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ONE-PARAMETER HOMOTHETIC MOTION IN THE MINKOWSKI 3-SPACE

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Abstract. A one-parameter homothetic motion in three-dimensional Minkowski space is defined by means of the Hamilton operators. We study some properties of this motion and show that it has only one pole point at every instant t. We also obtain the Darboux vector of the homothetic motion in E_1^3 and show that it can be written as multiplication of two split quaternions.

Keywords: Split quaternion, Hamilton operator, Homothetic motion, pole point

Mathematics Subject Classification: 15A33

1. Introduction

Split quaternions, H', or coquaternions are elements of a 4-dimensional associative algebra introduced by James Cockle in 1849. These quaternions are identified with the semi-Euclidean space E_2^4. A split quaternion can be applied to rotation in Minkowski 3-space. Some algebraic properties of Hamilton operators for split quaternions are considered in[2] where these quaternions have been expressed in terms of 4×4 matrices by means of these operators. By De Moivre's formula, we obtained any powers of these matrices [3]. Homothetic motions with aid of the Hamilton operators in four-dimensional semi-Euclidean space E_2^4 are studied in [1]. It is found that this motion also has only one pole point at every instant t. Tosu and *et. al* [6] investigated the one-parameter homothetic motion of a rigid body in 3-dimensional Lorentz space. Also, it is shown that this motion is regular in space-like and time-like regions and has only one instantaneous rotation centre at all time t.

In this paper, a Hamilton motion is defined in three-dimensional Minkowski space E_1^3 by means of the Hamilton operators and it is shown that it is a homothetic motion. We study some properties of this motion and show that it has only one pole point at every instant t. Therefore, the darboux vector of the motion is obtained and in special case ($N_\alpha = 1$) this vector can be written in multiplication of two split quaternions. Finally, we give some examples for more clarification.

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2. Preliminaries

We start with preliminaries on the geometry of 3-dimensional Minkowski space. The Minkowski 3−space E_1^3 is the Euclidean space E^3 provided with the inner product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in E^3$. We say that a Lorentzian vector $\overrightarrow{u}$ in E_1^3 is spacelike, lightlike or timelike if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L > 0, \langle \overrightarrow{u}, \overrightarrow{u} \rangle_L = 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L < 0$, respectively. The norm of the vector $\overrightarrow{u} \in E_1^3$ is defined by $\| \overrightarrow{u} \| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L|}$. The Lorentzian vector product $\overrightarrow{u} \wedge_L \overrightarrow{v}$ of $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as follows:

$$\overrightarrow{u} \wedge_L \overrightarrow{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The hyperbolic and Lorentzian unit spheres are

$$H_0^2 = \{ \overrightarrow{a} \in E_1^3 : \langle \overrightarrow{a}, \overrightarrow{a} \rangle_L = -1 \} \quad \text{and} \quad S_1^2 = \{ \overrightarrow{a} \in E_1^3 : \langle \overrightarrow{a}, \overrightarrow{a} \rangle_L = 1 \},$$

respectively.

Each rotation of Minkowski 3-space is represented by a rotation matrix with respect to standard basis. These matrices form the three-dimensional special orthogonal group

$$SO_1(3) = \{ A \in M_3(\mathbb{R}) : A^t \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \det A = 1 \}.$$

We can call these matrices semi-orthogonal matrices [5].

3. Split Quaternions Algebra

The semi-Euclidean 4-space with 2-index is represented with E_2^4. The inner product of this semi-Euclidean space is

