A NOTE ON DISCRETE FRAMES OF TRANSLATES IN $\mathbb{C}^N$

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Abstract. In this note, we present necessary and sufficient conditions with explicit frame bounds for a discrete system of translates of the form $\{T_k\phi\}_{k \in \mathbb{Z}}^N$ to be a frame for the unitary space $\mathbb{C}^N$.

Keywords: Frame, frames of translates, discrete system in $\mathbb{C}^N$.

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1. Introduction and Preliminaries

Motivated by discrete Gabor system in a finite dimensional complex space by Pfander [7], we give some frame properties of a family of the form $\{T_k\phi\}_{k \in \mathbb{Z}}^N$ (called family of translates) in $\mathbb{C}^N$. The discrete wavelet structure (and wave packet) in $\mathbb{C}^N$ studied by authors in [6, 8]. Frames of translates in $L^2(\mathbb{R})$ studied by Benedetto and Li [1], Christensen et al. [4] and Daubechies [5]. A family of translates can at most be a frame for a subspace of $L^2(\mathbb{R})$, but this is not the case in $\mathbb{C}^N$. In this paper, we prove necessary and sufficient conditions with explicit frame bounds for a discrete system of translates of the form $\{T_k\phi\}_{k \in \mathbb{Z}}^N$ to be a frame for $\mathbb{C}^N$. We also characterize generator functions associated with discrete frames of translates in $\mathbb{C}^2$.

First we recall some basic definitions and notations to make the paper self-contained. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. A countable sequence $\{f_k\}_{k \in I} \subset \mathcal{H}$ is called a frame (or Hilbert frame) for $\mathcal{H}$ if there exist constants $0 < \alpha_o \leq \beta_o < \infty$ such that

$$\alpha_o \|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq \beta_o \|f\|^2 \text{ for all } f \in \mathcal{H}. $$

Associated with the frame $\{f_k\}_{k \in I}$ for $\mathcal{H}$, the frame operator $S : \mathcal{H} \to \mathcal{H}$ given by

$$Sf = \sum_{k \in I} \langle f, f_k \rangle f_k, \quad f \in \mathcal{H}. $$

The operator $S$ is an invertible operator on $\mathcal{H}$. This gives the reconstruction formula for each $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k \in I} \langle S^{-1}f, f_k \rangle f_k = \sum_{k \in I} \langle f, S^{-1}f_k \rangle f_k. \quad (1)$$

Theorem 1.1. [3] A family of vectors $\{f_k\}_{k=1}^m \subset \mathbb{C}^N$ is a frame for $\mathbb{C}^N$ if and only if $\text{span}\{f_k\}_{k=1}^m = \mathbb{C}^N$.

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In the rest part of this section, we follow notations and definitions given in [7]. Let \( N \) be a positive integer. In the unitary space \( \mathbb{C}^N \) an arbitrary element \( x \) is represented by \( ((x(0), x(1), \ldots, x(N - 1))^T \), where \( x^T \) denotes the transpose of the vector \( x \). More precisely, we write
\[
\mathbb{C}^N = \{ (x(0), x(1), \ldots, x(N - 1))^T : x(i) \in \mathbb{C}, \ i \in \mathbb{Z}^N = \{0, 1, \ldots, N - 1\} \}.
\]

Let \( k \in \mathbb{Z}^N \). The translation operator \( T_k : \mathbb{C}^N \to \mathbb{C}^N \) is given by
\[
T_k((x(0), x(1), \ldots, x(N - 1))^T = (x(0 - k), x(1 - k), \ldots, x((N - 1) - k))^T,
\]
where substraction is over modulo \( N \).

For \( l \in \mathbb{Z}^N \), the modulation operator \( M_l : \mathbb{C}^N \to \mathbb{C}^N \) is defined as
\[
M_l((x(0), x(1), \ldots, x(N - 1))^T = (e^{2\pi i 0/N} x(0), e^{2\pi i 1/N} x(1), \ldots, e^{2\pi i (N-1)/N} x(N - 1))^T.
\]

The Fourier transform \( \mathcal{F} \) on \( \mathbb{C}^N \) is given pointwise as follows (see [7] at page 196):
\[
\mathcal{F}x(m) = \hat{x}(m) = \sum_{n \in \mathbb{Z}^N} x(n)e^{-2\pi imn/N}, m \in \mathbb{Z}^N,
\]

One of the major properties of the Fourier transform includes the Fourier inversion formula and the Parseval-Plancherel formula:

**Theorem 1.2.** [2, p. 197] The normalized harmonics \( \frac{1}{\sqrt{N}}e^{2\pi im\bullet/N}, m = 0, 1, \ldots, N - 1 \), form an orthonormal basis of \( \mathbb{C}^N \) and, hence, we have
\[
x = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \hat{x}(m)e^{2\pi im\bullet/N} \text{ and } \langle x, y \rangle = \frac{1}{N} \langle \hat{x}, \hat{y} \rangle, \ x, y \in \mathbb{C}^N.
\]

In matrix notation, the Fourier transform is represented by the Fourier matrix given by
\[
W_N = (\omega^{-rs})_{r,s=0}^{N-1}, \text{ where } \omega = e^{2\pi i/N}.
\]

2. DISCRETE FRAMES OF TRANSLATES

**Definition 2.1.** Let \( \phi \in \mathbb{C}^N \). A family of vectors \( \{T_k\phi\}_{k \in \mathbb{Z}^N} \) for \( \mathbb{C}^N \) is called a discrete frame of translates (in short DFT) for \( \mathbb{C}^N \) if there exists positive scalars \( a_o \leq b_o < \infty \) such that
\[
a_o \|x\|^2 \leq \sum_{k \in \mathbb{Z}^N} |\langle T_k\phi, x \rangle|^2 \leq b_o \|x\|^2 \text{ for all } x \in \mathbb{C}^N.
\]

The vector \( \phi \) is called a generator function (or scaling function) for DFT.

