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A NOTE ON DISCRETE FRAMES OF TRANSLATES IN $\mathbb{C}^N$

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Abstract. In this note, we present necessary and sufficient conditions with explicit frame bounds for a discrete system of translates of the form $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ to be a frame for the unitary space $\mathbb{C}^N$.

Keywords: Frame, frames of translates, discrete system in $\mathbb{C}^N$.

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1. Introduction and Preliminaries

Motivated by discrete Gabor system in a finite dimensional complex space by Pfander [7], we give some frame properties of a family of the form $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ (called *family of translates*) in $\mathbb{C}^N$. The discrete wavelet structure (and wave packet) in $\mathbb{C}^N$ studied by authors in [6, 8]. Frames of translates in $L^2(\mathbb{R})$ studied by Benedetto and Li [1], Christensen et al. [4] and Daubechies [5]. A family of translates can at most be a frame for a subspace of $L^2(\mathbb{R})$, but this is not the case in $\mathbb{C}^N$. In this paper, we prove necessary and sufficient conditions with explicit frame bounds for a discrete system of translates of the form $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ to be a frame for $\mathbb{C}^N$. We also characterize generator functions associated with discrete frames of translates in $\mathbb{C}^2$

First we recall some basic definitions and notations to make the paper self-contained. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle .,. \rangle$ linear in the first entry. A countable sequence $\{f_k\}_{k\in I} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist constants $0 < \alpha_o \leq \beta_o < \infty$ such that

$$\alpha_o\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq \beta_o\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

Associated with the frame $\{f_k\}_{k\in I}$ for $\mathcal{H}$, the frame operator $S : \mathcal{H} \to \mathcal{H}$ given by

$$Sf = \sum_{k\in I}\langle f, f_k\rangle f_k, \ f \in \mathcal{H}.$$

The operator S is an invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for each $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k\in I}\langle S^{-1}f, f_k\rangle f_k = \sum_{k\in I}\langle f, S^{-1}f_k\rangle f_k. \quad (1)$$

Theorem 1.1. [3] *A family of vectors $\{f_k\}_{k=1}^m \subset \mathbb{C}^N$ is a frame for $\mathbb{C}^N$ if and only if span$\{f_k\}_{k=1}^m = \mathbb{C}^N$.*

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In the rest part of this section, we follow notations and definitions given in [7]. Let N be a positive integer. In the unitary space $\mathbb{C}^N$ an arbitrary element x is represented by $((x(0), x(1), ..., x(N-1))^T$, where x^T denotes the transpose of the vector x. More precisely, we write

$$\mathbb{C}^N = \{(x(0), x(1), ..., x(N-1))^T : x(i) \in \mathbb{C},\ i \in \mathbb{Z}^N = \{0, 1, ..., N-1\}\}.$$

Let $k \in \mathbb{Z}^N$. The *translation operator* $T_k : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$T_k((x(0), x(1), ..., x(N-1))^T = (x(0-k), x(1-k), ..., x((N-1)-k))^T,$$

where substraction is over modulo N.
For $l \in \mathbb{Z}^N$, the *modulation operator* $M_l : \mathbb{C}^N \to \mathbb{C}^N$ is defined as

$$M_l((x(0), x(1), ..., x(N-1))^T = (e^{2\pi i l0/N}x(0), e^{2\pi i l1/N}x(1), ..., e^{2\pi i l(N-1)/N}x(N-1))^T.$$

The *Fourier transform* $\mathcal{F}$ on $\mathbb{C}^N$ is given pointwise as follows (see [7] at page 196):

$$\mathcal{F}x(m) = \widehat{x}(m) = \sum_{n \in \mathbb{Z}^N} x(n)e^{-2\pi imn/N}, m \in \mathbb{Z}^N,$$

One of the major properties of the Fourier transform includes the *Fourier inversion formula* and the *Parseval-Plancherel formula*:

Theorem 1.2. [2, p. 197] *The normalized harmonics* $\frac{1}{\sqrt{N}}e^{2\pi im(\bullet)/N}$, $m = 0, 1, ..., N-1$, *form an orthonormal basis of* $\mathbb{C}^N$ *and, hence, we have*

$$x = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \widehat{x}(m)e^{2\pi im(\bullet)/N} \text{ and } \langle x, y\rangle = \frac{1}{N}\langle \widehat{x}, \widehat{y}\rangle,\ x, y \in \mathbb{C}^N.$$

In matrix notation, the Fourier transform is represented by the Fourier matrix given by

$$W_N = (\omega^{-rs})_{r,s=0}^{N-1},\ \text{where } \omega = e^{2\pi i/N}.$$

2. Discrete Frames of Translates

Definition 2.1. *Let* $\phi \in \mathbb{C}^N$. *A family of vectors* $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ *for* $\mathbb{C}^N$ *is called a* discrete frame of translates *(in short DFT) for* $\mathbb{C}^N$ *if there exists positive scalars* $a_o \leq b_o < \infty$ *such that*

$$a_o\|x\|^2 \leq \sum_{k\in\mathbb{Z}^N} |\langle T_k\phi, x\rangle|^2 \leq b_o\|x\|^2 \text{ for all } x \in \mathbb{C}^N.$$

The vector ϕ is called a *generator function* (or *scaling function*) for DFT.