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} = -u_1 v_1 - u_2 v_2 + u_3 v_3 + u_4 v_4,$$

and we say that $\overrightarrow{u}$ is timelike, spacelike or lightlike if $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} < 0$, $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} > 0$ and $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} = 0$ for the vector $\overrightarrow{u} \in E_2^4$, respectively. Split quaternions H' are identified with the semi-Euclidean space E_2^4. Besides, the subspace of H' consisting of pure split quaternions H_0' is identified with the Minkowski 3-space. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector product.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, i, j, k\}$ satisfying the equalities $i^2 = -1$, $j^2 = k^2 = 1$ and

$$i * j = k = -j * i, \ j * k = -i = -k * j, k * i = j = -i * k.$$

Also, similar to the division algebra of quaternions, the split quaternion algebra is the even subalgebra of the Clifford algebra of the three-dimensional Lorentzian space. That is, the non-division algebra of split quaternions H' is isomorphic with the even subalgebra $Cl_{2,1}^+$ of the Clifford algebra $Cl_{\ 2,1}$ where $Cl_{2,1}^+$ has the basis

$$\{1, e_2 e_3 \to i, e_3 e_1 \to k, e_1 e_2 \to j\}.$$

Scalar and vector parts of split quaternion q are denoted by $S_q = a_0$ and $\overrightarrow{V}_q = a_1 i + a_2 j + a_3 k$, respectively. The split quaternion product of two quaternions $q = (a_0, a_1, a_2, a_3)$ and $p = (b_0, b_1, b_2, b_3)$ is defined as

$$q * p = S_q S_p - \langle \overrightarrow{V}_q, \overrightarrow{V}_p \rangle_L + S_q \overrightarrow{V}_p + S_p \overrightarrow{V}_q + \overrightarrow{V}_q \wedge_L \overrightarrow{V}_p$$

where $\langle , \rangle_L$ and $\wedge_L$ are the Lorentzian inner product and vector products, respectively.

Let $q = (a_0, a_1, a_2, a_3) = S_q + \overrightarrow{V}_q$ be a split quaternion. The conjugate of a split quaternion, denoted q^*, is defined as $q^* = S_q - \overrightarrow{V}_q$. We say that a split quaternion q is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively, where $I_q = q * q^* = q^* * q$. Obviously, $-I_q = -a_0^2 - a_1^2 + a_2^2 + a_3^2$ is identified with $\langle q, q \rangle_{E_2^4}$ for the split quaternion $q = (a_0, a_1, a_2, a_3)$.

The norm of $q = (a_0, a_1, a_2, a_3)$ is defined as $N_q = \sqrt{\left| a_0^2 + a_1^2 - a_2^2 - a_3^2 \right|}$. If $N_q = 1$ then q is called unit split quaternion [5].

Definition 3.1. *Let q be a split quaternion, then $\overset{+}{h}_q : H' \to H'$ and $\overset{-}{h}_q : H' \to H'$ are defined as follows:*

$$\overset{+}{h}_q(x) = qx, \quad \overset{-}{h}_q(x) = xq, \quad x \in H'$$

The Hamilton's operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \qquad (1)$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \qquad (2)$$

Theorem 3.1. *If q and p are split quaternions, then the following identities hold:*

(i) $qp = \overset{+}{H}(q)p$, $qp = \overset{-}{H}(p)q$, $\overset{+}{H}(q)\overset{-}{H}(q) = \overset{-}{H}(q)\,\overset{+}{H}(q)$

(ii) $\overset{+}{H}(q^*) = \varepsilon(\overset{+}{H}(q))^t\varepsilon$, $\overset{-}{H}(q^*) = \varepsilon(\overset{-}{H}(q))^t\varepsilon$, $\varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$.

(iii) $\det \overset{+}{H}(q) = \det \overset{-}{H}(q) = N_q^2$.

Proof. The proof can be found in [2]. □

4. Homothtic Motion in E_1^3

The one-parameter homothetic motions of a body in three-dimensional Minkowski space E_1^3 is generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where A is a semi-orthogonal matrix. The matrix $B = hA$ is called a homothetic matrix and Y, X and C are $n \times 1$ real matrices. The homothetic scalar h and the elements of A and C are continuously differentiable functions of a real parameter t. Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space R and the fixed space R_0, respectively. At the initial time $t = t_0$, we consider the coordinate systems of R and R_0 as coincident. To avoid the case of affine transformation we assume that

$$h = h(t) \neq cons.\ ,\ h(t) \neq 0.$$

and to avoid the case of a pure translation or a pure rotation, we also assume that

