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ON THE THIRD BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS IN A NON-REGULAR DOMAIN OF $\mathbb{R}^{N+1}$

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ABSTRACT. In this paper, we look for sufficient conditions on the lateral surface of the domain and on the coefficients of the boundary conditions of a $N-$space dimensional linear parabolic equation, in order to obtain existence, uniqueness and maximal regularity of the solution in a Hilbertian anisotropic Sobolev space when the right hand side of the equation is in a Lebesgue space. This work is an extension of solvability results obtained for a second order parabolic equation, set in a non-regular domain of $\mathbb{R}^3$ obtained in [1], to the case where the domain is cylindrical, not with respect to the time variable, but with respect to N space variables, $N > 1$.

Keywords: Parabolic equations, Non-regular domains, Robin conditions, Anisotropic Sobolev spaces.

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1. INTRODUCTION

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\}$$

where T is a finite positive number, while φ_1 and φ_2 are Lipschitz continuous real-valued functions defined on $[0, T]$, and such that

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0$$

for $t \in \,]0, T]$. For fixed positive numbers $b_i, i = 1, ..., N-1$, with $N > 1$, let Q be the $(N+1)$-dimensional domain defined by

$$Q = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times \prod_{i=1}^{N-1} \,]0, b_i[\,.$$

In Q, consider the boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f \in L^2(Q), \\ \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0, \ i = 1{,}2, \\ u|_{\partial Q \setminus (\Sigma_i \cup \Sigma_T)} = 0, \ i = 1{,}2, \end{cases} \tag{1}$$

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where $\Delta u = \sum_{k=1}^{N} \partial_{x_k}^2 u$, ∂Q is the of boundary of Q, Σ_i, $i = 1{,}2$ is the part of ∂Q where $x_1 = \varphi_i(t)$, $i = 1{,}2$, Σ_T is the part of ∂Q where $t = T$ and with the fundamental hypothesis $\varphi(0) = 0$.

The difficulty related to this kind of problems comes from this singular situation for evolution problems, i.e., φ_1 is allowed to coincide with φ_2 for $t = 0$, which prevent the domain Q to be transformed into a regular domain by means of a smooth transformation, see for example Sadallah [2]. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a set measure zero.

We are especially interested in the question of what sufficient conditions, as weak as possible, the functions φ_1, φ_2 and the coefficients β_i, $i = 1, 2$, must verify in order that Problem (1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_\gamma^{1,2}(Q) = \left\{ u \in H^{1,2}(Q) : u|_{\partial Q \setminus (\Sigma_i \cup \Sigma_T)} = \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0,\, i = 1{,}2 \right\}$$

with

$$H^{1,2}(Q) = \left\{ u \in L^2(Q) : \partial_t u, \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u \in L^2(Q), 1 \leq i_1 + i_2 + ... + i_N \leq 2 \right\}.$$

Note that the Robin type condition $\partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0$, $i = 1{,}2$ is a perturbation by β_i, $i = 1{,}2$ of the Neumann type one and it is well known that Dirichlet and Neumann type boundary conditions correspond to two extreme cases, namely $\beta_i = \infty$ and $\beta_i = 0$, $i = 1{,}2$, respectively. We can find in [3], [4], [5], [6], [7], [8] and [9] solvability results of this kind of problems with Dirichlet boundary conditions. In Nazarov [10], results for the Neumann problem in a conical domain were proved. We can find in Savaré [11] an abstract study for parabolic problems with mixed (Dirichlet-Neumann) lateral boundary conditions. The case of Robin type conditions in a non-rectangular domain is studied in [12].

The organization of this paper is as follows. In Section 2, we prove that Problem (1) admits a (unique) solution in the case of a truncated domain. In Section 3 we approximate Q by a sequence (Q_n) of such domains and we establish (for T small enough) a uniform estimate of the type

$$\|u_n\|_{H^{1,2}(Q_n)} \leq K \|f\|_{L^2(Q_n)},$$

where u_n is the solution of Problem (1) in Q_n and K is a constant independent of n. Finally, in Section 4 we prove the two main results of this paper.

The main assumptions on the functions φ_1, φ_2 and on the coefficients β_i, $i = 1, 2$, are

$$\varphi_i'(t)\varphi(t) \quad \to \quad 0 \qquad \text{as } t \to 0, \quad i = 1, 2. \tag{2}$$

The coefficients β_i, $i = 1, 2$ are real numbers such that

$$\beta_1 < 0 \text{ and } \beta_2 > 0, \tag{3}$$

$$(-1)^i \left(\beta_i - \frac{\varphi_i'(t)}{2} \right) \geq 0 \quad \text{a.e. } t \in]0, T[,\, i = 1{,}2. \tag{4}$$

2. Resolution of the problem (1) in truncated domains Q_n

In this section, we replace Q by $Q_n, n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$Q_n = \left\{ (t, x) \in Q : \frac{1}{n} < t < T \right\},$$

where $x = (x_1, x_2, ..., x_N)$.

Theorem 2.1. *Under the assumptions (3) and (4) on the functions of parametrization* φ_i *and on the coefficients* $\beta_i, i = 1, 2$, *and for each* $n \in \mathbb{N}^*$ *such that* $\frac{1}{n} < T$, *the following problem admits a (unique) solution* $u_n \in H^{1,2}(Q_n)$

$$\begin{cases} \partial_t u_n - \Delta u_n = f_n \in L^2(Q_n), \\ \partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,n}} = 0, \ i = 1,2, \\ u_n|_{\partial Q_n \setminus (\Sigma_{i,n} \cup \Sigma_{T,n})} = 0, \ i = 1,2. \end{cases} \tag{5}$$

Here

$$\Sigma_{i,n} = \left\{ (t, \varphi_i(t)) \in \mathbb{R}^2 : \frac{1}{n} < t < T \right\} \times \prod_{k=1}^{N-1}]0, b_k[\ , \ i = 1, \ 2$$

and $\Sigma_{T,n}$ *is the part of the boundary of* Q_n *where* $t = T$.