Remark 2.1. *It is well known that a frame of translates for* $L^2(\mathbb{R})$ *need not be a basis for* $L^2(\mathbb{R})$ *.On the other hand, a* DFT *for* $\mathbb{C}^N$ *contains exactly* N *vectors. Hence by using the fact that a spanning set of* $\mathbb{C}^N$ *with exactly* N *vectors is linearly independent, we get that every* DFT *is a basis for* $\mathbb{C}^N$. *From this we notice that* $\{T_k\phi\}_{k\in\mathbb{Z}^1}$ *is a frame for* $\mathbb{C}^1$ *if and only if* $\phi \neq 0$.

The following theorem gives a sufficient condition for a family of translates $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ to be a frame for $\mathbb{C}^N$.

Theorem 2.1. *Let* $\phi \in \mathbb{C}^N$. *Assume that*

$$A = \inf_{m\in\mathbb{Z}^N} \left[|\widehat{\phi}(m)|^2\right] > 0.$$

Then, $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ *is a DFT for* $\mathbb{C}^N$ *with frame bounds* A *and* $N\|\phi\|^2$.

Proof. By using the Parsevals-Plancherel formula, we compute

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} |\langle T_k\phi, x\rangle|^2 &= \sum_{k\in\mathbb{Z}^N} \langle T_k\phi, x\rangle\overline{\langle T_k\phi, x\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \langle \widehat{T_k\phi}, \widehat{x}\rangle\overline{\langle \widehat{T_k\phi}, \widehat{x}\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \langle M_{-k}\widehat{\phi}, \widehat{x}\rangle\overline{\langle M_{-k}\widehat{\phi}, \widehat{x}\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \left[\sum_{n\in\mathbb{Z}^N} \widehat{\phi}(n)e^{-2\pi ink/N}\overline{\widehat{x}(n)}\right]\left[\sum_{m\in\mathbb{Z}^N} \overline{\widehat{\phi}(m)}e^{2\pi imk/N}\widehat{x}(m)\right] \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \left[\sqrt{N}\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle \sqrt{N}\overline{\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle}\right] \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} N\left|\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\right|^2 \\
&= \frac{1}{N}\sum_{m\in\mathbb{Z}^N} |\widehat{\phi}(m)|^2|\widehat{x}(m)|^2 \\
&\geq \frac{1}{N}\inf_{m\in\mathbb{Z}^N}\left[|\widehat{\phi}(m)|^2\right]\sum_{m\in\mathbb{Z}^N} |\widehat{x}(m)|^2 \\
&= \frac{A}{N}\|\widehat{x}\|^2 \\
&= A\|x\|^2 \text{ for all } x \in \mathbb{C}^N.
\end{aligned}$$

Therefore, $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ satisfies lower frame inequality with bound A.

For the upper frame inequality, we compute

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} |\langle T_k\phi, x\rangle|^2 &\leq \sum_{k\in\mathbb{Z}^N} \|T_k\phi\|^2\|x\|^2 \\
&= \|x\|^2\sum_{k\in\mathbb{Z}^N} \|T_k\phi\|^2 \\
&= N\|\phi\|^2\|x\|^2.
\end{aligned}$$

Hence $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ is a *DFT* for $\mathbb{C}^N$ with frame bounds A and $N|\phi\|^2$. □

Next we prove a necessary condition for *DFT* in $\mathbb{C}^N$.

