TWMS J. App. Eng. Math. V.6, N.2, 2016, pp. 298-306

ON A SUM FORM FUNCTIONAL EQUATION

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Abstract. The general solutions of a sum form functional equation containing two unknown mappings, without imposing any regularity condition on them, have been obtained.

Keywords: functional equation, additive mapping, multiplicative mapping, logarithmic mapping.

Mathematics subject classification (2010): 39B22, 39B52.

1. Introduction

Functional equations appear in various branches of pure mathematics and applied mathematics, business mathematics, economics, information theory, thermodynamics, physics, engineering, and so on (see [1], [3], [4], [5])

For $n = 1, 2, \ldots$; let $\Gamma_n = \left\{ (p_1, \ldots, p_n) : p_i \geq 0,\ i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\}$ denote the set of all n-component complete discrete probability distributions with nonnegative elements. Let $\mathbb{R}$ denote the set of all real numbers; $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$; $]0, 1[= \{x \in \mathbb{R} : 0 < x < 1\}$ and $]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

By giving necessary motivations from statistics point of view, considering the first and second order moments of a specific random variable, Nath and Singh [7] derived the functional equation

$$\phi_2(pq) = q\phi_2(p) + p\phi_2(q) + 2\phi_1(p)\phi_1(q)$$

for all $p \in I$, $q \in I$; $\phi_2 : I \to \mathbb{R}$, $\phi_1 : I \to \mathbb{R}$ with $\phi_2(0) = 0$, $\phi_2(1) = 0$, $\phi_1(0) = 0$, $\phi_1(1) = 0$.

For all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, the authors [7] considered the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + c \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) \tag{A}$$

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§ Manuscript received: November 25, 2015; accepted: February 29, 2016.

in which $c \neq 0$ is a given real constant; $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ are unknown mappings. Clearly $f = \phi_2$ and $g = \phi_1$ satisfy (A) with $c = 2$. Keeping in view $\phi_1(0) = 0$, $\phi_1(1) = 0$, $\phi_2(0) = 0$, $\phi_2(1) = 0$ and the fact that $f = \phi_2$ and $g = \phi_1$, we have

$$\text{(i)} \quad f(0) = 0, \qquad \text{(ii)} \quad g(0) = 0 \tag{1}$$

and

$$\text{(i)} \quad f(1) = 0, \qquad \text{(ii)} \quad g(1) = 0. \tag{2}$$

Nath and Singh [7] obtained the general solutions of (A) by assuming (1) and (2); and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers.

The object of this paper is to **obtain the general solutions of (A)** for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers; **without assuming (1) and (2)**.

2. Some Preliminary Results

In this section, we mention some known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if $A(x+y) = A(x)+A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known (see Daróczy and Losonczi [2]) that if a mapping $a : I \to \mathbb{R}$ is additive on I, then there exists a unique mapping $A : \mathbb{R} \to \mathbb{R}$ which is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in \,]0, 1]$, $q \in \,]0, 1]$.

Result 2.1 ([6]). *Let $\psi : I \to \mathbb{R}$ be a mapping which satisfies the equation $\sum_{i=1}^{k} \psi(x_i) = c$ for all $(x_1, \ldots, x_k) \in \Gamma_k$; c a given real constant and $k \geq 3$ a fixed integer. Then there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = b(x) - \frac{1}{k}b(1) + \frac{c}{k}$ for all $x \in I$.*

Chaundy and Mcleod [1] considered the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \tag{B}$$

where $f : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; n and m being positive integers.

Result 2.2 ([6]). *If a mapping $f : I \to \mathbb{R}$ satisfies (B) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers, then f is of the form*

$$f(p) = \begin{cases} f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ f(0) & \text{if } p = 0, \end{cases}$$

where $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping; the mapping $D : \mathbb{R}\times \,]0, 1] \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and $D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q)$ for all $p \in \,]0, 1]$, $q \in \,]0, 1]$.