Proof. The uniqueness of the solution is easy to check, thanks to (4). Let us prove its existence. The change of variables

$$\Phi : (t, x) \longmapsto (t, y) = \left(t, \frac{x_1 - \varphi_1(t)}{\varphi(t)}, x' \right)$$

transforms Q_n into the cylinder $P_n = \left] \frac{1}{n}, T \right[\times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$. Here and in the sequel $x = (x_1, x_2, ..., x_N)$, $x' = (x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$. Putting

$$w_n(t, y) = u_n(t, x) \text{ and } g_n(t, y) = f_n(t, x),$$

then Problem (5) is transformed, in P_n into the variable-coefficient parabolic problem

$$\begin{cases} \partial_t w_n + a(t, y_1) \partial_{y_1} w_n - \dfrac{1}{b^2(t)} \partial^2_{y_1} w_n - \sum_{k=2}^N \partial^2_{y_k} w_n = g_n, \\ \partial_{y_1} w_n + \beta_i \varphi(t) w_n|_{\Sigma_{i,P_n}} = 0, \ i = 1,2, \\ w_n|_{\partial P_n \setminus (\Sigma_{i,P_n} \cup \Sigma_{T,P_n})} = 0, \ i = 1,2, \end{cases} \tag{6}$$

where $\Sigma_{1,P_n} =]0, T[\times \{0\} \times \prod_{k=1}^{N-1}]0, b_k[$, $\Sigma_{2,P_n} =]0, T[\times \{1\} \times \prod_{k=1}^{N-1}]0, b_k[$, $\Sigma_{T,P_n} = \{T\} \times]0, 1[\times \prod_{k=1}^{N-1}]0, b_k[$, $b(t) = \varphi(t)$ and $a(t, y_1) = -\dfrac{y_1 \varphi'(t) + \varphi_1'(t)}{\varphi(t)}$.

Since the functions a and φ are bounded when $t \in \left] \frac{1}{n}, T \right[$, then the above change of variables which is $(N+1)$-Lipschitz preserves the spaces $H^{1,2}$ and L^2. In other words

$$f_n \in L^2(Q_n) \Leftrightarrow g_n \in L^2(P_n), \ u_n \in H^{1,2}(Q_n) \Leftrightarrow w_n \in H^{1,2}(P_n).$$

In the sequel, the variables (t, y) will be denoted again by (t, x). Consider the simplified problem

$$\begin{cases} \partial_t w_n - \dfrac{1}{b^2(t)} \partial^2_{x_1} w_n - \sum_{k=2}^N \partial^2_{x_k} w_n = g_n, \\ \partial_{x_1} w_n + \beta_i \varphi(t) w_n|_{\Sigma_{i,P_n}} = 0, \ i = 1,2, \\ w_n|_{\partial P_n \setminus (\Sigma_{i,P_n} \cup \Sigma_{T,P_n})} = 0, \ i = 1,2. \end{cases} \tag{7}$$

Lemma 2.1. *For each* $n \in \mathbb{N}^*$ *such that* $\frac{1}{n} < T$ *and for every* $g_n \in L^2(P_n)$, *there exists a unique* $w_n \in H^{1,2}(P_n)$ *solution of (7).*

Proof. Since the coefficient $b(t)$ is continuous in $\overline{P_n}$, the optimal regularity result is given by Ladyzhenskaya-Solonnikov-Ural'tseva [13]. □

Lemma 2.2. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact*

$$a(t, x_1)\,\partial_{x_1} : H^{1,2}_\gamma(P_n) \longrightarrow L^2_\omega(P_n).$$

Here, for $i = 1, 2$

$$H^{1,2}_\gamma(P_n) = \{w_n \in H^{1,2}(P_n) : w_n|_{\partial P_n \setminus (\Sigma_{i,P_n} \cup \Sigma_{T,P_n})} = \partial_{x_1} w_n + \beta_i \varphi(t)\, w_n|_{\Sigma_{i,P_n}} = 0\}.$$

Proof. P_n has the "horn property" of Besov [14], so

$$\partial_{x_1} : H^{1,2}_\gamma(P_n) \longrightarrow H^{\frac{1}{2},1}(P_n),\ w_n \longmapsto \partial_{x_1} w_n,$$

is continuous. Since P_n is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(P_n)$ into $L^2(P_n)$, where

$$H^{\frac{1}{2},1}(P_n) = L^2\left(\frac{1}{n}, T; H^1\left(]0,1[\times \prod_{i=1}^{N-1}]0, b_i[\right)\right) \cap H^{\frac{1}{2}}\left(\frac{1}{n}, T; L^2\left(]0,1[\times \prod_{i=1}^{N-1}]0, b_i[\right)\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [15]. Consider the composition

$$\partial_{x_1} : H^{1,2}_\gamma(P_n) \to H^{\frac{1}{2},1}(P_n) \to L^2(P_n),\ w_n \mapsto \partial_{x_1} w_n \mapsto \partial_{x_1} w_n,$$

then, ∂_{x_1} is a compact operator from $H^{1,2}_\gamma(P_n)$ into $L^2(P_n)$. Since $a(.,.)$ is a bounded function for $\frac{1}{n} < t < T$, the operator $a\partial_{x_1}$ is also compact from $H^{1,2}_\gamma(P_n)$ into $L^2(P_n)$. □

Lemma 2.1 shows that the operator $\partial_t - \dfrac{1}{b^2(.)}\partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k}$ is an isomorphism from $H^{1,2}_\gamma(P_n)$ into $L^2(P_n)$. On the other hand, the operator $a\partial_{x_1}$ is compact (see Lemma 2.2). Consequently, the operator $\partial_t + a(.,.)\,\partial_{x_1} - \dfrac{1}{b^2(.)}\partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k}$ is a Fredholm operator from $H^{1,2}_\gamma(P_n)$ into $L^2(P_n)$. Thus the invertibility of $\partial_t + a(.,.)\,\partial_{x_1} - \dfrac{1}{b^2(.)}\partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k}$ follows from its injectivity.

Let $w_n \in H^{1,2}_\gamma(P_n)$ be a solution of

$$\partial_t w_n + a(t, x_1)\,\partial_{x_1} w_n - \frac{1}{b^2(t)}\partial^2_{x_1} w_n - \sum_{k=2}^N \partial^2_{x_k} w_n = 0$$

in P_n. We perform the inverse change of variable of Φ. Thus we set

$$u_n = w_n \circ \Phi.$$

It turns out that $u_n \in H^{1,2}_\gamma(Q_n)$, and

$$\partial_t u_n - \Delta u_n = 0, \text{in } Q_n.$$

In addition u_n fulfils the boundary conditions

$$\partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,n}} = u_n|_{\partial Q_n \setminus (\Sigma_{i,n} \cup \Sigma_{T,n})} = 0,\ i = 1{,}2,$$

which imply that u_n vanishes (see Theorem 4.1); this is the desired injectivity and ends the proof of Theorem 2.1.