$$\frac{d}{dt}(hA) \neq 0, \quad \frac{d}{dt}(C) \neq 0.$$

If we differentiate the equation $A^t \varepsilon A = \varepsilon$, we get

$$(\dot{A}^t \varepsilon) A + (A^t \varepsilon)\dot{A} = 0.$$

By choosing $A^t \varepsilon \dot{A} = \Omega$ and $\Omega^t = (\dot{A}^t \varepsilon) A$, we can see that

$$\Omega = \begin{bmatrix} 0 & \Omega_z & \Omega_y \\ -\Omega_z & 0 & \Omega_x \\ -\Omega_y & -\Omega_x & 0 \end{bmatrix}$$

is a anti-symmetric matrix in the sense of Lorentzien, *i.e.*, $\Omega = -\Omega^t$. And also we have, since $A^t = \varepsilon A^{-1} \varepsilon$ then

$$\begin{aligned} \Omega &= A^t \varepsilon \dot{A} = (\varepsilon A^{-1} \varepsilon) \varepsilon \dot{A} = \varepsilon A^{-1} \dot{A} \\ &\Rightarrow A^{-1} \dot{A} = \varepsilon \Omega \ \ [6]. \end{aligned}$$

5. Hamilton Motions in Minkowski 3-Space

Let us consider the curve $\alpha : I \subset \mathbb{R} \to E_2^4$ defined by

$$\alpha(t) = (a_0(t), a_1(t), a_2(t), a_3(t)), \text{ for every } t \in I. \qquad (3)$$

We suppose that $\alpha(t)$ is a differential curve of order r and it does not pass through the origin. Also, the map $F_\alpha : H_0' \to H_0'$ is defined as

$$F_\alpha(x) = \alpha * x * \alpha^*, \qquad x \in H_0' \qquad (4)$$

Using the definition of $\overset{+}{H}$, $\overset{-}{H}$ equation (4) is written as

$$F_\alpha(x) = x' = \overset{+}{H}(\alpha)\overset{-}{H}(\alpha^*)\ x.$$

From (1) and (2), we have

$$\overset{+}{H}(\alpha)\overset{-}{H}(\alpha^*) = \begin{bmatrix} a_0^2+a_1^2-a_2^2-a_3^2 & 0 & 0 & 0 \\ 0 & a_0^2+a_1^2+a_2^2+a_3^2 & 2(a_0a_3-a_1a_2) & -2(a_0a_2+a_1a_3) \\ 0 & 2(a_0a_3+a_1a_2) & a_0^2-a_1^2-a_2^2+a_3^2 & 2(a_0a_1+a_2a_3) \\ 0 & 2(a_1a_3-a_0a_2) & 2(a_0a_1-a_2a_3) & a_0^2-a_1^2+a_2^2-a_3^2 \end{bmatrix}.$$

This simplifies to

$$\overset{+}{H}(\alpha)\overset{-}{H}(\alpha) = \begin{bmatrix} h' & 0 \\ 0 & B \end{bmatrix},$$

where $h' = N_\alpha^2 = a_0^2+a_1^2-a_2^2-a_3^2$ and

$$B = [b_{ij}]_{3\times 3} = \begin{bmatrix} a_0^2+a_1^2+a_2^2+a_3^2 & 2(a_0a_3-a_1a_2) & -2(a_0a_2+a_1a_3) \\ 2(a_0a_3+a_1a_2) & a_0^2-a_1^2-a_2^2+a_3^2 & 2(a_0a_1+a_2a_3) \\ 2(a_1a_3-a_0a_2) & 2(a_0a_1-a_2a_3) & a_0^2-a_1^2+a_2^2-a_3^2 \end{bmatrix}.$$

For matrix B, we have $B^t\varepsilon B\varepsilon = h'^2 I_3$ and $\det B = h'^3$.

Definition 5.1. *The one parameter Hamilton motions of a body in Minkowski 3-space are generated by transformation*

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_0 \\ 0 \end{bmatrix}, \qquad (5)$$

where $B_{3\times 3}$ is above matrix. X, X_0 and C are $n\times 1$ real matrics, A and C are continuously differentiable functions of a real parameter t; X and X_0 correspond to the position vectors of the same point P.