Modified Form of Result 2.2. *If a mapping $f : I \to \mathbb{R}$ satisfies (B) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers, then f is of the form*

$$f(p) = f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) \tag{3}$$

for all $p \in I$; $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping; $D : \mathbb{R} \times I \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and

$$D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q) \tag{4}$$

for all $p \in I$, $q \in I$.

Using the fact that $a(1) = E(1, 1)$, it can be easily deduced from (4) that

$$a(1) + D(1, 1) = 0. \tag{5}$$

3. On the Functional Equation (A)

The main result of this paper is the following:

Theorem. *Let c be a nonzero given constant and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be mappings which satisfy the equation (A) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, for all $p \in I$, any general solution (f, g) of (A) is of the form*

$$\begin{cases} \text{(i)} \quad f(p) = f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) \\ \text{(ii)} \quad g(p) = A_1(p) + g(0); \end{cases} \tag{β_1}$$

or

$$\begin{cases} \text{(i)} \quad f(p) = f(0) + f(0)(nm - n - m)p + a(p) \\ \qquad\qquad + D(p, p) + \dfrac{1}{2}\, cp[\ell^*(p)]^2 \\ \text{(ii)} \quad g(p) = p\ell^*(p); \end{cases} \tag{β_2}$$

or

$$\begin{cases} \text{(i)} \quad f(p) = f(0) + f(0)(nm - n - m)p \\ \qquad\qquad + c\lambda^2[M(p) - p] + a(p) + D(p, p) \\ \text{(ii)} \quad g(p) = \lambda[M(p) - p]; \end{cases} \tag{β_3}$$

or

$$\begin{cases} \text{(i)} \quad f(p) = f(0) + \{f(0)(nm - n - m) \\ \qquad\qquad - c[g(1) + (n - 1)g(0)][g(1) + (m - 1)g(0)]\}p + a(p) + D(p, p) \\ \text{(ii)} \quad g(p) = A_2(p) + g(0); \quad g(1) + (m - 1)g(0) \neq 0 \end{cases} \tag{β_4}$$

where λ is an arbitrary nonzero real constant; $A_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are additive mappings such that $A_1(1) = -mg(0)$ and $A_2(1) = g(1) - g(0)$; $\ell^* : I \to \mathbb{R}$ is a logarithmic mapping which does not vanish identically on the open interval $]0, 1[$; $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; $a : \mathbb{R} \to \mathbb{R}$ and $D : \mathbb{R} \times I \to \mathbb{R}$ are as described in the Modified Form of Result 2.2.

Proof. Let us write (A) in the form

$$\sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} f(p_i q_j) - f(p_i) - p_i \sum_{j=1}^{m} f(q_j) - cg(p_i) \sum_{j=1}^{m} g(q_j) \right\} = 0.$$

By using Result 2.1 and proceeding as in [7], we can obtain

$$\left[\sum_{j=1}^{m} g(xq_j) - g(x) - (m-1)g(0)\right]\sum_{t=1}^{m} g(r_t)$$
$$= \left[\sum_{t=1}^{m} g(xr_t) - g(x) - (m-1)g(0)\right]\sum_{j=1}^{m} g(q_j) \tag{6}$$

as $c \neq 0$. Equation (6) is valid for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$, $(r_1, \ldots, r_m) \in \Gamma_m$; $m \geq 3$ being a fixed integer.

From now onwards, we divide our discussion into two cases.

Case 1. $\sum\limits_{t=1}^{m} g(r_t)$ vanishes identically on Γ_m, that is,

$$\sum_{t=1}^{m} g(r_t) = 0$$

for all $(r_1, \ldots, r_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ such that

$$g(p) = A_1(p) - \frac{1}{m}A_1(1) \tag{7}$$

for all $p \in I$. The substitution $p = 0$ in (7) gives $A_1(1) = -\,mg(0)$. Now (7) gives (β_1)(ii) with $A_1(1) = -\,mg(0)$. Utilizing this form of g in (A), we obtain the functional equation (B) for $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers.

By the Modified Form of Result 2.2, it follows that $f : I \to \mathbb{R}$ is of the form (β_1)(i). Thus, we have obtained the solution (β_1) of (A).