Lemma 2.3. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$W = \left\{u_n \in D\left(\left[\frac{1}{n}, T\right]; H^4\left(]0,1[\times \prod_{i=1}^{N-1}]0, b_i[\right)\right) : \partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,P_n}} = 0,\ i = 1{,}2\right\},$$

(see [15, p.13]*), is dense in*

$$H_{\gamma}^{1,2}(P_n) = \left\{ u_n \in H^{1,2}(P_n) : \partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,P_n}} = 0,\ i = 1,2 \right\}.$$

The above lemma is a particular case of [15, Theorem 2.1], from which, we can derive the following result in order to justify the calculus of the section 3.

Lemma 2.4. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$\left\{ u_n \in H^4(P_n) : u_n|_{\partial P_n \setminus \left(\Sigma_{i,P_n} \cup \Sigma_{T,P_n}\right)} = \partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,P_n}} = 0,\ i = 1,2 \right\}$$

is dense in the space

$$\left\{ u_n \in H^{1,2}(P_n) : u_n|_{\partial P_n \setminus \left(\Sigma_{i,P_n} \cup \Sigma_{T,P_n}\right)} = \partial_{x_1} u_n + \beta_i u_n|_{\Sigma_{i,P_n}} = 0,\ i = 1,2 \right\}.$$

Remark 2.1. *In Lemma 2.4, we can replace P_n by Q_n with the help of the change of variables defined above.*

3. A uniform estimate

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote by $u_n \in H^{1,2}(Q_n)$ the solution of Problem (5) in Q_n. Such a solution u_n exists by Theorem 2.1.

Theorem 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$ with T small enough, there exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{H^{1,2}(Q_n)} \le K \|f_n\|^2_{L^2(Q_n)} \le K \|f\|^2_{L^2(Q)},$$

where

$$\|u_n\|_{H^{1,2}(Q_n)} = \sqrt{\|\partial_t u_n\|^2_{L^2(Q_n)} + \|u_n\|^2_{L^2(Q_n)} + \sum_{\substack{i_1,\dots,i_N=0 \\ 1 \le i_1+\dots+i_N \le 2}}^{2} \left\|\partial_{x_1}^{i_1} \dots \partial_{x_N}^{i_N} u_n\right\|^2_{L^2(Q_n)}}.$$

In order to prove Theorem 3.1, we need some preliminary results. The proof of the following Lemma can be found in [1].

Lemma 3.1. *Under the assumption (3) on $(\beta_i)_{i=1,2}$, there exists a positive constant C_1 (independent of a and b) such that*

$$\left\|v^{(k)}\right\|^2_{L^2(a,b)} \le C_1 (b-a)^{2(2-k)} \left\|v^{(2)}\right\|^2_{L^2(a,b)},\ k = 0,\ 1,$$

for each $v \in H_{\gamma}^2(a,b)$, with

$$H_{\gamma}^2(a,b) = \left\{ v \in H^2(a,b) : v'(a) + \frac{\beta_1}{b-a} v(a) = 0, v'(b) + \frac{\beta_2}{b-a} v(b) = 0 \right\}.$$

Lemma 3.2. *For every $\epsilon > 0$ chosen such that $\varphi(t) \le \epsilon$, there exists a constant $C > 0$ independent of n, such that*

$$\left\|\partial_{x_1}^j u_n\right\|^2_{L^2(Q_n)} \le C\epsilon^{2(2-j)} \left\|\partial_{x_1}^2 u_n\right\|^2_{L^2(Q_n)},\ j = 0,1.$$

Proof. Replacing in Lemma 3.1 v by u_n and $]a,b[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^j u_n\right)^2 dx_1 &\le C\varphi(t)^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^2 u_n\right)^2 dx_1 \\ &\le C\epsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^2 u_n\right)^2 dx_1 \end{aligned}$$

where C is the constant of Lemma 3.1. Integrating with respect to t, then with respect to x_2, x_3,..., x_N, we obtain the desired estimates. □

Proposition 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$ with T small enough, there exists a constant $C > 0$ independent of n such that*

$$\|\partial_t u_n\|^2_{L^2(Q_n)} + \sum_{\substack{i_1,i_2,...,i_N=0\\ i_1+i_2+...+i_N=2}}^{2} \left\|\partial^{i_1}_{x_1}\partial^{i_2}_{x_2}...\partial^{i_N}_{x_N} u_n\right\|^2_{L^2(Q_n)} \le C \|f\|^2_{L^2(Q)} .$$

Then, Theorem 3.1 is a direct consequence of Lemma 3.2 and Proposition 3.1, since ϵ is independent of n.

Proof. **Step 1.** First, we estimate the inner products

$$\sum_{k=1}^{N} \langle \partial_t u_n, \partial^2_{x_k} u_n\rangle \text{ and } \langle \sum_{k=1}^{N} \partial^2_{x_k} u_n, \sum_{j=1}^{N} \partial^2_{x_j} u_n\rangle, k \neq j$$

in $L^2(Q_n)$ making use of the boundary conditions (particulary, of the relation $\partial_{x_1} u_n + \beta_i u_n = 0$ on the parts of the boundary of Q_n where $x_1 = \varphi_i(t)$, $i = 1,2$). We use these estimates (step2) when we develop the expression of $\|f_n\|^2_{L^2(Q_n)}$.

1) Estimation of $-2\langle \partial_t u_n, \partial^2_{x_1} u_n\rangle$: We have

$$\partial_t u_n \partial^2_{x_1} u_n \;=\; \partial_{x_1}(\partial_t u_n \partial_{x_1} u_n) - \tfrac{1}{2}\partial_t(\partial_{x_1} u_n)^2 .$$

Then

$$\begin{aligned} -2\langle \partial_t u_n, \partial^2_{x_1} u_n\rangle &= -2\int_{Q_n} \partial_{x_1}(\partial_t u_n \partial_{x_1} u_n)\, dt\, dx + \int_{Q_n} \partial_t (\partial_{x_1} u_n)^2\, dt\, dx \\ &= \int_{\partial Q_n} \left[(\partial_{x_1} u_n)^2 \nu_t - 2\partial_t u_n \partial_{x_1} u_n \nu_{x_1}\right] d\sigma, \end{aligned}$$

where $\nu_t, \nu_{x_1}, ..., \nu_{x_N}$ are the components of the unit outward normal vector at ∂Q_n and $dx = dx_1 dx_2 ... dx_N$. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \frac{1}{n}$, $x_k = 0, k = 2, ..., N$ and $x_k = b_{k-1}, k = 2, ..., N$ we have $u_n = 0$ and consequently $\partial_{x_1} u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_{x_1} u_n)^2\, dx$$

is nonnegative. On the parts of the boundary where $x_1 = \varphi_i(t)$, $i = 1,2$, we have

$$\nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}}, \; \nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}}$$

and

$$\partial_{x_1} u_n\left(t, \varphi_i(t), x'\right) + \beta_i u_n\left(t, \varphi_i(t), x'\right) = 0,\; i = 1,2.$$

