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THE CONNECTED DETOUR MONOPHONIC NUMBER OF A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ *detour monophonic path*. A set S of vertices of G is a *detour monophonic set* of G if each vertex v of G lies on an $x - y$ detour monophonic path, for some x and y in S. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. A *connected detour monophonic set* of G is a detour monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected detour monophonic set of G is the *connected detour monophonic number* of G and is denoted by $dm_c(G)$. We determine bounds for $dm_c(G)$ and characterize graphs which realize these bounds. It is shown that for positive integers r, d and $k \geq 6$ with $r < d$, there exists a connected graph G with monophonic radius r, monophonic diameter d and $dm_c(G) = k$. For each triple a, b, p of integers with $3 \leq a \leq b \leq p - 2$, there is a connected graph G of order p, $dm(G) = a$ and $dm_c(G) = b$. Also, for every pair a, b of positive integers with $3 \leq a \leq b$, there is a connected graph G with $m_c(G) = a$ and $dm_c(G) = b$, where $m_c(G)$ is the connected monophonic number of G.

Keywords: detour monophonic set, detour monophonic number, connected detour monophonic set, connected detour monophonic number.

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood*

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of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced and further studied in [1, 6]. The *detour distance* $D(u, v)$ between two vertices u and v in G is the length of a longest $u - v$ path in G. An $u - v$ path of length $D(u, v)$ is called an $u - v$ *detour* [2]. It is known that D is a metric on the vertex set V of G. The closed detour interval $I_D[x, y]$ consists of x, y, and all the vertices in some $x - y$ detour of G. For $S \subseteq V$, $I_D[S]$ is the union of the sets $I_D[x, y]$ for all $x, y \in S$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour distance, detour number were introduced and studied in [3, 4].

For a connected graph G of order at least two, a *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an *$x - y$ detour monophonic path.* A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is defined as the *monophonic number* of G, denoted by $m(G)$ [9]. A *connected monophonic set* of G is a monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected monophonic set of G is the *connected monophonic number* of G and is denoted by $m_c(G)$. The connected monophonic number of a graph was introduced and studied in [10]. A set S of vertices of G is a *detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic path, for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour number of a graph was introduced in [12] and further studied in [11].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. A vertex u in G is a *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$.

The monophonic distance was introduced and studied in [7, 8]. The following theorems will be used in the sequel.

Theorem 1.1. [10] *Each extreme vertex of a connected graph G belongs to every connected monophonic set of G.*

Theorem 1.2. [10] *Every cutvertex of a connected graph G belongs to every connected monophonic set of G.*

Theorem 1.3. [10] *For any nontrivial tree T of order p, $m_c(T) = p$.*

Theorem 1.4. [12] *Each extreme vertex of a connected graph G belongs to every detour monophonic set of G.*

Corollary 1.1. [12] *For the complete graph $K_p (p \geq 2)$, $dm(K_p) = p$.*

Corollary 1.2. [12] *If T is a tree with k endvertices, then $dm(T) = k$.*

Theorem 1.5. [12] *Let G be a connected graph with a cutvertex v and let S be a detour monophonic set of G. Then every component of $G - v$ contains an element of S.*

Theorem 1.6. [12] *Let G be a connected graph of order $p \geq 3$. Then $dm(G) = p - 1$ if and only if $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Connected Detour Monophonic Number

Definition 2.1. *A connected detour monophonic set of a graph G is a detour monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected detour monophonic set of G is the connected detour monophonic number of G and is denoted by $dm_c(G)$. A connected detour monophonic set of cardinality $dm_c(G)$ is called a dm_c-set of G.*

Example 2.1. *For the graph G in Figure 2.1, $S_1 = \{w, u, z\}$ and $S_2 = \{x, u, z\}$ are the minimum detour monophonic sets of G and so $dm(G) = 3$. Since the subgraph $G[S_i]$ is not connected, S_i is not a connected detour monophonic set of G for $i = 1, 2$. It is clear that $T = \{u, x, y, z\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = 4$.*

