varphi + \zeta(g)\varphi\\ &= (\zeta(f) + \zeta(g))\varphi.\end{aligned}$$

To end the linearity of ζ,

$$\zeta(\alpha f)\varphi = T_{\alpha f}(\varphi) = P(\alpha f\varphi) = \alpha P(f\varphi) = \alpha T_f(\varphi) = \alpha\zeta(f)\varphi.$$

2) ζ **is bounded.** Since $\|T_\varphi\| \leq \|P\|\,\|\varphi\|$, it follows that T_φ is bounded and hence $\zeta(\varphi)$ is bounded.

3) $\zeta(\varphi^*) = (\zeta(\varphi))^*$. For $y \in Y$,

$$\begin{aligned}\left\langle T_\varphi^*(y^*), y\right\rangle &= \langle y^*, T_\varphi(y)\rangle\\ &= \langle y^*, P(\varphi y)\rangle\\ &= \langle P^*(y^*), \varphi y\rangle\\ &= \langle P(y^*), \varphi y\rangle\\ &= \langle y^*, \varphi y\rangle\\ &= \langle \varphi^* y^*, y\rangle\\ &= \langle \varphi^* y^*, P(y)\rangle\\ &= \langle P(\varphi^* y^*), y\rangle\\ &= \langle T_{\varphi^*}(y^*), y\rangle,\end{aligned}$$

that is,

$$(T_\varphi)^* = T_{\varphi^*},$$

which implies that

$$\zeta(\varphi^*) = T_{\varphi^*} = (T_\varphi)^* = (\zeta(\varphi))^*.$$

□

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