TWMS J. App. Eng. Math. V.6, N.1, 2016, pp. 102-114

# CONSTRUCTION OF SHIFT INVARIANT $M$-BAND TIGHT FRAMELET PACKETS

FIRDOUS A. SHAH[1], §

Abstract. Framelets and their promising features in applications have attracted a great deal of interest and effort in recent years. In this paper, we outline a method for constructing shift invariant $M$-band tight framelet packets by recursively decomposing the multiresolution space $V_J$ for a fixed scale $J$ to level 0 with any combined mask $\mathbf{m} = [m_0, m_1, \ldots, m_L]$ satisfying some mild conditions.

Keywords: $M$-band wavelets; tight wavelet frame; framelet packet; extension principle; shift invariant; Fourier transform.

AMS Subject Classification: 42C40; 42C15; 65T60.

## 1. Introduction

The traditional wavelet frames provide poor frequency localization in many applications as they are not suitable for signals whose domain frequency channels are focused only on the middle frequency region. Therefore, in order to make more kinds of signals suited for analyzing by wavelet frames, it is necessary to extend the concept of wavelet frames to a library of wavelet frames, called *framelet packets* or *wavelet frame packets*. The original idea of framelet packets was introduced by Coifman et al.[4] to provide more efficient decomposition of signals containing both transient and stationary components. Chui and Li [3] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate orthogonal wavelets such that they may be used in a wider field. Other notable generalizations are the wavelet packets and $p$-framelet packets on the positive half-line $\mathbb{R}^+$ [15, 16], the non-stationary wavelet packets [13], the vector-valued wavelet packets [8], the $M$-band wavelet packets [10] and the tight framelet packets on $\mathbb{R}^d$ [12].

On the otherhand, the standard orthogonal wavelets are not also suitable for the analysis of high-frequency signals with relatively narrow bandwidth. To overcome this shortcoming, $M$-band orthonormal wavelets were created as a direct generalization of the 2-band wavelets [19]. The motivation for a larger $M(M > 2)$ comes from the fact that,

[1]Department of Mathematics, University of Kashmir, South Campus, Anantnag-192 101, Jammu and , Kashmir, India.
e-mail: fashah79@gmail.com;

§ Manuscript received: February 19, 2015; Accepted: January 04, 2016.
TWMS Journal of Applied and Engineering Mathematics, Vol.6, No.1; 

unlike the standard wavelet decomposition which results in a logarithmic frequency resolution, the $M$-band decomposition generates a mixture of logarithmic and linear frequency resolution and hence generates a more flexible tiling of the time-frequency plane than that resulting from 2-band wavelet. The other significant difference between 2-band wavelets and $M$-band wavelets in construction lies in the aspect that the wavelet vectors are not uniquely determined by the scaling vector and the orthonormal bases do not consist of dilated and shifted functions through a single wavelet, but consist of ones by using $M-1$ wavelets (see [1,6,11]). It is this point that brings more freedoms for optimal wavelet bases.

Tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. A catalyst for this development is the unitary extension principle (UEP) introduced by Ron and Shen [14], which provides a general construction of tight wavelet frames for $L^2(\mathbb{R}^n)$ in the shift-invariant setting, and included the pyramidal decomposition and reconstruction filter bank algorithms. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called *mother framelets.* The theory of tight wavelet frames has been extensively studied and well developed over the recent years. To mention only a few references on tight wavelet frames, the reader is referred to [2,5,7]. In the $M$-band setting, Han and Cheng [9] have provided the general construction of $M$-band tight wavelet frames on $\mathbb{R}$ by following the procedure of Daubechies et al.[5] via extension principles. They have presented a systematic algorithm for constructing tight wavelet frames generated by a given refinable function with dilation factor $M > 2$. Recently, Shah and Debnath [17] have introduced a general construction scheme for a class of stationary $M$-band tight framelet packets in $L^2(\mathbb{R})$ via extension principles.

In continuation to the investigation initiated by Shah and Debnath in [17], we provide an explicit description for the construction of a class of $M^{-J}$-shift invariant $M$-band tight framelet packets in $L^2(\mathbb{R})$ by recursively decomposing the multiresolution space $V_J$ for a fixed scale $J$ to level 0 with any combined MRA mask $\mathbf{m} = [m_0, m_1, \ldots, m_L]$ satisfying a much weaker condition $\sum_{\ell=0}^{L} |m_\ell(\xi)|^2 = 1$ than the unitary extension principle requirement $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, where $\mathcal{M}(\xi) = \left\{ m_\ell \left( \xi + \frac{2\pi p}{M} \right) \right\}_{\ell,p=0}^{M-1}$.

This paper is organized as follows. Section 2 concerns some basic concepts about $M$-band tight wavelet frames using extension principles. In Section 3, we prove a crucial lemma called the *splitting lemma* which splits a given $M$-band tight wavelet frame into $M^{-J}$-shift invariant $M$-band tight framelet packets and by virtue of this lemma, we prove our main results.