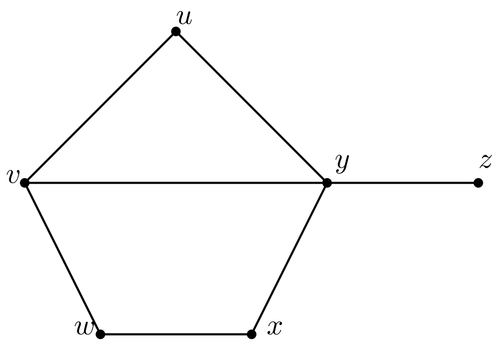

Figure 2.1: G

Theorem 2.1. *Each extreme vertex of a connected graph G belongs to every connected detour monophonic set of G.*

Proof. Since every connected detour monophonic set of G is a detour monophonic set of G, it follows from Theorem 1.4. □

Corollary 2.1. *For the complete graph $K_p (p \geq 2), dm_c(K_p) = p$.*

Theorem 2.2. *Let G be a connected graph with cutvertices and let S be a connected detour monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.*

Proof. Since every connected detour monophonic set of G is a detour monophonic set of G, it follows from Theorem 1.5. □

Theorem 2.3. *Every cutvertex of a connected graph G belongs to every connected detour monophonic set of G.*

Proof. Let v be any cutvertex of G and let $G_1, G_2, ..., G_r (r \geq 2)$ be the components of $G - v$. Let S be any connected detour monophonic set of G. Then by Theorem 2.2, S contains at least one element from each $G_i (1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

For a cutvertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at

v. Since every endblock B is a branch of G at some cutvertex, it follows from Theorem 2.2 that every minimum connected detour monophonic set of G contains at least one vertex from B that is not a cutvertex. Thus the following corollaries are consequences of Theorems 2.2 and 2.3.

Corollary 2.2. *If G is a connected graph with $k \geq 2$ endblocks, then $dm_c(G) \geq k + 1$.*

Corollary 2.3. *If k is the maximum number of blocks to which a vertex in a graph G belongs, then $dm_c(G) \geq k + 1$.*

Corollary 2.4. *For any nontrivial tree T of order p, $dm_c(T) = p$.*

Proof. It follows from Theorems 2.1 and 2.3. □

Theorem 2.4. *For the complete bipartite graph $G = K_{r,s}$ $(2 \leq r \leq s)$, $dm_c(G) = 3$ or 4.*

Proof. Let $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the partite sets of $K_{r,s}$. For $r = 2$, $S = X$ is the unique minimum detour monophonic set of G. Since $G[S]$ is not connected, and since $S' = S \cup \{y_i\}$ is a connected detour monophonic set of G for any $i(1 \leq i \leq s)$, we have $dm_c(G) = 3$.

Now, let $r \geq 3$. Let S be any set formed by taking two vertices from X and two vertices from Y. Then clearly, it is a minimum connnected detour monophonic set of G and so $dm_c(G) = 4$. □

Theorem 2.5. *For any connected graph G of order $p \geq 2$, $2 \leq dm_c(G) \leq p$.*

Proof. Since $V(G)$ is a connected detour monophonic set of G, it follows that $dm_c(G) \leq p$. Also it is clear that $dm_c(G) \geq 2$ and so $2 \leq dm_c(G) \leq p$. □

Theorem 2.6. *For a connected graph G of order $p \geq 2$, $2 \leq dm(G) \leq dm_c(G) \leq p$.*

Proof. Any detour monophonic set needs at least two vertices and so $dm(G) \geq 2$. Since every connected detour monophonic set of G is also a detour monophonic set of G, it follows that $dm(G) \leq dm_c(G)$. Also, since $V(G)$ induces a connected detour monophonic set of G, it is clear that $dm_c(G) \leq p$. □

Theorem 2.7. *For a connected graph G of order $p \geq 2$, $2 \leq m_c(G) \leq dm_c(G) \leq p$.*

Proof. Any connected monophonic set needs at least two vertices and so $m_c(G) \geq 2$. Since every connected detour monophonic set is also a connected monophonic set, it follows that $m_c(G) \leq dm_c(G)$. Also, since $V(G)$ induces a connected detour monophonic set of G, it is clear that $dm_c(G) \leq p$. □

Now we proceed to characterize graphs G for which the lower bound in Theorem 2.5 is attained.

