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ON GENERALIZATION OF WEIERSTRASS APPROXIMATION THEOREM FOR A GENERAL CLASS OF POLYNOMIALS AND GENERATING FUNCTIONS

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Abstract. Here, in this work we present a generalization of the Weierstrass Approximation Theorem for a general class of polynomials. Then we generalize it for two variable continuous function $F(x,t)$ and prove that on a rectangle $[a,b]\times(-1,1)$, $a\le x\le b, |t|<1, a,b,t\in\mathbb{R}$, it uniformly converges into a generating function.As a result,we are able to apply our theorems to derive a number of generating functions.

Keywords: Weierstrass Approximation Theorem, Bernstein Approximation theorem, generalization of Weierstrass Approximation Theorem, generating functions.

AMS Subject Classification: 41A30, 33C20.

1. Introduction

Weierstrass [5] has proved his Weierstrass Approximation Theorem in the year 1885 (see also, Schep [4]). Here we put that in the following general form:
Weierstrass Approximation Theorem.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R};;a,b\in\mathbb{R};;$ be a continuous function. Then f is on interval $[a,b]$ a uniform limit of polynomials.*

In other words, assume that the function f is continuous on bounded interval $[a,b]$, given any $\in>0$, there is a polynomial $P_n(n\ge 0)$ with sufficiently high degree n such that

$$|f(x)-P_n(x)|<\in, for\ a\le x\le b, a,b\in\mathbb{R}. \tag{1}$$

Now, to obtain a general proof of the theorem 1 we define a transformation formula for a bounded uniformly continuous function $f:\mathbb{R}\ \to\mathbb{R}$ such that $H_kf(x)=\frac{1}{k}\int_{-\infty}^{\infty}f(u)g(\frac{u-x}{k})du$, where $k>0$, and $g\left(x\right)$ is a probability density function defined by

$$\int_{-\infty}^{\infty}g(x)dx=1,\ \ otherwise\ g(x)=0. \tag{2}$$

Again, in this order we prove that

Theorem 1.2. *Let $f:\mathbb{R}\ \to\mathbb{R}$ be a bounded uniformly continuous function. Then H_kf converges uniformly to f as $k>0$.*

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Proof. Let $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(x) - f(y)| < \dfrac{\epsilon}{2}$, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Assume $|f(x)| \le M,\ \forall x \in \mathbb{R}$.

Again, $g(x)$ is a probability density function so that with the help of Eqn. (2) we may write

$$f(x) = \frac{1}{k}\int_{-\infty}^{\infty} f(x) g\left(\frac{u-x}{k}\right) du, k > 0. \tag{3}$$

Now let $k_0 > 0$ such that $0 < k \le k_0 < \frac{\epsilon\delta}{4MF\left(\frac{\delta}{k_0}\right)}$, where $F\left(\frac{\delta}{k_0}\right) = \int_{\frac{\delta}{k_0}}^{\infty}(v)\,g(v)\,dv < \infty$, then on using Eqns. (2) and (3), we get

$$\begin{aligned}
|H_{k_0}f(x) - f(x)| &\le \frac{1}{k_0}\int_{-\infty}^{\infty} |f(u) - f(x)| g\left(\frac{u-x}{k_0}\right) du|_{when|u-x|<\delta.} \\
&+ \frac{1}{k_0}\int_{|u-x|\ge\delta} |f(u) - f(x)| g\left(\frac{u-x}{k_0}\right) du \\
&\le \frac{\epsilon}{2} + \frac{1}{k_0}\int_{|u-x|\ge\delta} |f(u) - f(x)| g\left(\frac{u-x}{k_0}\right) du \\
&\le \frac{\epsilon}{2} + \frac{2M}{k_0}\int_{|u-x|\ge\delta} g\left(\frac{u-x}{k_0}\right) du
\end{aligned} \tag{4}$$

Again, let $\dfrac{u-x}{k_0} = v$ so that $du = k_0 dv$, $\left|\dfrac{u-x}{k_0}\right| = |v| \ge \dfrac{\delta}{k_0}$ and $\dfrac{k_0\,|v|}{\delta} \ge 1$, also $|v| = \begin{cases} -v, v < 0, \\ v, v > 0;; \end{cases}$ then the inequality (4) becomes

$$|H_{k_0}f(x) - f(x)| \le \frac{\epsilon}{2} + 2M\int_{|v|\ge\frac{\delta}{k_0}} g(v)\,dv \le \frac{\epsilon}{2} + \frac{2Mk_0}{\delta}\int_{|v|\ge\frac{\delta}{k_0}} |v|\,g(v)\,dv$$

$$= \frac{\epsilon}{2} + \frac{2Mk_0}{\delta}\int_{-\infty}^{-\frac{\delta}{k_0}} (-v)\,g(v)\,dv + \frac{2Mk_0}{\delta}\int_{\frac{\delta}{k_0}}^{\infty} (v)\,g(v)\,dv$$

$$= \frac{\epsilon}{2} - \frac{2Mk_0}{\delta}\int_{\infty}^{\frac{\delta}{k_0}} (v)\,g(-v)\,dv + \frac{2Mk_0}{\delta}\int_{\frac{\delta}{k_0}}^{\infty} (v)\,g(v)\,dv \tag{5}$$