Consequently the corresponding boundary integral is

$$\begin{aligned} I_{n,k} &= (-1)^{k+1} \int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^{T} \varphi_k'(t) \left[\partial_{x_1} u_n(t, \varphi_k(t), x')\right]^2 dt dx', \; k = 1, 2, \\ J_{n,k} &= (-1)^k\, 2 \int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^{T} \beta_k (\partial_t u_n . u_n)(t, \varphi_k(t), x')\, dt dx', \; k = 1, 2, \end{aligned}$$

where $dx' = dx_2 ... dx_N$. Then, we have

$$-2\langle \partial_t u_n, \partial^2_{x_1} u_n\rangle \ge -|I_{n,1}| - |I_{n,2}| - |J_{n,1}| - |J_{n,2}| . \tag{8}$$

2) Estimation of $-2\sum_{k=2}^{N}\langle \partial_t u_n, \partial^2_{x_k} u_n\rangle$: We have

$$\partial_t u_n \partial^2_{x_k} u_n \;=\; \partial_{x_k}(\partial_t u_n \partial_{x_k} u_n) - \tfrac{1}{2}\partial_t(\partial_{x_k} u_n)^2 .$$

Then

$$\begin{aligned}-2\langle \partial_t u_n, \partial_{x_k}^2 u_n\rangle &= -2\int_{Q_n} \partial_{x_k}\left(\partial_t u_n \partial_{x_k} u_n\right)\, dt\, dx + \int_{Q_n} \partial_t \left(\partial_{x_k} u_n\right)^2\, dt\, dx \\ &= \int_{\partial Q_n} \left[\left(\partial_{x_k} u_n\right)^2 \nu_t - 2\partial_t u_n \partial_{x_k} u_n \nu_{x_k}\right] d\sigma.\end{aligned}$$

On the part of the boundary where $t = \frac{1}{n}$, $x_k = 0, k = 2, ..., N$ and $x_k = b_{k-1}, k = 2, ..., N$ we have $u_n = 0$ and consequently $\partial_{x_k} u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$, $\nu_{x_k} = 0, k = 2, ..., N$ and $\nu_t = 1$. The corresponding boundary integral

$$\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \left(\partial_{x_k} u_n\right)^2\, dx$$

is nonnegative. On the parts of the boundary of Q_n where $x_1 = \varphi_i(t)$, $i = 1, 2$, we have $\nu_{x_1} = \frac{(-1)^i}{\sqrt{1+(\varphi_i')^2(t)}}$, $\nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}}$ and $\nu_{x_k} = 0, k = 2, ..., N$. Consequently the corresponding boundary integral is

$$M_{n,j} = (-1)^{j+1} \int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^{T} \varphi_j'(t) \left[\partial_{x_k} u_n\left(t, \varphi_j(t), x'\right)\right]^2 dt dx', \, j = 1, 2.$$

Then, we have

$$-2\langle \partial_t u_n, \partial_{x_k}^2 u_n\rangle \geq M_{n,1} + M_{n,2},\, k = 2, ..., N. \tag{9}$$

3) Estimation of $2\sum_{k=2}^N \langle \partial_{x_1}^2 u_n, \partial_{x_k}^2 u_n\rangle$: We have

$$\partial_{x_1}^2 u_n.\partial_{x_k}^2 u_n = \partial_{x_1}\left(\partial_{x_1} u_n.\partial_{x_k}^2 u_n\right) - \partial_{x_k}\left(\partial_{x_1} u_n.\partial_{x_1}\partial_{x_k} u_n\right) + \left(\partial_{x_1}\partial_{x_k} u_n\right)^2.$$

Then

$$\begin{aligned}2\langle \partial_{x_1}^2 u_n, \partial_{x_k}^2 u_n\rangle &= 2\int_{Q_n} \partial_{x_1}\left(\partial_{x_1} u_n.\partial_{x_k}^2 u_n\right) dt\, dx - 2\int_{Q_n} \partial_{x_k}\left(\partial_{x_1} u_n.\partial_{x_1}\partial_{x_k} u_n\right) dt\, dx \\ &\quad +2\int_{Q_n} \left(\partial_{x_1}\partial_{x_k} u_n\right)^2 dt\, dx \\ &= 2\int_{Q_n} \left(\partial_{x_1}\partial_{x_k} u_n\right)^2 dt\, dx \\ &\quad +2\int_{\partial Q_n} \left[\partial_{x_1} u_n \partial_{x_k}^2 u_n \nu_{x_1} - \partial_{x_1} u_n.\partial_{x_1}\partial_{x_k} u_n \nu_{x_k}\right] d\sigma.\end{aligned}$$

On the part of the boundary where $t = \frac{1}{n}$, $x_k = 0, k = 2, ..., N$ and $x_k = b_{k-1}, k = 2, ..., N$ we have $u_n = 0$ and consequently $\partial_{x_k} u_n = 0$. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$, $\nu_{x_k} = 0, k = 2, ..., N$ and $\nu_t = 1$. The corresponding boundary integral vanishes. On the parts of the boundary of Q_n where $x_1 = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}},\ \nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}} \text{ and } \nu_{x_k} = 0, k = 2, ..., N$$

and

$$\partial_{x_1} u_n\left(t, \varphi_i(t), x'\right) + \beta_i u_n\left(t, \varphi_i(t), x'\right) = 0,\ i = 1{,}2.$$

Consequently, the corresponding boundary integral is

$$H_{n,j} = (-1)^j\, 2\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^{T} \beta_k \left[\partial_{x_k} u_n\left(t, \varphi_j(t), x'\right)\right]^2 dt dx', \, j = 1, 2.$$

Then, we have

$$2\langle \partial_{x_1}^2 u_n, \partial_{x_k}^2 u_n\rangle = 2\left\|\partial_{x_1}\partial_{x_k} u_n\right\|_{L^2(Q_n)}^2 + H_{n,1} + H_{n,2}. \tag{10}$$

Summing up the estimates (9) and (10) and using the hypothesis (4), we obtain

$$-2\langle \partial_t u_n, \partial_{x_k}^2 u_n\rangle + 2\langle \partial_{x_1}^2 u_n, \partial_{x_k}^2 u_n\rangle \geq 2\left\|\partial_{x_1}\partial_{x_k} u_n\right\|_{L^2(Q_n)}^2, k = 2, ..., N. \tag{11}$$