Case 2. $\sum\limits_{t=1}^{m} g(r_t)$ does not vanish identically on Γ_m.

In this case, there exists a probability distribution $(r_1^*, \ldots, r_m^*) \in \Gamma_m$ such that

$$\sum_{t=1}^{m} g(r_t^*) \neq 0. \tag{8}$$

Setting $r_t = r_t^*$, $t = 1, \ldots, m$ in (6) and using (8), we obtain

$$\sum_{j=1}^{m} g(xq_j) - g(x) - (m-1)g(0)$$
$$= \left[\sum_{t=1}^{m} g(r_t^*)\right]^{-1}\left[\sum_{t=1}^{m} g(xr_t^*) - g(x) - (m-1)g(0)\right]\sum_{j=1}^{m} g(q_j). \tag{9}$$

Define a mapping $M : I \to \mathbb{R}$ as

$$M(x) = \left[\sum_{t=1}^{m} g(r_t^*)\right]^{-1}\left[\sum_{t=1}^{m} g(xr_t^*) - g(x) - (m-1)g(0)\right] \tag{10}$$

for all $x \in I$. Now, from (9) and (10), it follows that

$$\sum_{j=1}^{m} g(xq_j) = M(x)\sum_{j=1}^{m} g(q_j) + g(x) + (m-1)g(0). \tag{11}$$

From (10), it is easy to conclude that

$$M(0) = 0. \tag{12}$$

The substitution $x = 1$, in (10), gives

$$1 - M(1) = [g(1) + (m-1)g(0)] \left[\sum_{t=1}^{m} g(r_t^*) \right]^{-1}. \tag{13}$$

Let us write (11) in the form

$$\sum_{j=1}^{m} \{g(xq_j) - M(x)g(q_j) - [g(x) + (m-1)g(0)]q_j\} = 0.$$

By Result 2.1, there exists a mapping $E : I \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$g(xq) - M(x)g(q) - [g(x) + (m-1)g(0)]q = E(x;q) - \frac{1}{m}E(x;1). \tag{14}$$

Equation (14) holds for all $x \in I$ and $q \in I$. The substitution $q = 0$ in it gives (using $E(x;0) = 0$)

$$E(x;1) = mg(0)[M(x) - 1] \tag{15}$$

for all $x \in I$. From (14) and (15), we obtain

$$g(xq) - M(x)[g(q) - g(0)] - [g(x) + (m-1)g(0)]q - g(0) = E(x;q). \tag{16}$$

Case 2.1. $E(x;q) \equiv 0$ on $I \times I$.
In this case, $E(x;1) = 0$. So, (15) gives

$$mg(0) = mg(0)M(x) \tag{17}$$

for all $x \in I$. Since the left hand side of (17) is independent of the variable x, $x \in I$, it follows that

$$mg(0)M(x) = mg(0)M(q) \tag{18}$$

for all $x \in I$ and $q \in I$. Also, from (16) and the fact that $E(x;q) \equiv 0$ on $I \times I$, we obtain

$$g(xq) - g(0) = M(x)[g(q) - g(0)] + [g(x) + (m-1)g(0)]q \tag{19}$$

for all $x \in I$ and $q \in I$. The left hand side of (19) is symmetric in x and q. Hence, so should be its right hand side. This fact gives rise to the equation

$$\begin{aligned} &M(x)[g(q) - g(0)] + [g(x) + (m-1)g(0)]q \\ &\quad = M(q)[g(x) - g(0)] + [g(q) + (m-1)g(0)]x. \end{aligned} \tag{20}$$

Making use of (18), (20) gives rise to the equation

$$[g(q) + (m-1)g(0)][M(x) - x] = [g(x) + (m-1)g(0)][M(q) - q] \tag{21}$$

valid for all $x \in I$ and $q \in I$.