Theorem 5.1. *The Hamilton motion determined by equation* (5) *is a homothetic motion in E_1^3.*

Proof. The matrix B can be represented as

$$B = h\begin{bmatrix} \frac{b_{11}}{h} & \frac{b_{21}}{h} & \frac{b_{31}}{h} \\ \frac{b_{21}}{h} & \frac{b_{22}}{h} & \frac{b_{32}}{h} \\ \frac{b_{31}}{h} & \frac{b_{32}}{h} & \frac{b_{33}}{h} \end{bmatrix} = hA,$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$,

$$t \;\to\; h(t) = a_0^2(t)+a_1^2(t)-a_2^2(t)-a_3^2(t).$$

So, we finde $A^T\varepsilon A\varepsilon = A\varepsilon A^T\varepsilon = I_3$ and $\det A = 1$, *i.e.* $A \in SO_1(3)$. Thus B is a homothetic matrix, and equation (5) determines a homothetic motion. □

6. Pole point and pole curves of the motion

To find the pole point, we have to solve the equation

$$\dot{B}X + \dot{C} = 0. \qquad (6)$$

Any solution of equation (6) is a pole point of the motion at that instant in R_0. Since $\dot{B}$ is regular, the equation (6) has only one solution, *i.e.* $X_0 = (-\dot{B}^{-1})\dot{C}$ at every instant t. This pole point in the fix system as

$$X = B(-\dot{B}^{-1}\dot{C}) + C.$$

Theorem 6.1. *During the homothetic motion the pole curves slide and roll upon each others and the number of the sliding-rolling of the motion is h.*

Example 6.1. *Let $\alpha: I \subset \mathbb{R} \to E_2^4$ be a curve given by*

$$t \to \alpha(t) = (\cosh t \ , t, \ \sinh t, -1)\,, \quad \textit{for every } t \in I.$$

$\alpha(t)$ is a differentiable regular of order r. Because, $\alpha(t)$ does not pass though the origin, the matrix B can be represented as

$$\begin{aligned} B &= \begin{bmatrix} 1+t^2+\sinh^2 t.\cosh^2 t & 2(\cosh t - t\sinh t) & -2(\cosh t.\sinh t + t) \\ 2(\cosh t + t\sinh t) & 2-t^2 & 2(t\cosh t + \sinh t) \\ 2(t - \cosh t.\sinh t) & 2(t\cosh t - \sinh t) & -1-t^2+\sinh^2 t.\cosh^2 t \end{bmatrix} \\ &= t^2\, A, \end{aligned}$$

where $h(t) = t^2$, $A \in SO_1(3)$. Thus $\alpha(t)$ satisfies all conditions of the above theorems.

7. Darboux Vector of the Motion

In Euclidean 3-space, Yaylı [7] has showed the the Darboux vector of the homothetic motion which is defined by the Hamilton operators, can be written as multiplication of two real quaternions. In this section, we obtain the Darboux vector of the homothetic motion in the Minkowski 3-space and show that it can be written as multiplication of two split quaternions.

Suppose that $\alpha(t)$ is a curve as defined in (3). The Darboux matrix in the homothetic motion defined by homothetic matrix B, is

$$\Omega = B^t\, \varepsilon \dot{B}$$

So we obtain

$$\Omega = \frac{2}{h'} \begin{bmatrix} \frac{\dot{h}'}{2} & a_0\dot{a}_3 - \dot{a}_0 a_3 - \dot{a}_1 a_2 + a_1\dot{a}_2 & \dot{a}_1 a_3 - a_1\dot{a}_3 - \dot{a}_0 a_2 + a_0\dot{a}_2 \\ -(a_0\dot{a}_3 - \dot{a}_0 a_3 - \dot{a}_1 a_2 + a_1\dot{a}_2) & \frac{\dot{h}'}{2} & \dot{a}_2 a_3 - a_2\dot{a}_3 - \dot{a}_0 a_1 + a_0\dot{a}_1 \\ -(\dot{a}_1 a_3 - a_1\dot{a}_3 - \dot{a}_0 a_2 + a_0\dot{a}_2) & -(\dot{a}_2 a_3 - a_2\dot{a}_3 - \dot{a}_0 a_1 + a_0\dot{a}_1) & \frac{\dot{h}'}{2} \end{bmatrix}.$$