## 2. $M$-Band Tight Wavelet Frames

We begin this section by reviewing some major concepts concerning $M$-band wavelet frames. In the rest of this paper,, we use $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$ to denote the sets of all natural numbers, non-negative integers, integers and real numbers, respectively.

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as usual by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-i\xi x} dx, \quad \xi \in \mathbb{R}$$

and its inverse is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\, e^{i\xi x} d\xi, \quad x \in \mathbb{R}.$$

For given $\Psi := \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$, define the $M$-band wavelet system

$$X(\Psi) := \left\{\psi_{\ell,j,k} : j, k \in \mathbb{Z}, 1 \le \ell \le L\right\} \tag{2.1}$$

where $\psi_{\ell,j,k} = M^{j/2}\psi_\ell(M^j. - k)$. The wavelet system $X(\Psi)$ is called a *$M$-band wavelet frame*, or simply a *$M$-band framelet system*, if there exist positive numbers $0 < A \le B < \infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\|f\|_2^2 \le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\langle f, \psi_{\ell,j,k}\rangle\right|^2 \le B\|f\|_2^2. \tag{2.2}$$

The largest $A$ and the smallest $B$ for which (2.2) holds are called wavelet frame bounds. A wavelet frame is a *tight wavelet frame* if $A$ and $B$ are chosen so that $A = B = 1$ and then generators $\psi_1, \psi_2, \dots, \psi_L$ are often referred as *$M$-band framelets*. Moreover, if only the upper bound holds in the above inequality, then $X(\Psi)$ is said to be a *Bessel sequence* with Bessel constant $B$.

Next we give a brief overview of MRA-based construction of wavelet frames associated with the dilation factor $M > 2$. As we know that the $M$-band tight framelets can be constructed by the unitary extension principle (UEP) (see [5]), which uses the multiresolution analysis (MRA). The MRA often starts from a refinable function $\varphi$. A compactly supported function $\varphi$ is said to be $M$-refinable if it satisfies a refinement equation

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_0[k]\, \varphi(Mx - k). \tag{2.3}$$

for some $h_0 \in l^2(\mathbb{Z})$. By taking Fourier transform at both sides of (2.3), we have

$$\hat{\varphi}(\xi) = m_0\left(\frac{\xi}{M}\right) \hat{\varphi}\left(\frac{\xi}{M}\right), \tag{2.4}$$

where

$$m_0(\xi) = \frac{1}{M} \sum_{k\in\mathbb{Z}} h_0[k] e^{ik\xi},$$

is a $2\pi$-periodic measurable function in $L^\infty[-\pi, \pi]$ and is often called the *refinement symbol* of $\varphi$. We further assume that:

$$\lim_{\xi\to 0} \hat{\varphi}(\xi) = 1, \ \xi \in \mathbb{R}, \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \in L^\infty[-\pi, \pi]. \tag{2.5}$$

For a compactly supported refinable function $\varphi \in L^2(\mathbb{R})$, let $V_0$ be the closed shift invariant space generated by $\{\varphi(. - k) : k \in \mathbb{Z}\}$ and $V_j = \left\{\varphi(M^j.) : \varphi \in V_0\right\}$, $j \in \mathbb{Z}$. It is known that when $\varphi$ is compactly supported, then $\{V_j\}_{j\in\mathbb{Z}}$ forms a multiresolution analysis (see [5]). Recall that a multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ that satisfies: (i) $V_j \subset V_{j+1}, j \in \mathbb{Z}$, (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (see [6]).

Let $\Psi := \{\psi_1, \dots, \psi_L\} \subset V_1$, then

$$\hat{\psi}_\ell(\xi) = m_\ell\left(\frac{\xi}{M}\right) \hat{\varphi}\left(\frac{\xi}{M}\right), \tag{2.6}$$

where

$$m_\ell(\xi) = \frac{1}{M} \sum_{k\in\mathbb{Z}} h_\ell[k] e^{ik\xi}, \quad \ell = 1, \dots, L$$

are the $2\pi$-periodic measurable functions in $L^\infty[-\pi, \pi]$ and are called the *framelet symbols*.

Han and Cheng [9] gave a complete characterization of the $M$-band tight wavelet frames via the unitary extension principle. The following is the fundamental tool they gave to construct $M$-band tight wavelet frames.

**Theorem 2.1[9].** Suppose that the refinable function $\varphi$ and the framelet symbols $m_0, m_1, \dots, m_L$ satisfy (2.3)–(2.6). Define $\psi_1, ..., \psi_L$ by (2.6). Let $\mathcal{M}(\xi) = \left\{m_\ell\left(\xi + \frac{2\pi p}{M}\right)\right\}_{\ell,p=0}^{M-1}$ such that $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, for $a.e\ \xi \in \sigma(V_0) := \left\{\xi \in [-\pi, \pi] : \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \neq 0\right\}$, then $M$-band wavelet system $X(\Psi)$ forms a tight wavelet frame for $L^2(\mathbb{R})$ with frame bound 1.