Indeed, for $k = 2, ..., N$ we have

$$\sum_{j=1}^{2} M_{n,j} + H_{n,j} = \sum_{j=1}^{2} \int_0^{b_{N-1}} .. \int_0^{b_1} \int_{\frac{1}{n}}^{T} (-1)^k \left(2\beta_j - \varphi_j'(t)\right) \left[\partial_{x_k} u_n \left(t, \varphi_j(t), x'\right)\right]^2 dt dx',$$

which is nonnegative, thanks to the hypothesis (4). By a similar argument, we obtain

$$\begin{array}{l} 2\langle \partial_{x_2}^2 u_n, \partial_{x_k}^2 u_n \rangle \geq 2 \left\| \partial_{x_2} \partial_{x_k} u_n \right\|_{L^2(Q_n)}^2, k = 3, ..., N, \\ 2\langle \partial_{x_3}^2 u_n, \partial_{x_k}^2 u_n \rangle \geq 2 \left\| \partial_{x_3} \partial_{x_k} u_n \right\|_{L^2(Q_n)}^2, k = 4, ..., N, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 2\langle \partial_{x_{N-1}}^2 u_n, \partial_{x_N}^2 u_n \rangle \geq 2 \left\| \partial_{x_{N-1}} \partial_{x_N} u_n \right\|_{L^2(Q_n)}^2. \end{array} \tag{12}$$

Step 2. Estimation of $I_{n,k}$, $J_{n,k}$: We have

$$\begin{aligned} \|f_n\|_{L^2(Q_n)}^2 &= \langle \partial_t u_n - \textstyle\sum_{k=1}^N \partial_{x_k}^2 u, \partial_t u_n - \sum_{k=1}^N \partial_{x_k}^2 u \rangle \\ &= \|\partial_t u_n\|_{L^2(Q_n)}^2 + \textstyle\sum_{k=1}^N \left\|\partial_{x_k}^2 u_n\right\|_{L^2(Q_n)}^2 \\ &\quad -2 \textstyle\sum_{k=1}^N \langle \partial_t u_n, \partial_{x_k}^2 u_n \rangle + 2 \sum_{k=2}^N \langle \partial_{x_1}^2 u_n, \partial_{x_k}^2 u_n \rangle \\ &\quad +2 \textstyle\sum_{k=3}^N \langle \partial_{x_2}^2 u_n, \partial_{x_k}^2 u_n \rangle + ... + 2\langle \partial_{x_{N-1}}^2 u_n, \partial_{x_N}^2 u_n \rangle. \end{aligned}$$

It is the reason for which we look for an estimate of the type

$$|I_{n,1}| + |I_{n,2}| + |J_{n,1}| + |J_{n,2}| \leq K\epsilon \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2.$$

A. Estimation of $I_{n,k}$, $k = 1,2$ □

Lemma 3.3. *There exists a constant $K > 0$ independent of n such that*

$$|I_{n,k}| \leq K\epsilon \left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2, \quad k = 1,2.$$

Proof. We convert the boundary integral $I_{n,1}$ into a surface integral by setting

$$\begin{aligned} \left[\partial_{x_1} u_n (t, \varphi_1(t), x')\right]^2 &= -\tfrac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} \left[\partial_{x_1} u_n(t,x)\right]^2 \Big|_{x_1 = \varphi_1(t)}^{x_1 = \varphi_2(t)} \\ &= -\textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_{x_1} \left\{ \tfrac{\varphi_2(t) - x_1}{\varphi(t)} \left[\partial_{x_1} u_n(t,x)\right]^2 \right\} dx_1 \\ &= \textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} [-2 \tfrac{\varphi_2(t) - x_1}{\varphi(t)} \partial_{x_1} u_n(t,x) \partial_{x_1}^2 u_n(t,x) + \tfrac{1}{\varphi(t)} \left[\partial_{x_1} u_n\right]^2] dx_1. \end{aligned}$$

Then, we have

$$\begin{aligned} I_{n,1} &= \textstyle\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_1'(t) \left[\partial_{x_1} u_n (t, \varphi_1(t), x')\right]^2 \ dt dx' \\ &= \textstyle\int_{Q_n} \tfrac{\varphi_1'(t)}{\varphi(t)} \left(\partial_{x_1} u_n\right)^2 dt dx + 2 \int_{Q_n} \tfrac{\varphi_2(t) - x_1}{\varphi(t)} \varphi_1'(t) \left(\partial_{x_1} u_n\right) \left(\partial_{x_1}^2 u_n\right) \ dt dx. \end{aligned}$$

Thanks to Lemma 3.2, we can write

$$\textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1} u_n(t,x)\right]^2 dx_1 \leq C \left[\varphi(t)\right]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^2 u_n(t,x)\right]^2 dx_1.$$

Therefore

$$\textstyle\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1} u_n(t,x)\right]^2 \tfrac{|\varphi_1'|}{\varphi} dx_1 \leq C \left|\varphi_1'\right| \varphi \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^2 u_n(t,x)\right]^2 dx_1,$$

consequently,

$$|I_{n,1}| \leq C \int_{Q_n} \left|\varphi_1'\right| \varphi \left(\partial_{x_1}^2 u_n\right)^2 dt dx + 2 \int_{Q_n} \left|\varphi_1'\right| \left|\partial_{x_1} u_n\right| \left|\partial_{x_1}^2 u_n\right| \ dt dx,$$

since $\left|\frac{\varphi_2(t)-x_1}{\varphi(t)}\right| \leq 1$. Using the inequality

$$2\left|\varphi_1'\partial_{x_1}u_n\right|\left|\partial_{x_1}^2 u_n\right| \leq \epsilon\left(\partial_{x_1}^2 u_n\right)^2+\tfrac{1}{\epsilon}\left(\varphi_1'\right)^2\left(\partial_{x_1}u_n\right)^2$$

for all $\epsilon > 0$, we obtain

$$|I_{n,1}| \leq C\int_{Q_n}\left|\varphi_1'\right|\varphi(t)\left(\partial_{x_1}^2 u_n\right)^2 dtdx+\int_{Q_n}[\epsilon\left(\partial_{x_1}^2 u_n\right)^2+\frac{1}{\epsilon}\left(\varphi_1'\right)^2\left(\partial_{x_1}u_n\right)^2]dtdx.$$