Theorem 2.8. *Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $dm_c(G) = 2$.*

Proof. If $G = K_2$, then $dm_c(G) = 2$. Conversely, let $dm_c(G) = 2$. Let $S = \{u, v\}$ be a minimum connected detour monophonic set of G. Then uv is an edge. If $G \neq K_2$, there exists a vertex w different from u and v. Then w can not lie on any $u - v$ detour monophonic path, so that S is not a detour monophonic set, which is a contradiction. Thus $G = K_2$. □

Theorem 2.9. *If G is a connected graph of order $p \geq 2$ with every vertex of G is either a cutvertex or an extreme vertex, then $dm_c(G) = p$.*

Proof. It follows from Theorems 2.1 and 2.3. ☐

Remark 2.1. *The converse of the Theorem 2.9 is not true. For the graph G given in Figure 2.2, $S = V(G)$ is the unique minimum connected detour monophonic set of G and so $dm_c(G) = p$, but the vertex x is neither a cutvertex nor an extreme vertex of G.*

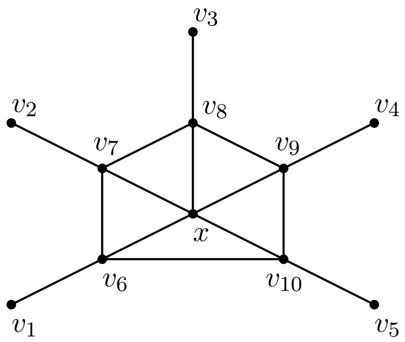

Figure 2.2: G

We leave the following problem as an open question.

Problem 2.1. *Characterize graphs G for which (i) $m_c(G) = dm_c(G)$ and (ii) $dm(G) = dm_c(G)$.*

Theorem 2.10. *If G is a connected non-complete graph of order $p \geq 2$ such that it has a minimum cutset consisting of κ vertices, then $dm_c(G) \leq p - \kappa(G) + 1$.*

Proof. If G is non-complete, it is clear that $1 \leq \kappa(G) \leq p - 2$. Let $U = \{u_1, u_2, ..., u_\kappa\}$ be a minimum cutset of G. Let $G_1, G_2, ..., G_r (r \geq 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every vertex $u_i (1 \leq i \leq \kappa)$ is adjacent to at least one vertex of G_j for every $j (1 \leq j \leq r)$. It is clear that S is a detour monophonic set of G and $G[S]$ is not connected. Also, it is clear that $G[S \cup \{x\}]$ is a connected detour monophonic set for any vertex x in U so that $dm_c(G) \leq p - \kappa(G) + 1$. ☐

Remark 2.2. *The bound in Theorem 2.10 is sharp. For any tree T of order $p \geq 2$, $dm_c(T) = p$. Also, $\kappa(T) = 1$, $p - \kappa(T) + 1 = p$. Thus $dm_c(T) = p - \kappa(T) + 1$.*

Corollary 2.5. *If G is a connected non-complete graph of order $p \geq 2$ having no cutvertices, then $dm_c(G) \leq p - 1$.*

Proof. Since $\kappa(G) \geq 2$, the result follows from Theorem 2.10. ☐

Theorem 2.11. *If G is a nontrivial connected graph of order p and monophonic diameter $d = p - 1$, then $dm_c(G) \geq p - d + 1$.*

Proof. For any graph G, $dm_c(G) \geq 2$. Since $d = p - 1$, we have $p - d + 1 = 2$ and so $dm_c(G) \geq p - d + 1$. ☐

Remark 2.3. *The converse of Theorem 2.11 is not true. For the graph G given in Figure 2.3, $p = 8$ and monophonic diameter $d = 2$ so that $p - d + 1 = 7$. Also by Theorem 2.9, $dm_c(G) = 8$. Thus $dm_c(G) > p - d + 1$, but $d \neq p - 1$.*

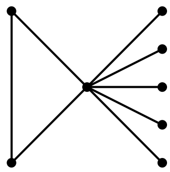

Figure 2.3: G

Theorem 2.12. *Let G be a connected graph of order $p \geq 2$ such that every vertex v of G is either an endvertex or a cutvertex, then $dm_c(G) \geq p - d + 1$, where d is the monophonic diameter of G.*