Now, we introduce the notion of *$M$-band quasi-affine system* from level $J$. A $M$-band quasi-affine system from level $J$ is defined as:

**Definition 2.2.** Let $\Psi = \{\psi_1, \dots, \psi_L\}$ be a finite set of functions in $L^2(\mathbb{R})$. An $M$-band quasi-affine system from level $J$ is defined as

$$X^{q,J}(\Psi) = \left\{\psi^{q,J}_{\ell,j,k} : j, k \in \mathbb{Z}, \ell = 1, 2, \dots, L\right\}$$

where $\psi^{q,J}_{\ell,j,k}$ is defined by

$$\psi^{q,J}_{\ell,j,k} = \begin{cases} M^{j/2}\psi_\ell(M^j. - k), & j \geq J \\ M^{j-J/2}\psi_\ell\big(M^j(. - M^{-J}k)\big), & j < J \end{cases} \tag{2.7}$$

The concepts of affine and quasi-affine frames are closely related. The $M$-band quasi-affine system is obtained by oversampling the $M$-band affine system starting from level $J-1$ and downward to a $M^{-J}$-shift invariant system. Hence, the whole quasi-affine system is a $M^{-J}$-shift invariant system. This was first observed by Ron and Shen in [14] under some decay assumptions in order to convert a non-shift invariant affine system to a shift invariant system.

**Theorem 2.3.** The $M$-band affine system $X(\Psi)$ given by (2.1) is a tight frame if and only if $M$-band quasi-affine system $X^{q,J}(\Psi)$ given by (2.7) is a tight frame (or tight $M^{-J}$-quasi-affine frame).

## 3. Shift-invariant $M$-Band Tight Framelet Packets

For each $j \in \mathbb{Z}$, we define

$$V_j = \overline{\text{span}}\{\varphi_{j,k} : k \in \mathbb{Z}\},$$

and

$$W_{j,\ell} = \overline{\text{span}}\{\psi_{\ell,j,k} : k \in \mathbb{Z}\}, \ \ell = 0, 1, \dots, L.$$

Then, in view of tight frame decomposition, we have

$$V_j = V_{j-1} + \sum_{\ell=1}^{L} W_{j-1,\ell}. \tag{3.1}$$

It is immediate from the above decomposition that these $L+1$ spaces are in general not orthogonal. Therefore, by the repeated applications of (3.1), we can further split the $V_j$ spaces as:

$$V_j = V_{j-1} + \sum_{\ell=1}^{L} W_{j-1,\ell} = V_{j-2} + \sum_{r=j-2}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell} = \cdots = V_{j_0} + \sum_{r=j_0}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell} = \sum_{r=-\infty}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell}. \tag{3.2}$$

Next, we prove a splitting lemma which play a key role in the construction of shift invariant $M$-band tight framelet packets. By this lemma, we can split a given $M$-band tight wavelet frame into $M^{-J}$*-shift invariant $M$-band tight framelet packets.*

**Lemma 3.1.** *Let $g \in L^2(\mathbb{R})$, $j \leq J$ and $\{g_{j,k}^{q,J} : k \in \mathbb{Z}\}$ be a Bessel's sequence in $L^2(\mathbb{R})$ with bound $B_q$ i.e.,*

$$\sum_{k \in \mathbb{Z}} \left| \left\langle f, g_{j,k}^{q,J} \right\rangle \right|^2 \leq B_q \|f\|_2^2, \quad \textit{for any } f \in L^2(\mathbb{R}). \tag{3.3}$$

*Let $m_\ell, \ell = 0, 1, \dots, L$ be the framelet symbols associated with the refinable function $\varphi$ and the tight framelets $\psi_\ell, \ell = 1, \dots, L$ satisfying*

$$\sum_{\ell=0}^{L} \left| m_\ell(\xi) \right|^2 = 1. \tag{3.4}$$

*For $\ell = 0, 1, \dots, L$, define*

$$g_{\ell,j-1,k}^{q,J}(x) = \sum_{n \in \mathbb{Z}} m_\ell(n)\, g_{j,k}^{q,J}(x - M^{-j} n), \ k \in \mathbb{Z} \tag{3.5}$$

$$S_\ell^{q,J} = \overline{span}\{g_{\ell,j-1,k}^{q,J} : k \in \mathbb{Z}\}, \tag{3.6}$$

*and $S^{q,J} = \overline{span}\{g_{j,k}^{q,J} : k \in \mathbb{Z}\}$. Then* *(i). For $\ell = 0, 1, \dots, L$ and $k \in \mathbb{Z}$, we have*

$$\left\| g^{q,J}_{\ell,j-1,k} \right\|_2 \le \left\| g^{q,J}_{j,k} \right\|_2$$

*and*

$$\sum_{\ell=0}^{L} \left\| g^{q,J}_{\ell,j-1,k} \right\|_2 = \left\| g^{q,J}_{j,k} \right\|_2 .$$