Further, if in the Eqn. (5), the probability density function $g(v)$ is even function then $g(-v) = g(v)$ and for odd $g(v)$ it is $g(-v) = -g(v)$, therefore for $\int_{\frac{\delta}{k_0}}^{\infty}(v)\,g(v)\,dv = F\left(\frac{\delta}{k_0}\right) < \infty$, we get

$$|H_{k_0}f(x) - f(x)| \le \begin{cases} \dfrac{\epsilon}{2} + \left(\dfrac{4Mk_0}{\delta}\right) F\left(\dfrac{\delta}{k_0}\right), & g(v) \text{ is even function,} \\ \dfrac{\epsilon}{2}, g(v) \text{ is odd function.} \end{cases} \tag{6}$$

Now, in Eqn. (6) we set $k_0 = \frac{\epsilon\delta}{8MF\left(\frac{\delta}{\epsilon\delta_0}\right)}$ to get

$$|H_{k_0}f(x) - f(x)| \le \begin{cases} \epsilon, g(v) \text{ is even function,} \\ \frac{\epsilon}{2}, g(v) \text{ is odd function.} \end{cases} \tag{7}$$

Finally, in Eqn. (7) as $\epsilon \to 0$, $H_{k_0}f(x)$ converges uniformly to $f(x)$. □

2. The Weierstrass Approximation Theorem for a General Class of Polynomials

Here, we extend the function f to a bounded uniformly continuous function on $\mathbb{R}$. This is accomplished here with the add of the following extended function.

Let $\frac{f(x)-f(a)}{x-a} = f(a)$ on open interval $(a-1,a)$, and $\frac{f(x)-f(b)}{x-b} = -f(b)$ on open interval $(b,b+1)$, and $f(x)=0$ for all $x \in \mathbb{R} \backslash [a-1,b+1]$.

Particularly, we consider $R>0$ such that $R \in \mathbb{R}$ and $f(x)=0$ for $|x|>R$. Let $\epsilon>0$ and M such that $|f(x)| \leq M, \forall\ x \in \mathbb{R}$. Then by Theorem 2 there exists $k_0>0$ such that for all $x \in \mathbb{R}$, we have

$$|H_{k_0} f(x) - f(x)| < \frac{\epsilon}{2}. \tag{8}$$

Again, $f(u)=0$ for $|u|>R,\ R>0$, then we may write

$$H_{k_0} f(x) = \frac{1}{k_0} \int_{-R}^{R} f(u) g\left(\frac{u-x}{k_0}\right) du,\ where\ k_0>0. \tag{9}$$

Further letting $g(x) = \lim_{n\to\infty} \frac{1}{C} \sum_{m=0}^{n} (-n)_m A_{n,m} (x)^{2m}$, where, $A_{n,m} (\forall n \geq 0, \forall m \geq 0)$ is a bounded sequence, $(-n)_m = \frac{(-1)^m n!}{n-m!}, 0 \leq m \leq n$, and for $m>n$, there is $(-n)_m = 0$, and C is any constant which may be found on satisfying the equation (2). Also for all $|x| \leq R$, and all $|u| \leq R$, we have $|u-x| \leq 2R$, so that $\left|\frac{u-x}{k_0}\right| \leq \frac{2R}{k_0}$, hence then on closed interval $\left[-\frac{2R}{k_0}, \frac{2R}{k_0}\right]$, we get

$$\left| \frac{1}{k_0} g\left(\frac{u-x}{k_0}\right) - \frac{1}{Ck_0} \sum_{m=0}^{n} (-n)_m\ A_{n,m} \left(\frac{u-x}{k_0}\right)^{2m} \right| < \frac{\epsilon}{4RM} \tag{10}$$

Thus with the help of Eqns. (9) and (10), we have

$$\left| H_{k_0} f\,(x) - \frac{1}{Ck_0} \int_{-R}^{R} f\,(u) \sum_{m=0}^{n} (-n)_m\ A_{n,m} \left(\frac{u-x}{k_0}\right)^{2m} du \right| < \frac{\epsilon}{2} \tag{11}$$