Lemma 3.2 yields

$$\tfrac{1}{\epsilon}\int_{Q_n}\left(\varphi_1'\right)^2\left(\partial_{x_1}u_n\right)^2 dtdx \leq C\tfrac{1}{\epsilon}\int_{Q_n}\left(\varphi_1'\right)^2\varphi(t)^2\left(\partial_{x_1}^2 u_n\right)^2 dtdx.$$

Thus, there exists a constant $K > 0$ independent of n such that

$$\begin{aligned}|I_{n,1}| &\leq C\int_{Q_n}\left[\left|\varphi_1'\right|\varphi(t)+\tfrac{1}{\epsilon}\left(\varphi_1'\right)^2\varphi(t)^2\right]\left(\partial_{x_1}^2 u_n\right)^2 dtdx+\int_{Q_n}\epsilon\left(\partial_{x_1}^2 u_n\right)^2 dtdx\\ &\leq K\epsilon\int_{Q_n}\left(\partial_{x_1}^2 u_n\right)^2 dtdx,\end{aligned}$$

because $\left|\varphi_1'\varphi(t)\right| \leq \epsilon$. The inequality

$$|I_{n,2}| \leq K\epsilon\left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2,$$

can be proved by a similar argument.

B. Estimation of $J_{n,k}$, $k = 1,2$: We have

$$\begin{aligned}J_{n,1} &= -2\int_0^{b_{N-1}}...\int_0^{b_1}\int_{\frac{1}{n}}^T\beta_1\partial_t u_n\left(t,\varphi_1(t),x'\right).u_n\left(t,\varphi_1(t),x'\right)dtdx'\\ &= -\int_0^{b_{N-1}}...\int_0^{b_1}\int_{\frac{1}{n}}^T\beta_1\left[\partial_t u_n^2\left(t,\varphi_1(t),x'\right)\right]dtdx'.\end{aligned}$$

By setting, for each fixed x' in $\prod_{i=1}^{N-1}]0,b_i[$, $h(t)=u_n^2\left(t,\varphi_1(t),x'\right)$, we obtain

$$\begin{aligned}J_{n,1} &= -\int_0^{b_{N-1}}...\int_0^{b_1}\int_{\frac{1}{n}}^T\beta_1.\left[h'(t)-\varphi_1'(t)\partial_{x_1}u_n^2\left(t,\varphi_1(t),x'\right)\right]dtdx'\\ &= \int_0^{b_{N-1}}...\int_0^{b_1}\int_{\frac{1}{n}}^T\beta_1.\varphi_1'(t)\partial_{x_1}u_n^2\left(t,\varphi_1(t),x'\right)dtdx'+\int_0^{b_{N-1}}...\int_0^{b_1}-\beta_1.h(t)|_{\frac{1}{n}}^T dx'.\end{aligned}$$

Since β_1 is negative and $u_n^2\left(\frac{1}{n},\varphi_1\left(\frac{1}{n}\right),x'\right)=0$, we have $\int_0^{b_{N-1}}...\int_0^{b_1}-\beta_1.h(t)|_{\frac{1}{n}}^T dx' \geq 0$. The last boundary integral in the expression of $J_{n,1}$ can be treated by a similar argument used in Lemma 3.3. So, we obtain the existence of a positive constant K independent of n, such that

$$\left|\int_0^{b_{N-1}}...\int_0^{b_1}\int_{\frac{1}{n}}^T\beta_1.\varphi_1'(t)\partial_{x_1}u_n^2\left(t,\varphi_1(t),x'\right)dtdx'\right| \leq K\epsilon\left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2,$$

and consequently,

$$|J_{n,1}| \geq -K\epsilon\left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2. \tag{13}$$

By a similar method and using the fact that β_2 is positive and $u_n^2\left(\frac{1}{n},\varphi_2\left(\frac{1}{n}\right),x'\right)=0$, we obtain the existence of a positive constant K independent of n, such that

$$|J_{n,2}| \geq -K\epsilon\left\|\partial_{x_1}^2 u_n\right\|_{L^2(Q_n)}^2. \tag{14}$$

Summing up the estimates (8), (11), (12), (13), (14) and making use of Lemma 3.2, we then obtain

$$\begin{aligned}
\|f_n\|^2_{L^2(Q_n)} \geq\ & \|\partial_t u_n\|^2_{L^2(Q_n)} + \textstyle\sum_{k=1}^N \left\|\partial^2_{x_k} u_n\right\|^2_{L^2(Q_n)} - 4K\epsilon \left\|\partial^2_{x_1} u_n\right\|^2_{L^2(Q_n)} \\
& +2\textstyle\sum_{k=2}^N \|\partial_{x_1}\partial_{x_k} u_n\|^2_{L^2(Q_n)} + 2\textstyle\sum_{k=3}^N \|\partial_{x_2}\partial_{x_k} u_n\|^2_{L^2(Q_n)} \\
& +2\textstyle\sum_{k=4}^N 2\|\partial_{x_3}\partial_{x_k} u_n\|^2_{L^2(Q_n)} + ... + 2\left\|\partial_{x_{N-1}}\partial_{x_N} u_n\right\|^2_{L^2(Q_n)} \\
\geq\ & \|\partial_t u_n\|^2_{L^2(Q_n)} + (1-4K_4\epsilon)\left\|\partial^2_{x_1} u_n\right\|^2_{L^2(Q_n)} + \textstyle\sum_{k=2}^N \left\|\partial^2_{x_k} u_n\right\|^2_{L^2(Q_n)} \\
& +2\textstyle\sum_{k=2}^N \|\partial_{x_1}\partial_{x_k} u_n\|^2_{L^2(Q_n)} + 2\textstyle\sum_{k=3}^N \|\partial_{x_3}\partial_{x_k} u_n\|^2_{L^2(Q_n)} \\
& +2\textstyle\sum_{k=4}^N 2\|\partial_{x_3}\partial_{x_k} u_n\|^2_{L^2(Q_n)} + ... + 2\left\|\partial_{x_{N-1}}\partial_{x_N} u_n\right\|^2_{L^2(Q_n)}.
\end{aligned}$$

Then, it is sufficient to choose ϵ such that $(1-4K\epsilon) > 0$, to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|^2_{L^2(Q_n)} \geq K_0 \left(\|\partial_t u_n\|^2_{L^2(Q_n)} + \sum_{\substack{i_1,i_2,...,i_N=0 \\ i_1+i_2+...+i_N=2}}^{2} \left\|\partial^{i_1}_{x_1}\partial^{i_2}_{x_2}...\partial^{i_N}_{x_N} u_n\right\|^2_{L^2(Q_n)} \right).$$

But $\|f_n\|_{L^2(Q_n)} \leq \|f\|_{L^2(Q)}$, then, there exists a constant $C > 0$, independent of n satisfying

$$\|\partial_t u_n\|^2_{L^2(Q_n)} + \sum_{\substack{i_1,i_2,...,i_N=0 \\ i_1+i_2+...+i_N=2}}^{2} \left\|\partial^{i_1}_{x_1}\partial^{i_2}_{x_2}...\partial^{i_N}_{x_N} u_n\right\|^2_{L^2(Q_n)} \leq C\|f_n\|^2_{L^2(Q_n)} \leq C\|f\|^2_{L^2(Q)}.$$

This ends the proof of Proposition 3.1. □

4. Main results

We are now able to prove the main results of the paper.