*(ii). For any sequence $d \in l^2(\mathbb{Z})$, there exists $L+1$ sequences $\{d_\ell\}_{\ell=0}^{L}$, defined by*

$$d_\ell(k) = \sum_{n\in\mathbb{Z}} \overline{m}_\ell^{J-j}(n)\, d(n+k),\ \ k \in \mathbb{Z} \tag{3.7}$$

*such that*

$$\sum_{k\in\mathbb{Z}} |d(k)|^2 = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}} |d_\ell(k)|^2, \tag{3.8}$$

*and*

$$\sum_{k\in\mathbb{Z}} d(k)\, g^{q,J}_{j,k} = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}} d_\ell(k)\ g^{q,J}_{\ell,j-1,k}. \tag{3.9}$$

*(iii). In particular for any $f \in L^2(\mathbb{R})$, let $d(k) = \langle f, g^{q,J}_{j,k} \rangle$, $k \in \mathbb{Z}$, then $d \in l^2(\mathbb{Z})$ and* (3.7) – (3.9) *gives*

$$d_\ell(k) = \left\langle f, g^{q,J}_{\ell,j-1,k} \right\rangle, \quad k \in \mathbb{Z}, \ell = 0, 1, \dots, L, \tag{3.10}$$

$$\sum_{k\in\mathbb{Z}} \left| \left\langle f, g^{q,J}_{j,k} \right\rangle \right|^2 = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}} \left| \left\langle f, g^{q,J}_{\ell,j-1,k} \right\rangle \right|^2, \tag{3.11}$$

*and*

$$\sum_{k\in\mathbb{Z}} \left\langle f, g^{q,J}_{j,k} \right\rangle g_{j,k} = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}} \left\langle f, g^{q,J}_{\ell,j-1,k} \right\rangle g^{q,J}_{\ell,j-1,k} \tag{3.12}$$

*respectively.*

*(iv). For $\ell = 0, 1, \dots, L$, the system $\left\{ g^{q,J}_{\ell,j-1,k} : k \in \mathbb{Z} \right\}$ is a Bessel's sequence with the space decomposition*

$$S^{q,J} = S^{q,J}_0 + S^{q,J}_1 + \cdots + S^{q,J}_L. \tag{3.13}$$

*Proof.* (i) Equation (3.5) can be recast in frequency domain as

$$\hat{g}^{q,J}_{\ell,j-1,k}(\xi) = m_\ell\left(M^{-j}\xi\right) \hat{g}^{q,J}_{j,k}(\xi), \quad \ell = 0, 1, \dots, L. \tag{3.14}$$

By invoking condition (3.4) and using Parseval's formula, we obtain

$$\begin{aligned}
\sum_{\ell=0}^{L}\left\|g^{q,J}_{\ell,j-1,k}\right\|_2 &= \sum_{\ell=0}^{L}\frac{1}{2\pi}\int_{\mathbb{R}}\left|m_\ell\left(M^{-j}\xi\right)\hat{g}^{q,J}_{j,k}\left(\xi\right)\right|^2 d\xi\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\hat{g}^{q,J}_{j,k}\left(\xi\right)\right|^2\sum_{\ell=0}^{L}\left|m_\ell\left(M^{-j}\xi\right)\right|^2 d\xi\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\hat{g}^{q,J}_{j,k}\left(\xi\right)\right|^2 d\xi\\
&= \left\|g^{q,J}_{j,k}\right\|_2.
\end{aligned}$$

Hence, it follows that $\left\|g^{q,J}_{\ell,j-1,k}\right\|_2 \le \left\|g^{q,J}_{j,k}\right\|_2$.

(ii) Taking Fourier transform on both sides of (3.7), we obtain

$$\hat{d}_\ell(\xi) = \overline{m}_\ell\left(M^{J-j}\xi\right)\hat{d}\left(\xi\right), \quad \ell = 0, 1, \ldots, L. \tag{3.15}$$

Again invoking condition (3.4) and using (3.15), we obtain

$$\sum_{\ell=0}^{L}\left|\hat{d}_\ell(\xi)\right|^2 = \sum_{\ell=0}^{L}\left|m_\ell\left(M^{J-j}\xi\right)\right|^2\left|\hat{d}\left(\xi\right)\right|^2 = \left|\hat{d}\left(\xi\right)\right|^2\sum_{\ell=0}^{L}\left|m_\ell\left(M^{J-j}\xi\right)\right|^2 = \left|\hat{d}\left(\xi\right)\right|^2.$$

Therefore, it follows that

$$\sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}}\left|d_\ell(k)\right|^2 = \sum_{k\in\mathbb{Z}}\left|d(k)\right|^2,$$

and hence (3.8) is proved.