Theorem 2.2. *Let $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ be a DFT for $\mathbb{C}^N$ with bounds A and B. Then,*

$$A \leq \sum_{m\in\mathbb{Z}^N} |\widehat{\phi}(m)|^2 \leq B. \tag{2}$$

Proof. Let $x \in \mathbb{C}^N$ be arbitrary. Then, by using the Parseval-Plancheral formula and Cauchy-Scharwtz inequality, we compute

$$\begin{aligned}
A\|x\|^2 &\leq \sum_{k\in\mathbb{Z}^N} |\langle T_k\phi, x\rangle|^2 \\
&\leq \sum_{k\in\mathbb{Z}^N} \|T_k\phi\|^2\|x\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x\|^2 \sum_{k\in\mathbb{Z}^N} \|T_k\phi\|^2 \\
&= N\|\phi\|^2\|x\|^2 \\
&= \|\widehat{\phi}\|^2\|x\|^2 \\
&= \|x\|^2 \sum_{m\in\mathbb{Z}^N} |\widehat{\phi}(m)|^2. \qquad (3)
\end{aligned}
$$

Choose $x \in \mathbb{C}^N$ such that $\|x\|^2 = 1$, then by (3), we have

$$A \;\leq \sum_{m\in\mathbb{Z}^N} |\widehat{\phi}(m)|^2.$$

Next we prove upper inequality in (2) by contradiction method. Assume that $\sum_{m\in\mathbb{Z}^N} |\widehat{\phi}(m)|^2 \; > B$. Then, there exist $m' \in \mathbb{Z}^N$ such that

$$\sup_{m\in\mathbb{Z}^N} (|\widehat{\phi}(m)|^2) = |\widehat{\phi}(m')|^2 \text{ and } N|\widehat{\phi}(m')|^2 \; > B.$$

Choose $x \in \mathbb{C}^N$ such that $\widehat{x}(m) = 0$ for $m \neq m'$ and $\widehat{x}(m) = \widehat{\phi}(m')$ for $m = m'$.
We compute

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}^N} |\langle T_k\phi, x\rangle|^2 &= \sum_{k\in\mathbb{Z}^N} \langle T_k\phi, x\rangle\overline{\langle T_k\phi, x\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \langle \widehat{T_k\phi}, \widehat{x}\rangle\overline{\langle \widehat{T_k\phi}, \widehat{x}\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \langle M_{-k}\widehat{\phi}, \widehat{x}\rangle\overline{\langle M_{-k}\widehat{\phi}, \widehat{x}\rangle} \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \left[\sum_{n\in\mathbb{Z}^N} \widehat{\phi}(n)e^{-2\pi ink/N}\overline{\widehat{x}(n)} \sum_{m\in Z^N} \overline{\widehat{\phi}(m)}e^{2\pi imk/N}\widehat{x}(m)\right] \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} \left[\sqrt{N}\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle \sqrt{N}\overline{\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle}\right] \\
&= \frac{1}{N^2}\sum_{k\in\mathbb{Z}^N} N\left|\left\langle \widehat{\phi}\,\overline{\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\right|^2 \\
&= \frac{1}{N}\sum_{m\in\mathbb{Z}^N} \left|\widehat{\phi}(m)\;\overline{\widehat{x}(m)}\right|^2 \\
&= \frac{1}{N}|\widehat{\phi}(m')\overline{\widehat{\phi}(m')}|^2 \\
&= \frac{1}{N}|\widehat{\phi}(m')|^2|\widehat{\phi}(m')|^2 \\
&> N\frac{B}{N}|\widehat{\phi}(m')|^2 \\
&= B\|\widehat{x}\|^2 \\
&= NB\|x\|^2 \\
&\geq B\|x\|^2.
\end{aligned}
$$

This shows that B is not an upper bound for $\{T_k\phi\}_{k\in\mathbb{Z}^N}$, a contradiction. Hence we must have $\sum_{m\in\mathbb{Z}^N}|\widehat{\phi}(m)|^2 \le B$. This completes the proof. □

We now demonstrate by a concrete example that condition given in Theorem 2.2 is not sufficient.

Example 2.1. *Let $N > 1$. Choose $\phi = (1, 1, ..., 1)^T \in \mathbb{C}^N$. Then, by definition of pointwise Fourier transform, we have*

$$\widehat{\phi}(0) = 1.\phi(0) + 1.\phi(1) + ... + 1.\phi(N-1) = N.$$

This gives $\sum_{m\in\mathbb{Z}^N}|\widehat{\phi}(m)|^2 = |\phi(0)|^2 + \sum_{m\in\mathbb{Z}^N\setminus\{0\}}|\widehat{\phi}(m)|^2 > 0$. Therefore, there exist $A, B > 0$ such that

$$A \le \sum_{m\in\mathbb{Z}^N}|\widehat{\phi}(m)|^2 \le B.$$

Hence condition (2) given in Theorem 2.2 is satisfied. On the other hand, the family of vectors $\{T_k\phi\}_{k\in\mathbb{Z}^N} = \{(1, 1, ..., 1)^T\}$ is not a frame for $\mathbb{C}^N$ (see Theorem 1.1).