We investigate the Darboux matrix in special case $h' = 1$. In the case, we have

$$\Omega = 2\begin{bmatrix} 0 & a_0\dot{a}_3 - \dot{a}_0a_3 - \dot{a}_1a_2 + a_1\dot{a}_2 & \dot{a}_1a_3 - a_1\dot{a}_3 - \dot{a}_0a_2 + a_0\dot{a}_2 \\ -(\dot{a}_0a_3 - a_0\dot{a}_3 + \dot{a}_1a_2 - a_1\dot{a}_2) & 0 & \dot{a}_2a_3 - a_2\dot{a}_3 - \dot{a}_0a_1 + a_0\dot{a}_1 \\ -(\dot{a}_1a_3 - a_1\dot{a}_3 - \dot{a}_0a_2 + a_0\dot{a}_2) & -(\dot{a}_2a_3 - a_2\dot{a}_3 - \dot{a}_0a_1 + a_0\dot{a}_1) & 0 \end{bmatrix}.$$

Darboux vector corresponds to skew-symmetric matrix Ω is defined by

$$\overrightarrow{\Omega} = (\Omega_x, \Omega_y, \Omega_z).$$

Therefore, Darboux vector of the motion is obtained

$$\overrightarrow{\Omega} = 2(\dot{a}_2a_3 - a_2\dot{a}_3 - \dot{a}_0a_1 + a_0\dot{a}_1, \dot{a}_1a_3 - a_1\dot{a}_3 - \dot{a}_0a_2 + a_0\dot{a}_2, a_0\dot{a}_3 - \dot{a}_0a_3 - \dot{a}_1a_2 + a_1\dot{a}_2).$$

This vector can be written in Multiplication of split quaternions as

$$\overrightarrow{\Omega} = 2(\dot{\alpha} * \alpha^*).$$

Example 7.1. *Suppose that the curve given as*

$$\begin{aligned} \alpha \quad &: \quad I \subset \mathbb{R} \to E_2^4 \\ t \quad &\to \quad \alpha(t) = \frac{1}{2}\left(\sqrt{2}\cosh t, \sqrt{3}, \sqrt{2}\sinh t, -1\right), \quad \text{for every } t \in I. \end{aligned}$$

$\alpha(t)$ is a differentiable regular of order r. Because, $\alpha(t)$ does not pass though the origin, the matrix B can be represented as

$$B = \begin{bmatrix} 1 + \frac{1}{2}(\cosh^2 t + \sinh^2 t) & -\frac{1}{\sqrt{2}}(\cosh t - \sqrt{3}\sinh t) & -\cosh t.\sinh t + \frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{2}}(\cosh t + \sqrt{3}\sinh t) & 0 & \frac{1}{\sqrt{2}}(\sqrt{3}\cosh t - \sinh t) \\ -\frac{1}{2}(\cosh t.\sinh t + \sqrt{3}) & \frac{1}{\sqrt{2}}(\sqrt{3}\cosh t + \sinh t) & \frac{1}{2}(\cosh^2 t + \sinh^2 t) - 1 \end{bmatrix}$$

B is a homothetic matrix and is defined a homothetic motion. Darboux vector of this motion is

$$\overrightarrow{\Omega} = (-\frac{1}{\sqrt{2}}(\cosh t + \sqrt{3}\sinh t), \ \frac{1}{2}, \frac{1}{\sqrt{2}}(\sqrt{3}\cosh t + \sinh t)).$$

CONCLUSION

A motion in 3-dimensional Minkowski space defined by using a spatial curve, it is shown that, this motion is a homothetic motion. We investigated some properties of this motion and showed that it has only one pole point at every instant t .

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