Also, the Fourier transform of (3.9) yields

$$\hat{d}\left(M^{-J}\xi\right)\hat{g}^{q,J}_{j,0}\left(\xi\right) = \sum_{\ell=0}^{L}\hat{d}_\ell\left(M^{-J}\xi\right)\hat{g}^{q,J}_{\ell,j-1,0}\left(\xi\right). \tag{3.16}$$

Therefore, in order to show that (3.9) holds, it suffices to show that the above equality holds i.e., $L.H.S = R.H.S$ which can be verified by using (3.14) and (3.15), so we have

$$\begin{aligned}
R.H.S &= \sum_{\ell=0}^{L}\hat{d}\left(M^{-J}\xi\right)\overline{\hat{m}}_\ell(M^{-j}\xi)\,\hat{m}_\ell(M^{-j}\xi)\,\hat{g}^{q,J}_{j,0}\left(\xi\right)\\
&= \hat{d}\left(M^{-J}\xi\right)\hat{g}^{q,J}_{j,0}\left(\xi\right)\sum_{\ell=0}^{L}\left|\hat{m}_\ell(M^{-j}\xi)\right|^2\\
&= L.H.S,
\end{aligned}$$

and hence, we get the desired result (3.9).

(iii). For $j \leq J$, $\left\{g_{j,k}^{q,J} : k \in \mathbb{Z}\right\}$ is a Bessel's sequence in $L^2(\mathbb{R})$, therefore it follows that $d \in l^2(\mathbb{Z})$. Moreover, if we obtain (3.10), then (3.11) and (3.12) are direct consequences of (3.8) and (3.9), respectively. Thus, it suffices to show that (3.10) is true and hence by (3.7), we obtain

$$\begin{aligned} d_\ell(k) &= \sum_{n\in\mathbb{Z}} \overline{m}_\ell^{J-j}(n) d(k+n),\ k \in \mathbb{Z} \\ &= \sum_{n\in\mathbb{Z}} \overline{m}_\ell^{J-j}(n) \left\langle f, g_{j,k+n}^{q,J} \right\rangle \\ &= \sum_{n\in\mathbb{Z}} \overline{m}_\ell(n) \left\langle f, g_{j,k+M^{J-j}n}^{q,J} \right\rangle \\ &= \left\langle f, \sum_{n\in\mathbb{Z}} m_\ell(n)\, g_{j,k+M^{J-j}n}^{q,J} \right\rangle \\ &= \left\langle f, g_{\ell,j-1,k}^{q,J} \right\rangle, \quad \ell = 0,1,\ldots,L, \end{aligned}$$

and hence part (iii) of the lemma is proved.

(iv). For any $f \in L^2(\mathbb{R})$, Eqs. (3.3) and (3.11) gives

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left| \left\langle f, g_{\ell,j-1,k}^{q,J} \right\rangle \right|^2 &\leq \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}} \left| \left\langle f, g_{\ell,j-1,k}^{q,J} \right\rangle \right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left| \left\langle f, g_{j,k}^{q,J} \right\rangle \right|^2 \\ &\leq B_q \|f\|^2, \quad \ell = 0,1,\ldots,L. \end{aligned}$$

Furthermore, from Eq. (3.5), we can get

$$g_{\ell,j-1,k}^{q,J} = \sum_{n\in\mathbb{Z}} m_\ell(n)\, g_{j,k+M^{J-j}n}^{q,J}.$$

Consequently, $S_0^{q,J} + S_1^{q,J} + \cdots + S_L^{q,J} \subseteq S^{q,J}$. On the other hand, by taking $d$ to be finitely support sequences in (3.9), we can obtain $S^{q,J} \subseteq S_0^{q,J} + S_1^{q,J} + \cdots + S_L^{q,J}$. Hence, Lemma 3.1 is proved completely. □

Next, we shall show the construction of the *basic shift invariant $M$-band framelet packets* for $L^2(\mathbb{R})$ by means of the unitary extension principle. To do this, let $X(\Psi)$ be the $M$-band tight wavelet frame for $L^2(\mathbb{R})$ constructed via UEP in an MRA $\{V_j : j \in \mathbb{Z}\}$ generated by the refinable function $\varphi$. We construct the shift invariant $M$-band framelet packets by recursively decomposing the MRA space $V_J$ for a fixed scale $J$ to level 0 with any combined MRA mask $\mathbf{h} = [h_0, h_1, \ldots, h_L]$ satisfying the condition (3.4) which is much weaker than the UEP requirement $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$.