Let $\{f_k\}_{k\in I}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k\in I}$ for $\mathcal{H}$ satisfying

$$f = \sum_{k\in I}\langle f, g_k\rangle f_k \text{ for all } f \in \mathcal{H} \tag{4}$$

is called a *dual frame* of $\{f_k\}_{k\in I}$. Let S be the frame operator for $\{f_k\}_{k\in I}$. Then, the family of vectors $\{S^{-1}f_k\}_{k\in I}$ is a frame for $\mathcal{H}$ and satisfies (4) (see equation (1)). The frame $\{S^{-1}f_k\}_{k\in I}$ is called the *canonical dual* frame of $\{f_k\}_{k\in I}$. The following theorem shows that the canonical dual of DFT in $\mathbb{C}^N$ have the same structure.

Theorem 2.3. *Suppose that $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ is a DFT for $\mathbb{C}^N$ with frame operator S. Then, the canonical dual frame of $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ is $\{T_kS^{-1}\phi\}_{k\in\mathbb{Z}^N}$.*

Proof. First we show that frame operator S commutes with translation operator. For any $k' \in \mathbb{Z}^N$ and $\psi \in \mathbb{C}^N$, we compute

$$\begin{aligned}
T_{k'}S\psi &= T_{k'}\sum_{k\in\mathbb{Z}^N}\langle\psi, T_k\phi\rangle T_k\phi \\
&= \sum_{k\in\mathbb{Z}^N}\langle\psi, T_k\phi\rangle T_{k'}T_k\phi \\
&= \sum_{k\in\mathbb{Z}^N}\langle\psi, T_k\phi\rangle T_{(k'+k)}\phi \\
&= \sum_{k\in\mathbb{Z}^N}\langle\psi, T_{(k-k')}\phi\rangle T_k\phi \\
&= \sum_{k\in\mathbb{Z}^N}\langle\psi, T_{-k'}T_k\phi\rangle T_k\phi \\
&= \sum_{k\in\mathbb{Z}^N}\langle T_{k'}\psi, T_k\phi\rangle T_k\phi \\
&= ST_{k'}\psi.
\end{aligned}$$

Therefore, the frame operator S commutes with translation operator. This gives

$$S^{-1}T_k\phi = (T_k^{-1}S)^{-1}\phi = (T_{-k}S)^{-1}\phi = (ST_{-k})^{-1}\phi = T_{-k}^{-1}S^{-1}\phi = T_kS^{-1}\phi$$

Hence the canonical dual frame of $\{T_k\phi\}_{k\in\mathbb{Z}^N}$ is $\{T_kS^{-1}\phi\}_{k\in\mathbb{Z}^N}$. The theorem is proved. □

To conclude the paper, we characterize generator functions for DFT in $\mathbb{C}^2$.

Theorem 2.4. *For $\phi = (x(0), x(1))^T \in \mathbb{C}^2$, a family of vectors $\{T_k\phi\}_{k\in\mathbb{Z}^2}$ is a DFT for $\mathbb{C}^2$ if and only if $(x(0))^2 \neq (x(1))^2$.*

Proof. First suppose that $\{T_k\phi\}_{k\in\mathbb{Z}^2}$ is a DFT for $\mathbb{C}^2$. Then, $\phi \neq 0$. Let us write $\phi = (x(0), x(1))^T = (a, b)$, where $a = x(0)$ and $b = x(1)$. Without loss of generality, let $a \neq 0$. Let, if possible, $a^2 = b^2$. Then, for $c_1 = \frac{-b}{a}, c_2 = 1 \neq 0$, we have

$$c_1(a, b)^T + c_2(b, a)^T = (\frac{-b}{a}a + b, \frac{-b^2}{a} + a)^T = (0, 0)^T.$$

which contradicts the linear independence of $\{T_k\phi\}_{k\in\mathbb{Z}^2}$. Hence $a^2 \neq b^2$.

For the converse part, assume that $a^2 \neq b^2$, where a and b are same as in forward part. Then, both a and b can not be zero. Without loss of generality, let $a \neq 0$. Let $c_1, c_2 \in \mathbb{C}$ be such that $c_1(a, b)^T + c_2(b, a)^T = 0$. Then, $c_1a + c_2b = 0$ and $c_1b + c_2a = 0$. This gives $c_1 = \frac{-c_2b}{a}$ and $\frac{(-b^2+a^2)c_2}{a} = 0$. By using that $a^2 \neq b^2$, we obtain $c_2 = c_1 = 0$. Therefore, $\{T_k\phi\}_{k\in\mathbb{Z}^2} = \{(a, b)^T, (b, a)^T\}$ is linearly independent and hence (by using Theorem 1.1) form a DFT for $\mathbb{C}^2$. □

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