4.1. Local in time result.

Theorem 4.1. *Assume that the functions of parametrization $\varphi_i, i = 1,2$ and the coefficients $\beta_i, i = 1,2$ fulfil conditions (2), (3) and (4). Then, for T small enough, the heat operator $L = \partial_t - \Delta$ is an isomorphism from $H^{1,2}_\gamma(Q)$ into $L^2(Q)$.*

Proof. **1) Injectivity of the operator L:** Let us consider $u \in H^{1,2}_\gamma(Q)$ a solution of the problem (1) with a null right-hand side term. So,

$$\partial_t u - \Delta u = 0 \text{ in } Q.$$

In addition u fulfils the boundary conditions

$$u|_{\partial Q \setminus (\Sigma_i \cup \Sigma_T)} = 0 \text{ and } \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0,\ i = 1,2.$$

Using Green formula, we have

$$\int_Q (\partial_t u - \Delta u)\, u\, dt\, dx = \int_{\partial Q} \left(\frac{1}{2}|u|^2 \nu_t - \sum_{k=1}^N \partial_{x_k} u.u\nu_{x_k} \right) d\sigma + \int_Q \sum_{k=1}^N |\partial_{x_k} u|^2\, dt\, dx$$

where ν_t, ν_{x_1}, ..., ν_{x_N} are the components of the unit outward normal vector at ∂Q. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q where $t = 0$, $x_k = 0, k = 2,...,N$ and $x_k = b_{k-1}, k = 2,...,N$ we have $u = 0$ and consequently the corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = \nu_{x_2} = ... = \nu_{x_N} = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$A = \frac{1}{2}\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 (T, x)\, dx$$

is nonnegative. On the part of the boundary where $x_1 = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2 (t)}}, \nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2 (t)}}, \nu_{x_k} = 0, k = 2, ..., N$$

and

$$\partial_{x_1} u \left(t, \varphi_i(t), x'\right) + \beta_i u \left(t, \varphi_i(t), x'\right) = 0, i = 1, 2.$$

Consequently the corresponding boundary integral is

$$\sum_{i=1}^{2} \int_0^{b_{N-1}} .. \int_0^{b_1} \int_0^T (-1)^i \left(\beta_i - \frac{\varphi_i'(t)}{2}\right) u^2 \left(t, \varphi_i(t), x'\right) dt dx'.$$

Then, we obtain

$$\begin{aligned} \int_Q (\partial_t u - \Delta u)\, u \; dt dx &= \textstyle\sum_{i=1}^{2} \int_0^{b_{N-1}} .. \int_0^{b_1} \int_0^T (-1)^i \left(\beta_i - \frac{\varphi_i'(t)}{2}\right) u^2 (t, \varphi_i(t), x')\, dt dx' \\ &\quad + \frac{1}{2}\textstyle\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} u^2 (T, x) \; dx + \int_Q \sum_{k=1}^{N} |\partial_{x_k} u|^2 \, dt dx. \end{aligned}$$

Consequently $\int_Q (\partial_t u - \Delta u)\, u \; dt \; dx = 0$ yields the equality $\int_Q \sum_{k=1}^N |\partial_{x_k} u|^2 \, dt dx = 0$, because

$$\sum_{i=1}^{2} \int_0^{b_{N-1}} ... \int_0^{b_1} \int_0^T (-1)^i \left(\beta_i - \frac{\varphi_i'(t)}{2}\right) u^2 \left(t, \varphi_i(t), x'\right) dt dx' \geq 0$$

thanks to the hypothesis (4). This implies that $\sum_{k=1}^N |\partial_{x_k} u|^2 = 0$ and consequently $\Delta u = 0$. Then, the hypothesis $\partial_t u - \Delta u = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions and the fact that $\beta_i \neq 0$, $i = 1, 2$ imply that $u = 0$.

2) Surjectivity of the operator L: Choose a sequence Q_n, $n = 1, 2, ...$ of reference domains (see section 2). Then we have $Q_n \to Q$, as $n \to \infty$.

Consider the solution $u_n \in H^{1,2}(Q_n)$ of the Robin problem (5) in Q_n. Such a solution u_n exists by Theorem 2.1. Let $\widetilde{u_n}$ the 0−extension of u_n to Q. Then, in virtue of Theorem 3.1, we know that there exists a constant C such that

$$\|\widetilde{u_n}\|_{L^2(Q)} + \left\|\widetilde{\partial_t u_n}\right\|_{L^2(Q)} + \sum_{\substack{i_1, i_2, ..., i_N = 0 \\ 1 \leq i_1 + i_2 + ... + i_N \leq 2}}^{2} \left\|\widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u_n}\right\|_{L^2(Q_n)}^2 \leq C \|f\|_{L^2(Q)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u_n}$ for $1 \leq i_1 + i_2 + ... + i_N \leq 2$ are bounded functions in $L^2(Q)$. So for a suitable increasing sequence of integers n_k, $k = 1, 2, ...$, there exist functions

$$u, \; v \text{ and } v_{i_1, i_2, ..., i_N} \; 1 \leq i_1 + i_2 + ... + i_N \leq 2$$

in $L^2(Q)$ with $1 \leq i_1 + i_2 + ... + i_N \leq 2$ such that

$$\widetilde{u_{n_k}} \rightharpoonup u, \; \widetilde{\partial_t u_{n_k}} \rightharpoonup v, \; \widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u_{n_k}} \rightharpoonup v_{i_1, i_2, ..., i_N},$$

weakly in $L^2(Q)$ as $k \to \infty$. Clearly,

$$v = \partial_t u, \; v_{i_1, i_2, ..., i_N} = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u \;, \; 1 \leq i_1 + i_2 + ... + i_N \leq 2$$

in the sense of distributions in Q and so in $L^2(Q)$. Finally, $u \in H^{1,2}(Q)$ and $\partial_t u - \Delta u = f$ in Q. On the other hand, the solution u satisfies the boundary conditions

$$u|_{\partial Q \setminus (\Sigma_i \cup \Sigma_T)} = 0 \text{ and } \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0,\ i = 1{,}2,$$

since

$$\forall n \in \mathbb{N}^*,\ u|_{Q_n} = u_n.$$

This proves the existence of solution to Problem (1) and ends the proof of Theorem 4.1. □