In the first step, we decompose $V_J := \overline{\text{span}}\{\varphi_{J,k} : k \in \mathbb{Z}\}$ associated with the combined mask $\mathbf{m}_J = [m_\mathbf{r} : \mathbf{r} \in \Lambda_1]$ satisfying the condition (3.4), where $\Lambda_1$ is a $J$-tuple index set defined by

$$\Lambda_1 = \big\{(r_J, r_{J-1}, \ldots, r_1) : 0 \le r_J \le \mathcal{J},\, r_{J-1} = \cdots = r_1 = 0\big\},$$

in which $\mathcal{J}$ is a positive constant. By invoking Lemma 3.1, we can decompose $V_J$ into spaces $W^{q,J}_{J-1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_1$, where

$$\begin{aligned} W^{q,J}_{J-1,\mathbf{r}} &= \overline{\text{span}}\left\{\omega^{q,J}_{\mathbf{r},J-1,k} : k \in \mathbb{Z}\right\}, \\ \omega^{q,J}_{\mathbf{r},J-1,k}(x) &= \sum_{n\in\mathbb{Z}} m_\mathbf{r}[n]\, \varphi_{J,k}(x - M^{-J}n), \quad k \in \mathbb{Z}. \end{aligned}$$

Then, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \big|\langle f, \varphi_{J,k}\rangle\big|^2 = \sum_{\mathbf{r}\in\Lambda_1}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},J-1,k}\right\rangle\right|^2.$$

At the second level of decomposition, by Lemma 3.1, each space $W^{q,J}_{J-1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_1$ is decomposed with a combined UEP mask $\mathbf{m}_{J-1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda_2^\mathbf{r}]$ satisfying the condition (3.4), where $\Lambda_2^\mathbf{r}$ is a subset of $\Lambda_2$ defined by

$$\Lambda_2^\mathbf{r} = \big\{\mathbf{r}' \in \Lambda_2 : \mathbf{r}'(1) = \mathbf{r}(1)\big\}$$

and $\Lambda_2$ is a $J$-tuple index set defined by

$$\Lambda_2 = \left\{(r_J, r_{J-1}, \ldots, r_1) : 0 \le r_J \le \mathcal{J},\, 0 \le r_{J-1} \le \mathcal{J}^{(r_J)},\, r_{J-2} = \cdots = r_1 = 0\right\},$$

in which $\mathcal{J}^{(r_J)}$ is a positive constant for each $(r_J)$ into spaces $W^{q,J}_{J-2,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_2^\mathbf{r}$, where

$$\begin{aligned} W^{q,J}_{J-2,\mathbf{r}'} &= \overline{\text{span}}\left\{\omega^{q,J}_{\mathbf{r}',J-2,k} : k \in \mathbb{Z}\right\}, \\ \omega^{q,J}_{\mathbf{r}',J-2,k}(x) &= \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega^{q,J}_{\mathbf{r},J-1,k}(x - M^{-J+1}n),\ k \in \mathbb{Z}. \end{aligned}$$

Thus, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},J-1,k}\right\rangle\right|^2 = \sum_{\mathbf{r}'\in\Lambda_2^\mathbf{r}}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r}',J-2,k}\right\rangle\right|^2.$$

Generally, at the $p$-th level $(2 \le p \le J)$ of decomposition, by Lemma 3.1, each space $W^{q,J}_{J-p+1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{p-1}$ is decomposed with a combined UEP mask $\mathbf{m}_{J-p+1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda^{\mathbf{r}}_p]$ satisfying the condition (3.4), where $\Lambda^{\mathbf{r}}_p$ is a subset of $\Lambda_p$ defined by

$$\Lambda^{\mathbf{r}}_p = \left\{\mathbf{r}' \in \Lambda_p : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for } 1 \le n \le p-1\right\} \tag{3.17}$$

and $\Lambda_p$ is a $J$-tuple index set defined by

$$\Lambda_p = \Big\{(r_J, r_{J-1}, \ldots, r_1) : 0 \le r_J \le \mathcal{J},\ 0 \le r_{J-s} \le \mathcal{J}^{(r_J, r_{J-1}, \ldots, r_{J-s+1})}, \\ 1 \le s \le p, r_{J-p} = \cdots = r_1 = 0\Big\},$$

in which $\mathcal{J}^{(r_J, r_{J-1}, \ldots, r_{J-s+1})}$ is a positive constant for each $(r_J, r_{J-1}, \ldots, r_{J-s+1})$ into spaces $W^{q,J}_{J-p,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda^{\mathbf{r}}_2$, where

$$\begin{aligned} W^{q,J}_{J-p,\mathbf{r}'} &= \overline{\operatorname{span}}\left\{\omega^{q,J}_{\mathbf{r}',J-p,k} : k \in \mathbb{Z}\right\}, \\ \omega^{q,J}_{\mathbf{r}',J-p,k}(x) &= \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega^{q,J}_{\mathbf{r},J-p+1,k}(x - M^{-J+p-1}n),\ k \in \mathbb{Z}. \end{aligned}$$

Therefore for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},J-p+1,k}\right\rangle\right|^2 = \sum_{\mathbf{r}'\in\Lambda^{\mathbf{r}}_p} \sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r}',J-p,k}\right\rangle\right|^2.$$