Proof. By Theorem 2.9, $dm_c(G) = p$. Since $d \geq 1$, it follows that $dm_c(G) \geq p - d + 1$. □

Theorem 2.13. *For any positive integers r, d and $k \geq 6$ with $r < d$, there exists a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $dm_c(G) = k$.*

Proof. We prove this theorem by considering two cases.
Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be the cycle of order $d + 2$. Let G be the graph obtained by adding $k - 3$ new vertices $u_1, u_2, \ldots, u_{k-3}$ to C_{d+2} and joining each of the vertices $u_1, u_2, \ldots, u_{k-3}, v_3, v_4, \ldots, v_{d+1}$ to the vertex v_1. The graph G is shown in Figure 2.4. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1$, $e_m(v_2) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Now, $u_1, u_2, ..., u_{k-3}, v_2, v_{d+2}$ are the extreme vertices and v_1 is the only cutvertex of G. Let $S = \{u_1, u_2, ..., u_{k-3}, v_2, v_{d+2}, v_1\}$. Since S is a connected detour monophonic set of G, it follows from Theorem 2.1 and Theorem 2.3 that $dm_c(G) = k$.

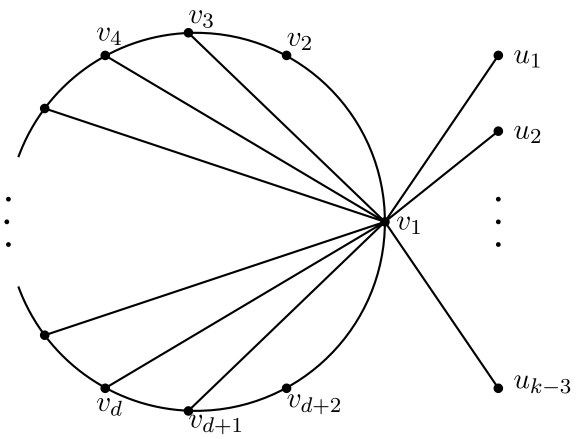

Figure 2.4: G

Case 2. $r \geq 2$. Let $C : v_1, v_2, ..., v_{r+2}, v_1$ be the cycle of order r+2 and $W = K_1 + C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, ..., u_{d+2}\}$. Let H be the graph obtained from C and W by identifying v_1 of C and the central vertex of W. Now add $k - 6$ new vertices $w_1, w_2, ..., w_{k-6}$ to the graph H and join each $w_i(1 \leq i \leq k-6)$ to the vertex v_1 and obtain the graph G of Figure 2.5. It is easy to verify that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Now, $w_1, w_2, ..., w_{k-6}$ are the endvertices and v_1 is the only cutvertex of G. Let $S = \{w_1, w_2, ..., w_{k-6}, v_1\}$. By, Theorem 2.1 and Theorem 2.3, every connected detour monophonic set of G contains S. It is clear that S is not a connected detour monophonic set of G. Also, $S \cup \{x_1, x_2, x_3, x_4\}$

where $x_j (1 \leq j \leq 4) \in V(G) - S$ is not a connected detour monophonic set of G. Let $T = S \cup \{u_1, u_2, u_{d+2}, v_2, v_{r+2}\}$. It is easy to verify that T is a connected detour monophonic set of G and so $dm_c(G) = k$. □

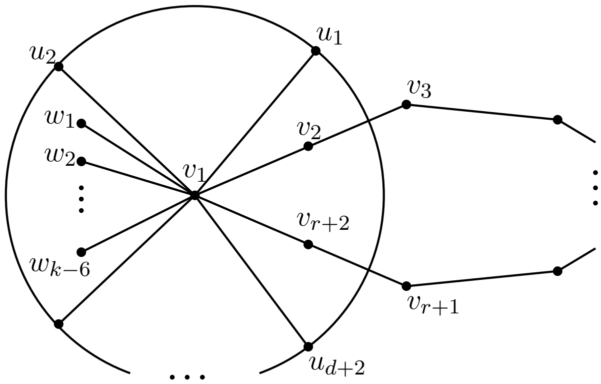

Figure 2.5: G

Problem 2.2. *For any three positive integers r, d and $k \geq 6$ with $r = d$ does there exist a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $dm_c(G) = k$?*