4.1.1. *Global in time result.* In the case where T is not in the neighborhood of zero, we set $Q = D_1 \cup D_2 \cup \Sigma_{T_1}$ (T_1 small enough) where

$$D_1 = \{(t,x) \in Q : 0 < t < T_1\},\ D_2 = \{(t,x) \in Q : T_1 < t < T\},$$

$$\Sigma_{T_1} = \left\{(T_1, x_1) \in \mathbb{R}^2 : \varphi_1(T_1) < x_1 < \varphi_2(T_1)\right\} \times \prod_{i=1}^{N-1}]0, b_i[.$$

In the sequel, f stands for an arbitrary fixed element of $L^2(Q)$ and $f_i = f|_{D_i}$, $i = 1, 2$. Theorem 4.1 applied to the non-regular domain D_1, shows that there exists a unique solution $v_1 \in H^{1,2}(D_1)$ of the problem

$$\begin{cases} \partial_t v_1 - \Delta v_1 = f_1 \in L^2(D_1), \\ \partial_{x_1} v_1 + \beta_i v_1|_{\Sigma_{i,1}} = 0,\ i = 1{,}2, \\ v_1|_{\partial D_1 \setminus (\Sigma_{i,1} \cup \Sigma_{T_1})} = 0,\ i = 1{,}2, \end{cases} \tag{15}$$

$\Sigma_{i,1}$ are the parts of the boundary of D_1 where $x_1 = \varphi_i(t)$, $i = 1{,}2$.

Lemma 4.1. *If* $u \in H^{1,2}\left(]0,T[\times]0,1[\times \prod_{i=1}^{N-1}]0,b_i[\right)$, *then* $u|_{t=0} \in H^1(\gamma_0)$, $u|_{x_1=0} \in H^{\frac{3}{4}}(\gamma_1)$ *and* $u|_{x_1=1} \in H^{\frac{3}{4}}(\gamma_2)$, *where* $\gamma_0 = \{0\} \times]0,1[\times \prod_{i=1}^{N-1}]0,b_i[$, $\gamma_1 =]0,T[\times \{0\} \times \prod_{i=1}^{N-1}]0,b_i[$ *and* $\gamma_2 =]0,T[\times \{1\} \times \prod_{i=1}^{N-1}]0,b_i[$.

The above lemma is a particular case of [15, Theorem 2.1, Vol.2]. The transformation $(t,x) \longmapsto (t,y) = (t, \varphi(t) x_1 + \varphi_1(t), x')$, leads to the following lemma:

Lemma 4.2. *If* $u \in H^{1,2}(D_2)$, *then* $u|_{\Sigma_{T_1}} \in H^1(\Sigma_{T_1})$, $u|_{x_1=\varphi_i(t)} \in H^{\frac{3}{4}}(\Sigma_{i,2})$, *where* $\Sigma_{i,2}, i = 1, 2$ *are the parts of the boundary of* D_2 *where* $x_1 = \varphi_i(t)$.

Hereafter, we denote the trace $v_1|_{\Sigma_{T_1}}$ by ψ which is in the Sobolev space $H^1(\Sigma_{T_1})$ because $v_1 \in H^{1,2}(D_1)$ (see Lemma 4.2). Now, consider the following problem in D_2

$$\begin{cases} \partial_t v_2 - \Delta v_2 = f_2 \in L^2(Q_2), \\ v_2|_{\Sigma_{T_1}} = \psi, \\ \partial_{x_1} v_2 + \beta_i v_2|_{\Sigma_{i,2}} = 0,\ i = 1{,}2, \\ v_2|_{\partial D_2 \setminus (\Sigma_{i,2} \cup \Sigma_{T_1})} = 0\ ,\ i = 1{,}2, \end{cases} \tag{16}$$

$\Sigma_{i,2}$ are the parts of the boundary of D_2 where $x_1 = \varphi_i(t)$, $i = 1{,}2$. We use the following result, which is a consequence of [15, Theorem 4.3, Vol.2] to solve Problem (16).

Proposition 4.1. *Let* R *be the cylinder* $]0,T[\times]0,1[\times \prod_{i=1}^{N-1}]0,b_i[$, $f \in L^2(R)$ *and* $\psi \in H^1(\gamma_0)$. *Then, the problem*

$$\begin{cases} \partial_t u - \Delta u = f \text{ in } R, \\ u|_{\gamma_0} = \psi, \\ \partial_{x_1} u + \beta_i u|_{\gamma_i} = 0,\ i = 1{,}2, \\ u|_{\partial R \setminus (\gamma_0 \cup \gamma_i)} = 0,\ i = 1{,}2, \end{cases}$$

where $\gamma_0 = \{0\} \times]0,1[\times \prod_{i=1}^{N-1}]0,b_i[$, $\gamma_1 =]0,T[\times \{0\} \times \prod_{i=1}^{N-1}]0,b_i[$ *and* $\gamma_2 =]0,T[\times \{1\} \times \prod_{i=1}^{N-1}]0,b_i[$, *admits a (unique) solution* $u \in H^{1,2}(R)$.

Remark 4.1. *In the application of* [15, Theorem 4.3, Vol.2], *we can observe that there are not compatibility conditions to satisfy because* $\partial_{x_1}\psi$ *is only in* $L^2(\gamma_0)$.

Thanks to the transformation $(t,x) \longmapsto (t,y) = (t, \varphi(t)x_1 + \varphi_1(t), x')$, we deduce the following result:

Proposition 4.2. *Problem (16) admits a (unique) solution* $v_2 \in H^{1,2}(D_2)$.

So, the function u defined by

$$u = \begin{cases} v_1 \text{ in } D_1, \\ v_2 \text{ in } D_2, \end{cases}$$

is the (unique) solution of Problem (1) for an arbitrary T. Our second main result is

Theorem 4.2. *Under the assumptions (2), (3) and (4) on the functions of parametrization* φ_i *and the coefficients* $\beta_i, i = 1,2$, *Problem (1) admits a (unique) solution* $u \in H^{1,2}(Q)$.

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Arezki Kheloufi for the photography and short autobiography, see TWMS J. App. Eng. Math., V.5, N.1.