Case 2.1.1. $M(x) - x = 0$ for all $x \in I$.
In this case, $M(x) = x$ for all $x \in I$. Now, (17) gives $mg(0)(1 - x) = 0$ for all $x \in I$. Choosing $x = \frac{1}{2}$, we obtain $g(0) = 0$. Using $M(x) = x$ for all $x \in I$ and the fact that $g(0) = 0$, (19) gives the functional equation $g(xq) = xg(q) + qg(x)$ whose general solution is $g(x) = x\ell(x)$ for all $x \in I$; $\ell : I \to \mathbb{R}$ being any logarithmic mapping. If $\ell(x) = 0$ for all

$x \in I$, then $g(x) = 0$ for all $x \in I$. Consequently, $\sum_{t=1}^{m} g(r_t^*) = 0$ contradicting (8). So, g must be of the form (β_2)(ii). Making use of this form of g in (A), we obtain the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j \ell^*(p_i)\ell^*(q_j).$$

The above equation can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{f(p_i q_j) - \frac{1}{2}\,c p_i q_j [\ell^*(p_i q_j)]^2\right\} = \sum_{i=1}^{n}\left\{f(p_i) - \frac{1}{2}\,c p_i [\ell^*(p_i)]^2\right\} + \sum_{j=1}^{m}\left\{f(q_j) - \frac{1}{2}\,c q_j [\ell^*(q_j)]^2\right\}.$$

Define a mapping $f_1 : I \to \mathbb{R}$ as $f_1(p) = f(p) - \frac{1}{2} cp[\ell^*(p)]^2$ for all $p \in I$. Then making use of Modified Form of Result 2.2, it can be proved that f is of the form (β_2)(i). Thus, we have obtained the solution (β_2) of (A).

Case 2.1.2. $[M(x) - x] \not\equiv 0$ on I.

In this case, there exists an element $x_0 \in I$ such that $[M(x_0) - x_0] \neq 0$. Setting $x = x_0$ in (21), we obtain

$$g(q) = \lambda[M(q) - q] - (m-1)g(0) \tag{22}$$

where $\lambda = [M(x_0) - x_0]^{-1}[g(x_0) + (m-1)g(0)]$. If $\lambda = 0$, then (22) gives $g(q) = -(m-1)g(0)$ for all $q \in I$. From this, it follows that $g(0) = 0$ as $m \geq 3$. Now (22) gives $g(q) = 0$ for all $q \in I$. In particular, $\sum_{t=1}^{m} g(r_t^*) = 0$ contradicting (8). Hence, $\lambda \neq 0$. Putting $q = 0$ in (22) and using (12), it follows that $g(0) = 0$. Thus, (22) gives

$$g(q) = \lambda[M(q) - q], \quad \lambda \neq 0 \tag{23}$$

for all $q \in I$. From (19), and the fact that $g(0) = 0$, we obtain

$$g(xq) = M(x)g(q) + qg(x) \tag{24}$$

for all $x \in I$, $q \in I$. From (23) and (24), it follows that $M(xq) = M(x)M(q)$ for all $x \in I$, $q \in I$. Thus, M is a multiplicative mapping. But, we have to consider only those multiplicative mappings M which satisfy the condition (12). The possibility $M(x) \equiv 1$, $x \in I$, is ruled out as, in this case, $M(0) \neq 0$. Since $[M(x_0) - x_0] \neq 0$ for some $x_0 \in I$, it follows that $g(x_0) \neq 0$ for some $x_0 \in I$. Since $g(0) = 0$, the possibility $x_0 = 0$ is ruled out. So, $x_0 \in\,]0, 1]$. Consider $x_0 = 1$. This means $g(1) \neq 0$. Hence, by (23), $M(1) \neq 1$. But, M is multiplicative. So, $M(x)[M(1) - 1] = 0$. Since $M(1) \neq 1$, it follows that $M(x) = 0$ for all $x \in I$. Consequently, (23) gives $g(q) = -\lambda q$ for all $q \in I$ with $\lambda \neq 0$ which is included in (β_4)(ii) upon choosing $A_2(q) = -\lambda q$ (as $g(0) = 0$) with $A_2(1) = g(1) = -\lambda \neq 0$. Now proceeding as in the Case 2.1.1, the corresponding form of f is

$$f(p) = f(0) + \{f(0)(nm - n - m) - c[g(1)]^2\}p + a(p) + D(p, p)$$

which is included in (β_4)(i).