Theorem 2.14. *If p, d and k are positive integers such that $2 \leq d \leq p-2$ and $3 \leq k \leq p$, then there exists a connected graph G of order p, monophonic diameter d and $dm_c(G) = k$.*

Proof. We prove this theorem by considering two cases.
Case 1. Let $d = 2$. First, let $k = 3$. Let $P_3 : v_1, v_2, v_3$ be the path of order 3. Now, add $p - 3$ new vertices $w_1, w_2, ..., w_{p-3}$ to P_3. Let G be the graph obtained by joining each $w_i (1 \leq i \leq p-3)$ to v_1 and v_3. The graph G is shown in Figure 2.6. Then G has order p and monophonic diameter $d = 2$. Clearly $S = \{v_1, v_2, v_3\}$ is a minimum connected detour monophonic set of G so that $dm_c(G) = k = 3$.

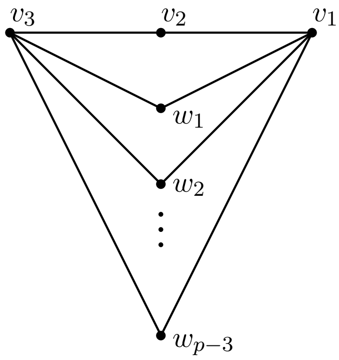

Figure 2.6: G

Now, let $4 \leq k \leq p$. Let K_{p-1} be the complete graph with the vertex set $\{w_1, w_2, ..., w_{p-k+1}, v_1, v_2, ..., v_{k-2}\}$. Now, add the new vertex x to K_{p-1} and let G be the graph obtained by joining x with each vertex $w_i (1 \leq i \leq p-k+1)$. The graph G is shown in Figure 2.7. Then G has order p and monophonic diameter $d = 2$. Let $S = \{v_1, v_2, ...v_{k-2}, x\}$ be the set of all extreme vertices of G. By Theorem 2.1, every connected detour monophonic set of G contains S. It is clear that S is a detour monophonic set of G. Since the induced subgraph $G[S]$ is not connected, $dm_c(G) \geq k$. For any vertex $v \in \{w_1, w_2, \ldots, w_{p-k+1}\}$, it is clear that $S \cup \{v\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = k$.

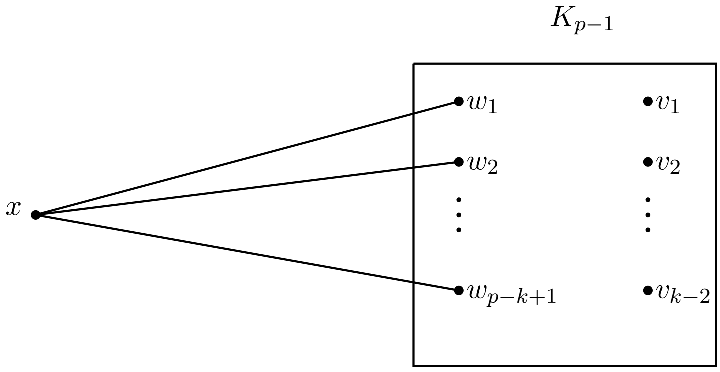

Figure 2.7: G

Case 2. $d \geq 3$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, ..., v_{d+2}, v_1$ be the cycle of order $d + 2$. Add $p-d-2$ new vertices $w_1, w_2, ..., w_{p-d-2}$ to C and join each vertex $w_i (1 \leq i \leq p-d-2)$ to both v_1 and v_3, thereby producing the graph G of Figure 2.8. Then G has order p and monophonic diameter d. It is clear that $S = \{v_3, v_4, v_5\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = 3 = k$.