In particular, at the $J$-th level of decomposition, by Lemma 3.1, each space $W^{q,J}_{1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{J-1}$ is decomposed with a combined UEP mask $\mathbf{m}_{1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda^{\mathbf{r}}_J]$ satisfying the condition (3.4), where $\Lambda^{\mathbf{r}}_J$ is a subset of $\Lambda_J$ defined by

$$\Lambda^{\mathbf{r}}_J = \left\{\mathbf{r}' \in \Lambda_J : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for } 1 \le n \le J-1\right\}$$

and $\Lambda_J$ is a $J$-tuple index set defined by

$$\Lambda_J = \left\{(r_J, r_{J-1}, \ldots, r_1) : 0 \le r_J \le \mathcal{J},\ 0 \le r_{J-s} \le \mathcal{J}^{(r_J, r_{J-1}, \ldots, r_{J-s+1})}, 1 \le s \le J\right\}, \tag{3.18}$$

in which $\mathcal{J}^{(r_J, r_{J-1}, \ldots, r_{J-s+1})}$ is a positive constant for each $(r_J, r_{J-1}, \ldots, r_{J-s+1})$ into spaces $W^{q,J}_{0,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda^{\mathbf{r}}_J$, where

$$\begin{aligned} W^{q,J}_{0,\mathbf{r}'} &= \overline{\operatorname{span}}\left\{\omega^{q,J}_{\mathbf{r}',0,k} : k \in \mathbb{Z}\right\}, \\ \omega^{q,J}_{\mathbf{r}',0,k}(x) &= \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega^{q,J}_{\mathbf{r},1,k}\left(x - M^{-1}n\right), \quad k \in \mathbb{Z}. \end{aligned}$$

Therefore for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},1,k}\right\rangle\right|^2 = \sum_{\mathbf{r}'\in\Lambda^{\mathbf{r}}_J} \sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r}',0,k}\right\rangle\right|^2.$$

Combining all the inner product equations in the above construction, we get

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_{J,k}\rangle|^2 = \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},0,k}\right\rangle\right|^2, \quad \text{for any } f \in L^2(\mathbb{R}). \tag{3.19}$$

In other words, we obtain another representation of $V_J$ as

$$V_J := \overline{\operatorname{span}}\left\{\omega^{q,J}_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J, k \in \mathbb{Z}\right\}.$$

**Theorem 3.2.** *For a given $M$-band tight wavelet frame $X(\Psi)$, the system*

$$\mathcal{F}^{q,J} = \left\{\omega^{q,J}_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J, k \in \mathbb{Z}\right\} \cup \left\{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geq J, k \in \mathbb{Z}\right\}$$

*is a $M^{-J}$-shift invariant $M$-band tight framelet packet for $L^2(\mathbb{R})$.*

*Proof.* Using (3.19) and the fact that $X(\Psi)$ is a tight wavelet frame for $L^2(\mathbb{R})$, we have

$$\begin{aligned}
\|f\|_2^2 &= \sum_{k\in\mathbb{Z}} |\langle f, \varphi_{J,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{\ell,j,k}\rangle|^2, \text{ for any } f \in L^2(\mathbb{R}) \\
&= \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \omega^{q,J}_{\mathbf{r},0,k}\right\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{\ell,j,k}\rangle|^2.
\end{aligned}$$

This completes the proof of Theorem 3.2.

As in the stationary case constructed above, we can obtain a library of tight $M$-band framelet packets of $L^2(\mathbb{R})$ by partitioning $\Lambda_J$ into disjoint subsets of the form

$$I_{j,\mathbf{r}} = \left\{(r_J, \ldots, r_{j+1}, r'_j, \ldots, r'_1) \in \Lambda_J : \mathbf{r} = (r_J, \ldots, r_{j+1}, 0, \ldots, 0) \in \Lambda_{J-j}\right\},$$

i.e.,

$$\Gamma_J = \left\{I_{j,\mathbf{r}} : \bigcup I_{j,\mathbf{r}} = \Lambda_J\right\}. \tag{3.20}$$

**Theorem 3.3.** *Let $\Gamma_J$ be a disjoint partition $\Lambda_J$, where $\Lambda_J$ and $\Gamma_J$ are defined in* (3.18) *and* (3.20)*, respectively. Then the system*

$$\mathcal{F}^{q,J}_{\Gamma_J} = \left\{\omega^{q,J}_{\mathbf{r},j,k} : I_{j,\mathbf{r}} \in \Gamma_J, k \in \mathbb{Z}\right\} \bigcup \left\{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geq J, k \in \mathbb{Z}\right\}$$

*also generates a tight frame for $L^2(\mathbb{R})$.*

*Proof.* Since $\Gamma_J$ is a disjoint partition of $\Lambda_J$, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{k\in\mathbb{Z}}\left|\left\langle f, \omega^{q,J}_{\mathbf{r},j,k}\right\rangle\right|^2 = \sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{\mathbf{r}'\in I_{j,\mathbf{r}}}\sum_{k\in\mathbb{Z}}\left|\left\langle f, \omega^{q,J}_{\mathbf{r}',0,k}\right\rangle\right|^2$$

$$= \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}}\left|\left\langle f, \omega^{q,J}_{\mathbf{r},0,k}\right\rangle\right|^2.$$

By applying Theorem 3.2, Theorem 3.3 is proved.