Now we consider the case when $x_0 \in\,]0, 1[$. In this case, we must have $g(0) = 0$ and also $g(1) = 0$. Now, from (23), it follows $M(1) = 1$.

Now we prove that M is not additive. To the contrary, suppose $M : I \to \mathbb{R}$ is additive. Then, for all $(r_1, \ldots, r_m) \in \Gamma_m$, using (23) and $M(1) = 1$, we have

$$\sum_{t=1}^{m} g(r_t) = \lambda \left[\sum_{t=1}^{m} M(r_t) - 1 \right] = \lambda[M(1) - 1] = 0$$

contradicting (8). So, M is not additive. Thus, the solution (β_3)(ii) stands obtained in which M is a multiplicative mapping with $M(0) = 0$, $M(1) = 1$ and M is not additive.

Now, making use of (β_3)(ii) in (A) and proceeding as in the Case 2.1.1, we can obtain (β_3)(i). Thus the solution (β_3) follows.

Case 2.2. $E(x; q) \not\equiv 0$ on $I \times I$.

In this case, there exists an element $(x^*, q^*) \in I \times I$ such that $E(x^*; q^*) \neq 0$. Now we prove that

$$\begin{aligned} r = [E(x^*; q^*)]^{-1} \{ &E(x^*; q^* r) + M(x^*) E(q^*; r) - E(x^* q^*; r) \\ &+ [M(x^*) M(q^*) - M(x^* q^*)][g(r) - g(0)] + rmg(0)[M(x^*) - 1] \} \end{aligned} \tag{25}$$

holds for all $r \in I$. Using (16), we have

$$\begin{aligned} &g((x^* q^*) r) - r q^* [g(x^*) + (m-1) g(0)] - r M(x^*)[g(q^*) - g(0)] - g(0) \\ &\quad = E(x^* q^*; r) + M(x^* q^*)[g(r) - g(0)] + r E(x^*; q^*) + rmg(0) \end{aligned} \tag{26}$$

and

$$\begin{aligned} &g(x^*(q^* r)) - q^* r [g(x^*) + (m-1) g(0)] - r M(x^*)[g(q^*) - g(0)] - g(0) \\ &\quad = E(x^*; q^* r) + M(x^*) E(q^*; r) + M(x^*) M(q^*)[g(r) - g(0)] + rmM(x^*) g(0). \end{aligned} \tag{27}$$

Since the left hand sides of (26) and (27) are same, we get

$$\begin{aligned} &E((x^* q^*); r) + M(x^* q^*)[g(r) - g(0)] + r E(x^*; q^*) + rmg(0) \\ &\quad = E(x^*; q^* r) + M(x^*) E(q^*; r) + M(x^*) M(q^*)[g(r) - g(0)] + rmM(x^*) g(0). \end{aligned} \tag{28}$$

Using the fact that $E(x^*; q^*) \neq 0$, (25) follows from (28).

Let us write (25) as

$$\begin{aligned} &r - [E(x^*; q^*)]^{-1} \{ E(x^*; q^* r) + M(x^*) E(q^*; r) - E(x^* q^*; r) + rmg(0)[M(x^*) - 1] \} \\ &\quad = [E(x^*; q^*)]^{-1} [M(x^*) M(q^*) - M(x^* q^*)][g(r) - g(0)]. \end{aligned} \tag{29}$$

Case 2.2.1. $[M(x^*) M(q^*) - M(x^* q^*)] \neq 0$.

In this case, (29) gives

$$g(r) = A_1(r) + g(0), \quad 0 \leq r \leq 1, \tag{30}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is a mapping defined as

$$\begin{aligned} A_1(t) = [M(x^*) M(q^*) - M(x^* q^*)]^{-1} \{ &t E(x^*; q^*) - E(x^*; q^* t) \\ &- M(x^*) E(q^*; t) + E(x^* q^*; t) - tmg(0)[M(x^*) - 1] \} \end{aligned} \tag{31}$$

for all $t \in \mathbb{R}$. Since $E : I \times \mathbb{R} \to \mathbb{R}$ is additive in the second variable, it follows that $A_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Putting $r = 1$ in (31) and using (15), it turns out that $A_1(1) = -mg(0)$. From (8), (30) and the fact that $A_1(1) = -mg(0)$, we observe that

$$\begin{aligned} 0 \neq \sum_{t=1}^{m} g(r_t^*) &= \sum_{t=1}^{m} [A_1(r_t^*) + g(0)] \\ &= A_1(1) + mg(0) = -mg(0) + mg(0) = 0 \end{aligned}$$

a contradiction. So, this case is not possible.