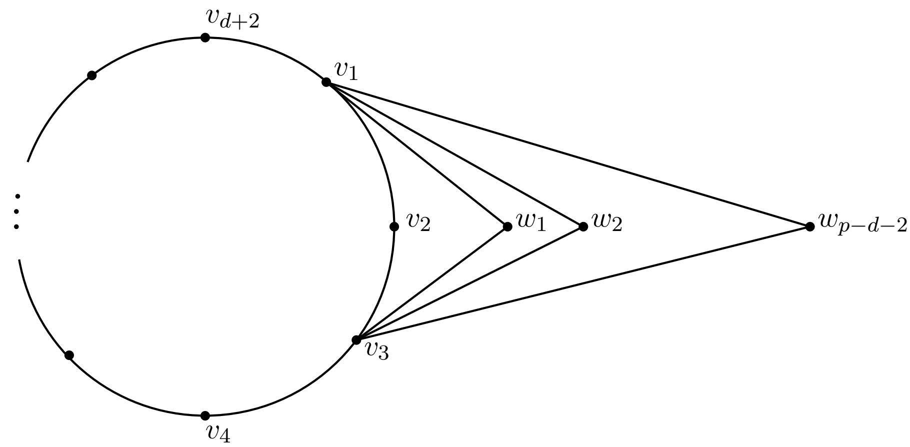

Figure 2.8: G

Now, let $k \geq 4$. Let $P_{d+1} : v_0, v_1, ..., v_d$ be a path of length d. Add $p - d - 1$ new vertices $w_1, w_2, ..., w_{p-k}, u_1, u_2, ..., u_{k-d-1}$ to P_{d+1} and join $w_1, w_2, ..., w_{p-k}$ to both v_0 and v_2 and join $u_1, u_2, ..., u_{k-d-1}$ to v_{d-1}, thereby producing the graph G of Figure 2.9.

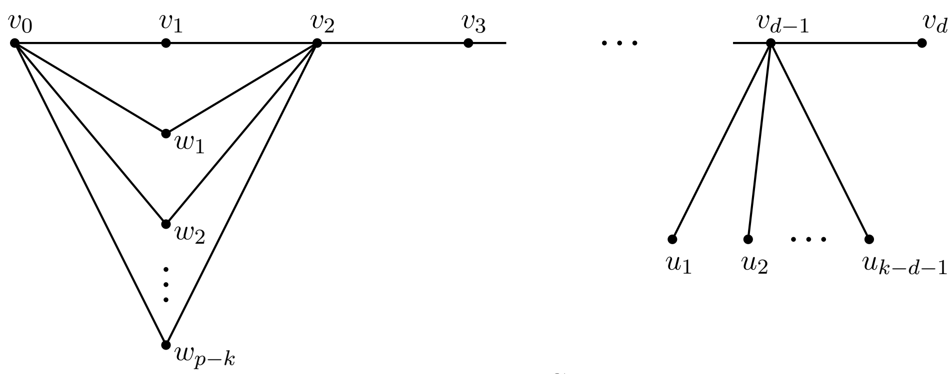

Figure 2.9: G

Then G has order p and monophonic diameter d. Let $S = \{v_2, v_3, ..., v_{d-1}, v_d, u_1, u_2, ..., u_{k-d-1}\}$ be the set of all cutvertices and endvertices of G. By Theorem 2.1 and Theorem 2.3, every connected detour monophonic set of G contains S. It is clear that S is not a connected detour monophonic set of G. It is easily seen that $S \cup \{v_0, v_1\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = k$. □

In view of Theorem 2.6, we have the following realization theorem.

Theorem 2.15. *If p, a and b are positive integers such that $3 \leq a \leq b \leq p-2$, then there exists a connected graph G of order p, $dm(G) = a$ and $dm_c(G) = b$.*

Proof. We prove this theorem by considering two cases.
Case 1. $3 \leq a = b \leq p-2$. Let K_{a-2} be the complete graph with the vertex set $\{w_1, w_2, ..., w_{a-2}\}$ and $C_4 : x, y, z, w, x$ be the cycle of order 4. Let H be the graph obtained from K_{a-2} and C_4 by joining each $w_i (1 \leq i \leq a-2)$ to the vertices y and z in C_4. Let G be the graph obtained from H by adding $p-a-2$ new vertices $v_1, v_2, \ldots, v_{p-a-2}$ to the graph H and join each $v_i (1 \leq i \leq p-a-2)$ to x and z. The graph G is shown in Figure 2.10.