Now in Eqn. (11), we define a general class of polynomials in the form $P_n\,(x) = \frac{1}{Ck_0} \int_{-R}^{R} f\,(u) \sum_{m=0}^{n} (-n)_m\ A_{n,m} \left(\frac{u-x}{k_0}\right)^{2m} du$ to get

$$|H_{k_0} f\,(x) - P_n\,(x)| < \frac{\epsilon}{2} \tag{12}$$

Finally, with the help of Eqns. (8) and (12) we find

$$|f\,(x) - P_n\,(x)| < |(H_{k_0} f\,(x) - P_n\,(x)) - (H_{k_0} f\,(x) - f\,(x))|$$

$$< |H_{k_0} f(x) - P_n(x)| + |H_{k_0} f(x) - f(x)| < \epsilon. \tag{13}$$

Hence, f is on interval $[a,b]$ a uniform limit of polynomials. **(The Weierstrass Approximation Theorem).**

Remark 2.1. *In Eqns.* (10) *and* (12) *set* $A_{n,m} = \frac{(n-m)!}{n!m!}$, *then* $C = \sqrt{\pi}$ *.Thus we get the results equivalent to the results given by*

$$g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad \text{and } P_n(x) = \frac{1}{k_0\sqrt{\pi}} \int_{-R}^{R} f(u) \sum_{m=0}^{n} \frac{(-1)^m}{m!} \left(\frac{u-x}{k_0}\right)^{2m} du \tag{14}$$

due to Schep [4].

Remark 2.2. *In Eqn.* (10) *set* $A_{n,m} = Cf\left(\frac{m}{n}\right)\frac{(-1)^m}{m!}$, *and replace* x *by* $\sqrt{\frac{x}{(1-x)}}$, *then it becomes Bernstein Approximation theorem [2] in the form*

$$|f(x) - Q_n(x)| < \in, where, Q_n(x) = x^n P_n(x). \tag{15}$$

3. Generalization of Weierstrass Approximation Theorem

Theorem 3.1. *Let* $F(x,t) : [a,b] \times (-1,1) \to \mathbb{R}$, $a \leq x \leq b, |t| < 1, a, b, t \in \mathbb{R}$, *be a two variable continuous function, then on the rectangle* $[a,b] \times (-1,1)$ $F(x,t)$ *is a generating function* $\sum_{n=0}^{\infty} P_n(x)t^n$, *where* $P_n(x)$ *is a polynomial defined in Eqn.* (12).

Proof. Since for $|t| < 1$, we have $|1-t| < 2$ then, from Eqn. (10) on closed interval $\left[-\frac{2R}{k_0}, \frac{2R}{k_0}\right]$, we write

$$\left|\frac{1}{k_0} g\left(\frac{u-x}{k_0}\right)\frac{1}{(1-t)} - \frac{1}{Ck_0}\left(\sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} (-n)_m A_{n,m} \left(\frac{u-x}{k_0}\right)^{2m}\right)\right| < \frac{\epsilon}{8RM} \tag{16}$$

Then by the Eqn. (16), we may write

$$\left|H_{k_0} f(x) \frac{1}{(1-t)} - \frac{1}{Ck_0}\left(\sum_{n=0}^{\infty} t^n \int_{-R}^{R} f(u) \sum_{m=0}^{n} (-n)_m A_{n,m}\left(\frac{u-x}{k_0}\right)^{2m} du\right)\right| < \frac{\epsilon}{4} \tag{17}$$

Again let $|F(x,t)| < \frac{M}{2}$, $\forall (x,t) \in \{[a,b] \times (-1,1) : a, b \in \mathbb{R}\}$ as $|f(x)| < M, \forall x \in \mathbb{R}$ and let $\left|F(x,t) - f(x)\frac{1}{(1-t)}\right| < \frac{\epsilon}{2}$, then making an application of Eqns. (12) and (17), we write

$$\left|H_{k_0} f(x) \frac{1}{(1-t)} - \sum_{n=0}^{\infty} P_n(x) t^n\right| < \frac{\epsilon}{4} \tag{18}$$

Therefore, on using Theorem 1 and Eqn. (18), we get

$$\begin{aligned}
\left|F(x,t) - \sum_{n=0}^{\infty} P_n(x) t^n\right| &< \left|F(x,t) - H_{k_0} f(x)\frac{1}{(1-t)}\right| \\
&+ \left|H_{k_0} f(x)\frac{1}{(1-t)} - \sum_{n=0}^{\infty} P_n(x) t^n\right| \\
&< \left|F(x,t) - f(x)\frac{1}{(1-t)}\right| + \left|f(x)\frac{1}{(1-t)} - H_{k_0} f(x)\frac{1}{(1-t)}\right| \\
&+ \left|H_{k_0} f(x)\frac{1}{(1-t)} - \sum_{n=0}^{\infty} P_n(x) t^n\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

This result implies that

$$\left| F\left(x,t\right) - \sum_{n=0}^{\infty} P_n\left(x\right) t^n \right| < \epsilon \tag{19}$$