Case 2.2.2. $[M(x^*)M(q^*) - M(x^*q^*)] = 0$.
The substitution $r = 1$, in (29), gives

$$mg(0)[M(x^*)M(q^*) - M(x^*q^*)] = 0\,.$$

Since $m \geq 3$ is a fixed integer and $[M(x^*)M(q^*) - M(x^*q^*)] = 0$, it follows that $g(0)$ is an arbitrary real number. Now, let us put $x = 1$ in (16). We obtain

$$[g(q) - g(0)][1 - M(1)] = E(1; q) + [g(1) + (m-1)g(0)]q \tag{32}$$

for all $q \in I$.

Case 2.2.2.1. $1 - M(1) \neq 0$.
In this case, (13) gives $[g(1) + (m-1)g(0)] \neq 0$. Consequently, $[g(1) - g(0)] \neq -mg(0)$. Also, from (32),

$$g(q) = [1 - M(1)]^{-1}\{E(1; q) + [g(1) + (m-1)g(0)]q\} + g(0). \tag{33}$$

Let us define a mapping $A_2 : \mathbb{R} \to \mathbb{R}$ as

$$A_2(t) = [1 - M(1)]^{-1}\{E(1; t) + [g(1) + (m-1)g(0)]t\} \tag{34}$$

for all $t \in \mathbb{R}$. Then, $A_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Now, it follows from (33) and (34) that g is of the form (β_4)(ii) with $A_2(1) = [g(1) - g(0)]$. From (β_4)(ii) and (A), it follows that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j)& \\ + c\,[g(1) + (n-1)g(0)][g(1) + (m-1)g(0)]&\end{aligned} \tag{35}$$

with $[g(1) + (m-1)g(0)] \neq 0$. Now, proceeding as in the Case 2.1.1, it can be proved that f is of the form (β_4)(i). Thus, we have obtained the solution (β_4).

Case 2.2.2.2. $1 - M(1) = 0$.
In this case, (13) gives

$$g(1) + (m-1)g(0) = 0. \tag{36}$$

The mapping $g : I \to \mathbb{R}$, mentioned in (β_1)(ii), (β_2)(ii) and (β_3)(ii), satisfies (36). But, we have to consider only those solutions of (A) which meet the requirement $[M(x^*)M(q^*) - M(x^*q^*)] = 0$ for some $x^* \in I$ and $q^* \in I$. There is only one such solution, namely β_3(ii), as in this solution, the mapping M is multiplicative and thus the condition $[M(x^*)M(q^*) - M(x^*q^*)] = 0$ for some $x^* \in I$, $q^* \in I$, is met with. Also $M(1) = 1$ and $M(0) = 0$. So, (β_3)(ii) gives $g(1) = 0$ and $g(0) = 0$. Now, from (16), $g(0) = 0$ and the fact that M is multiplicative, it follows that $E(x; q) = 0$ for all $x \in I$, $q \in I$, thereby, contradicting the fact that $E(x^*; q^*) \neq 0$ for some $x^* \in I$, $q^* \in I$. So, in this case we do not get any new solution. □

Remark. The solutions (β_1), (β_2) and (β_3) are respective **nontrivial generalizations** of solutions (3.1), (3.2) and (3.3) of the Theorem ([7], pp. 86–87). **The solution** (β_4) **is absolutely a new solution**. The solution (3.1) is included in (β_1) but not in (β_4) as $g(1) + (m-1)g(0) \neq 0$.

Acknowledgement. The authors are grateful to the referee for his critical comments.

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