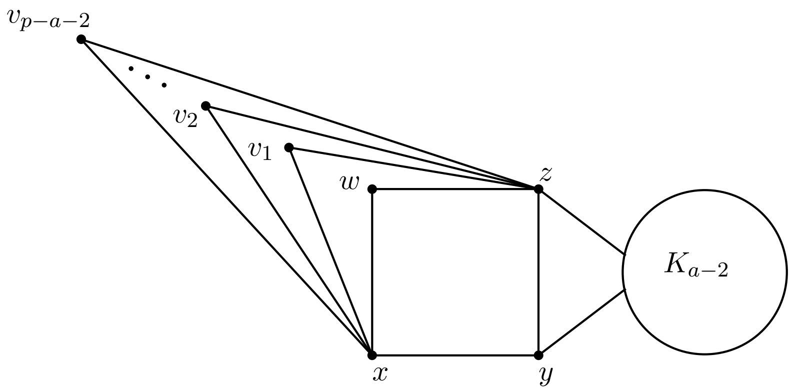

Figure 2.10: G

Let $S = \{w_1, w_2, \ldots, w_{a-2}\}$ be the set of all extreme vertices of G. By Theorem 1.4, every detour monophonic set contains S. It is clear that S is not a detour monophonic set of G. Also $S \cup \{v\}$, where $v \in V(G)-S$ is not a detour monophonic set of G. Since $S' = S \cup \{x, y\}$ is a detour monophonic set and $G[S']$ is also connected, we have $dm(G) = dm_c(G) = a$.
Case 2. $3 \leq a < b \leq p-2$. Let $P_{b-a+2} : u_1, u_2, ..., u_{b-a+2}$ be a path of length $b-a+1$. Add $p-b+a-2$ new vertices $w_1, w_2, ..., w_{p-b}, v_1, v_2, ..., v_{a-2}$ to P_{b-a+2} and join each w_i $(1 \leq i \leq p-b)$ to both u_1 and u_3 and join each $v_j (1 \leq j \leq a-2)$ to u_{b-a+1}, thereby producing the graph G of Figure 2.11. Then G has order p and $S = \{u_{b-a+2}, v_1, v_2, ..., v_{a-2}\}$ is the set of all endvertices of G. It is clear that S is not a detour monophonic set of G. Let $S' = S \cup \{u_1\}$. It is easy to verify that S' is a detour monophonic set of G and so $dm(G) = a$. Let $T = \{u_3, u_4, ..., u_{b-a+1}\}$ be the set of all cutvertices of G. By Theorem 2.1 and Theorem 2.3, every connected detour monophonic set of G contains $S \cup T$. Let $M = S \cup T$. It is clear that M is not a connected detour monophonic set of G. Also, $M \cup \{x\}$ where $x \in V(G) - M$ is not a connected detour monophonic set of G. Let $M' = M \cup \{u_1, u_2\}$. It is easily verified that M' is a minimum connected detour monophonic set of G and so $dm_c(G) = b$. □

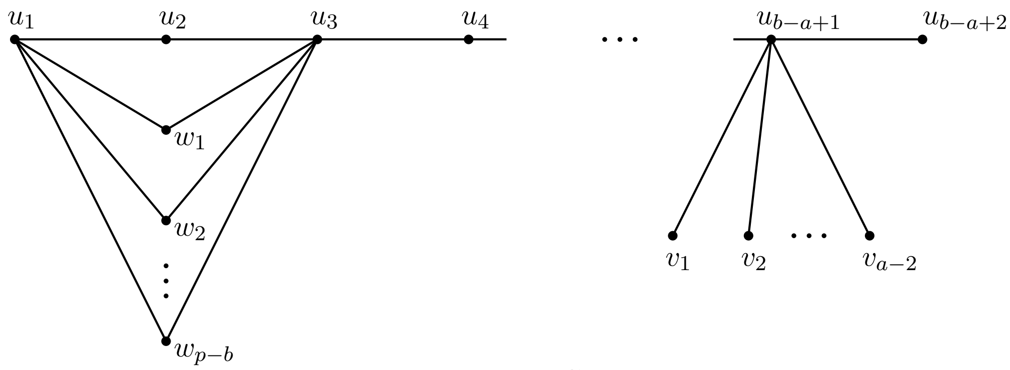

Figure 2.11: G

Theorem 2.16. *There does not exist a connected graph G of order $p \geq 2$ with $dm(G) = p-1$ and $dm_c(G) = p-1$.*

Proof. Since $dm(G) = p-1$, then, by Theorem 1.6, $G = K_1 + \cup m_j K_j$, where $m_j \geq 2$. Since every vertex of G is either a cutvertex or an extreme vertex of G, by Theorem 2.9, $dm_c(G) = p$, which is a contradiction. Therefore, there does not exist a connected graph G with $dm(G) = dm_c(G) = p-1$. □

In view of Theorem 2.7, we have the following realization theorem.