Therefore, on the rectangle $[a,b]\times(-1,1)$, $F\left(x,t\right)$ is a generating function $\sum_{n=0}^{\infty} P_n\left(x\right) t^n$, where $P_n\left(x\right)$ is a polynomial defined in Eqn. (12). □

4. Applications

The highly important role of the above theorems in deep consideration and the generating function description is pointed out rather precisely in this section.First we use the Theorem 3 and Eqn. (12) to get the following Theorem:

Theorem 4.1. *For any sequence $B_n\,(n \geq 0)$, the polynomial $P_n\left(x\right)$ defined by* $\frac{1}{Ck_0}\int_{-R}^{R} f(u) \sum_{m=0}^{n} (-n)_m A_{n,m} \left(\frac{u-x}{k_0}\right)^{2m} du$ *generates the generating function $F\left(x,\zeta\right)$ such that*

$$F\left(x,\zeta\right) = \frac{(1+\zeta)^{\alpha+1}}{Ck_0\left(1-\beta\zeta\right)} \sum_{n=0}^{\infty} B_n (-\zeta)^n \int_{-R}^{R} \left(\frac{u-x}{k_0}\right)^{2n} f\left(u\right) du \tag{20}$$

Here, it is provided that $A_{n,m} = \frac{\left(\alpha+\left(\beta+1\right)n\right)!}{n!\left(\alpha+\beta n+m\right)!} B_m$ *and* $\zeta = t(1+\zeta)^{\beta+1}$ *and* $\zeta\left(0\right) = 0.$

Proof. In definition of the polynomial set $A_{n,m} = \frac{\left(\alpha+\left(\beta+1\right)n\right)!}{n!\left(\alpha+\beta n+m\right)!} B_m$, then we get

$$P_n\left(x\right) = \frac{1}{Ck_0}\int_{-R}^{R} f(u) \sum_{m=0}^{n} (-1)^m \begin{pmatrix} \alpha+\left(\beta+1\right)n \\ n-m \end{pmatrix} B_m \left(\frac{u-x}{k_0}\right)^{2m} du \tag{21}$$

Therefore the polynomial given in Eqn. (21) generates the function

$$F(x,\zeta) = \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{Ck_0}\int_{-R}^{R} f(u) \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} (-1)^m \begin{pmatrix} \alpha+(\beta+1)n \\ n-m \end{pmatrix} B_m \left(\frac{u-x}{k_0}\right)^{2m} du \tag{22}$$

Now, in Eqn. (22) make an appeal to the Theorem due to Brown [1] on providing $\zeta = t(1+\zeta)^{\beta+1}$ and $\zeta\left(0\right) = 0$ we find the result (20). □

Example 4.1. *Define the function*
$f(x) = \begin{cases} \lambda,\ |x| \leq R \\ 0, |x| \geq R \end{cases}$, $A_{n,m} = \frac{(\alpha+(\beta+1)n)!}{n!(\alpha+\beta n+m)!} B_m$, *then on using the Theorem 4, we get generating function*

$$F(x,\zeta) = \frac{\lambda(1+\zeta)^{\alpha+1}}{C(1-\beta\zeta)} \left[\left(\frac{R+x}{k_0}\right) \sum_{n=0}^{\infty} B_n \left(-\zeta\left(\frac{R+x}{k_0}\right)^2\right)^n + \left(\frac{R-x}{k_0}\right) \sum_{n=0}^{\infty} B_n \left(-\zeta\left(\frac{R-x}{k_0}\right)^2\right)^n \right] \tag{23}$$

Example 4.2. *Define the function*
$f(x) = \begin{cases} \lambda, & |x| \leq R \\ 0, & |x| \geq R \end{cases}$, $A_{n,m} = \frac{(\alpha + (\beta+1)n)!}{n!(\alpha + \beta n + m)!} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m}$, *where for* $\alpha \neq 0, -1, -2, \ldots,$
$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$ *is a Pochhammer symbol. Then on using Theorem 4, we get the following generating function of classical generalized hypergeometric function* ${}_pF_q$

$$F(x,\zeta) = \frac{\lambda(1+\zeta)^{\alpha+1}}{C(1-\beta\zeta)} \left[\left(\frac{R+x}{k_0}\right) {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; ; \\ \beta_1, \ldots, \beta_q; ; \end{array} \left(-\zeta \left(\frac{R+x}{k_0} \right)^2 \right) \right] + \left(\frac{R-x}{k_0}\right) {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; ; \\ \beta_1, \ldots, \beta_q; ; \end{array} \left(-\zeta \left(\frac{R-x}{k_0} \right)^2 \right) \right] \right] \tag{24}$$

where for the definition of ${}_pF_q$ *,we refer [3].*

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