By Theorem 3.3, we can obtain various $M^{-J}$-shift invariant $M$-band tight framelet packets $\mathcal{F}^{q,J}_{\Gamma_J}$ from various disjoint partitions of $\Lambda_J$ . All such tight framelet packets $\mathcal{F}^{q,J}_{\Gamma_J}$ will be called *$M^{-J}$-shift invariant non-stationary $M$-band tight framelet packets.*

## References


[1] Bi,N., Dai,X. and Sun,Q., (1999), Construction of compactly supported $M$-band wavelets, Appl. Comput. Harmonic Anal., 6, pp. 113-131.

[2] Chai,A. and Shen,Z., (2007), Deconvolution: A wavelet frame approach, Numer. Math., 106, pp. 529-587.

[3] Chui,C.K. and Li,C., (1993), Non-orthogonal wavelet packets, SIAM J. Math. Anal., 24, pp. 712-738.

[4] Coifman,R.R., Meyer,Y. and Wickerhauser,M.V., (1992), Wavelet analysis and signal processing, In: M. B. Ruskai et al., eds., Wavelets and Their Applications. Jones and Bartlett, Boston, pp. 153-178.

[5] Daubechies,I., Han,B., Ron,A. and Shen,Z., (2003), Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmonic Anal., 14, pp. 1-46.

[6] Debnath,L. and Shah,F.A., (2003), Wavelet Transforms and Their Applications, Birkhäuser, New York.

[7] Han,B., (2012), Wavelets and framelets within the framework of non-homogeneous wavelet systems, In Springer Proceedings in Mathematics 13, Eds. Neamtu, M., Schumaker, L., Eds.; Springer, pp. 121-161.

[8] Han,J.C. and Chen,Q.J., (2008), Characterizations of a class of orthogonal multiple vector-valued wavelet packets, Int. J. Wavelets, Multiresolut. Inf. Process., 6(4), pp. 631-641.

[9] Han,J.C. and Cheng,Z., (2004), The construction of $M$-band tight wavelet frames, Proceedings of the Third International Conference on Machine Learning and Cybemetics, Shanghai, pp. 26-29.

[10] Jiankang,Z., Zheng,B. and Licheng,J., (1998), Theory of orthonormal $M$-band wavelet packets, J. Elect., 15, pp. 193-198.

[11] Lin,T., Xu,S., Shi,Q. and Hao,P., (2006), An algebraic construction of orthonormal $M$-band wavelets with perfect reconstruction, Appl. Comput. Harmonic Anal., 17, pp. 717-730.

[12] Lu,D. and Fan,Q., (2011), A class of tight framelet packets, Czechoslovak Math. J., 61, pp. 623-639.

[13] Nielsen,M., (2002), Highly nonstationary wavelet packets, Appl. Comput. Harmonic Anal., 12, pp. 209-229.

[14] Ron,A. and Shen,Z., (1997), Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal., 148, pp. 408-447.

[15] Shah,F.A., (2009), Construction of wavelet packets on $p$-adic field, Int. J. Wavelets, Multiresolut. Inf. Process., 7(5), pp. 553-565.

[16] Shah,F.A. and Debnath,L., (2011), $p$-Wavelet frame packets on a half-line using the Walsh-Fourier transform, Integ. Transf. Spec. Funct. 22(12), pp. 907-917.

[17] Shah,F.A. and Debnath,L., (2012), Explicit construction of $M$-band tight framelet packets, Analysis., 32, pp. 281-294.

[18] Shen,Z., (1995), Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$, SIAM J. Math. Anal., 26(4) , pp. 1061-1074.

[19] Steffen,P., Heller,N., Gopinath,R.A. and Burrus,C.S., (1993), Theory of regular $M$-band wavelet bases, IEEE Trans.Sig. Proces., 41, pp. 3497-3510.

**Firdous A. Shah** was born in 1979. He received his B. Sc. and M. Sc. Degrees from University of Kashmir in 2000 and 2002, respectively. He received his Doctorate degree in Mathematics from the Department of Mathematics, Jamia Millia Islamia, New Delhi, India in 2007. His primary research interests include basic theory of wavelets, wavelet packets, and numerical solution of differential and integral equations using wavelet methods. Dr. Shah has to his credit forty five research papers in Wavelet Analysis and their applications published in various international journals, co-authored a book entitled Wavelet Transforms and Their Applications, Birkhauser, Springer, New York, USA, 2015.