Theorem 2.17. *For every pair a, b of positive integers with $3 \leq a \leq b$, there is a connected graph G such that $m_c(G) = a$ and $dm_c(G) = b$.*

Proof. Case 1. $3 \leq a = b$. Let G be any tree of order a. Then by Theorem 1.3, $m_c(G) = a$ and Corollary 2.4, $dm_c(G) = b$.

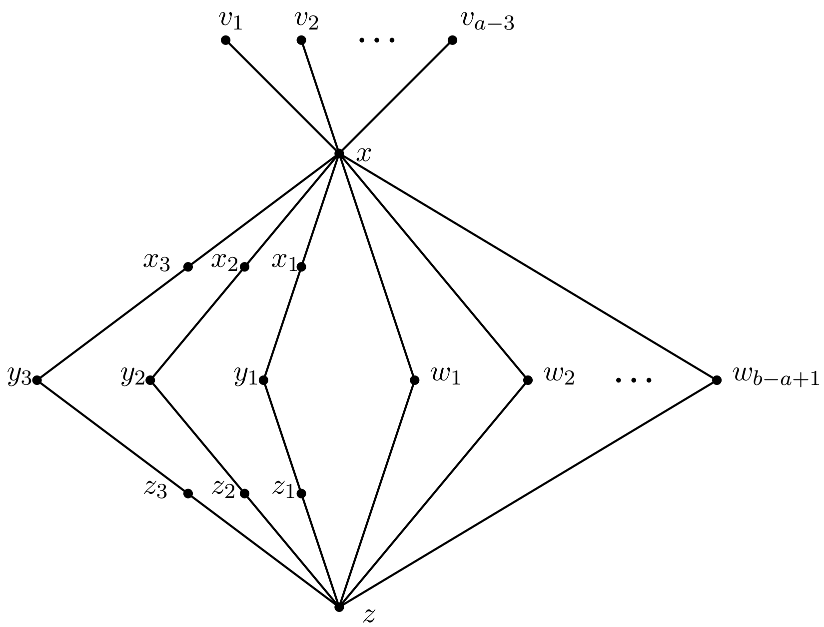

Figure 2.12: G

Case 2. $3 \leq a < b$. Let $P_i : x_i, y_i, z_i (1 \leq i \leq 3)$ be 3 copies of a path of length 2. Let G be the graph obtained by adding b new vertices $x, z, v_1, v_2, ..., v_{a-3}, w_1, w_2, ..., w_{b-a+1}$ and (i) joining each of the vertices $x_1, x_2, x_3, v_1, v_2, ...v_{a-3}, w_1, w_2, ..., w_{b-a+1}$ to x and (ii) joining each of the vertices $z_1, z_2, z_3, w_1, w_2, ..., w_{b-a+1}$ to z. The graph G is shown in Figure 2.12. Now, $\{v_1, v_2, \ldots, v_{a-3}\}$ is the set of all endvertices of G and x is the only cutvertex of G. Let $S = \{v_1, v_2, \ldots, v_{a-3}, x\}$. Clearly, by Theorem 1.1, Theorem 1.2, Theorem 2.1 and Theorem 2.3, every connected monophonic set and every connected detour monophonic set of G contains S. It is clear that S is not a monophonic set of G. Let $S' = S \cup \{z\}$. It is easily verified that S' is a monophonic set of G, which is not connected. Let $S'' = S' \cup \{w_i\}$ for some $1 \leq i \leq b - a + 1$. It is clear that S'' is a minimum connected monophonic set of G and so $m_c(G) = a$.

It is easily verified that $M = S \cup \{z, w_1, w_2, \ldots, w_{b-a+1}\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = b$. □

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