**Remark 2.1.** It is well known that a frame of translates for \( L^2(\mathbb{R}) \) need not be a basis for \( L^2(\mathbb{R}) \). On the other hand, a DFT for \( \mathbb{C}^N \) contains exactly \( N \) vectors. Hence by using the fact that a spanning set of \( \mathbb{C}^N \) with exactly \( N \) vectors is linearly independent, we get that every DFT is a basis for \( \mathbb{C}^N \). From this we notice that \( \{T_k\phi\}_{k \in \mathbb{Z}^1} \) is a frame for \( \mathbb{C}^1 \) if and only if \( \phi \neq 0 \).

The following theorem gives a sufficient condition for a family of translates \( \{T_k\phi\}_{k \in \mathbb{Z}^N} \) to be a frame for \( \mathbb{C}^N \).

**Theorem 2.1.** Let \( \phi \in \mathbb{C}^N \). Assume that
\[
A = \inf_{m \in \mathbb{Z}^N} \left[ |\hat{\phi}(m)|^2 \right] > 0.
\]
Then, \( \{T_k\phi\}_{k \in \mathbb{Z}^N} \) is a DFT for \( \mathbb{C}^N \) with frame bounds \( A \) and \( N\|\phi\|^2 \).
Proof. By using the Parsevals-Plancherel formula, we compute
\[
\sum_{k \in \mathbb{Z}^N} |\langle T_k \phi, x \rangle|^2 = \sum_{k \in \mathbb{Z}^N} \langle T_k \phi, x \rangle \langle T_k \phi, x \rangle
\]
\[
= \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \langle \hat{T}_k \phi, \hat{x} \rangle \langle \hat{T}_k \phi, \hat{x} \rangle
\]
\[
= \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \langle M_{-k} \hat{\phi}, \hat{x} \rangle \langle M_{-k} \hat{\phi}, \hat{x} \rangle
\]
\[
= \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \left[ \sum_{n \in \mathbb{Z}^N} \hat{\phi}(n)e^{-2\pi ink/N} \overline{\hat{x}(n)} \right] \left[ \sum_{m \in \mathbb{Z}^N} \overline{\hat{\phi}(m)}e^{2\pi imk/N} \hat{x}(m) \right]
\]
\[
= \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \sqrt{N} \left( \hat{\phi} \overline{\hat{x}}, \frac{1}{\sqrt{N}} e^{2\pi i \cdot k/N} \right) \sqrt{N} \left( \hat{\phi} \overline{\hat{x}}, \frac{1}{\sqrt{N}} e^{2\pi i \cdot k/N} \right)
\]
\[
= \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} N \left| \left( \hat{\phi} \overline{\hat{x}}, \frac{1}{\sqrt{N}} e^{2\pi i \cdot k/N} \right) \right|^2
\]
\[
= \frac{1}{N} \sum_{m \in \mathbb{Z}^N} \left| \hat{\phi}(m) \right|^2 \left| \hat{x}(m) \right|^2
\]
\[
\geq \frac{1}{N} \inf_{m \in \mathbb{Z}^N} \left| \hat{\phi}(m) \right|^2 \sum_{m \in \mathbb{Z}^N} \left| \hat{x}(m) \right|^2
\]
\[
= \frac{A}{N} \left\| \hat{x} \right\|^2
\]
\[
= A \left\| x \right\|^2 \text{ for all } x \in \mathbb{C}^N.
\]
Therefore, \( \{ T_k \phi \}_{k \in \mathbb{Z}^N} \) satisfies lower frame inequality with bound \( A \).

For the upper frame inequality, we compute
\[
\sum_{k \in \mathbb{Z}^N} |\langle T_k \phi, x \rangle|^2 \leq \sum_{k \in \mathbb{Z}^N} ||T_k \phi||^2 \left\| x \right\|^2
\]
\[
= ||x||^2 \sum_{k \in \mathbb{Z}^N} ||T_k \phi||^2
\]
\[
= N ||\phi||^2 \left\| x \right\|^2.
\]
Hence \( \{ T_k \phi \}_{k \in \mathbb{Z}^N} \) is a DFT for \( \mathbb{C}^N \) with frame bounds \( A \) and \( N ||\phi||^2 \). \( \square \)

Next we prove a necessary condition for DFT in \( \mathbb{C}^N \).

Theorem 2.2. Let \( \{ T_k \phi \}_{k \in \mathbb{Z}^N} \) be a DFT for \( \mathbb{C}^N \) with bounds \( A \) and \( B \). Then,
\[
A \leq \sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2 \leq B.
\]\( \text{(2)} \)

Proof. Let \( x \in \mathbb{C}^N \) be arbitrary. Then, by using the Parseval-Plancherel formula and Cauchy-Scharwz inequality, we compute
\[
A \left\| x \right\|^2 \leq \sum_{k \in \mathbb{Z}^N} |\langle T_k \phi, x \rangle|^2
\]
\[
\leq \sum_{k \in \mathbb{Z}^N} ||T_k \phi||^2 \left\| x \right\|^2
\]
\[ \|x\|_2^2 = \sum_{k \in \mathbb{Z}^N} \|T_k \phi\|_2^2 = N\|\phi\|_2^2\|x\|_2^2 \]

\[ = \|\hat{x}\|_2^2 \sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2. \quad (3) \]

Choose \(x \in \mathbb{C}^N\) such that \(\|x\|_2^2 = 1\), then by (3), we have

\[ A \leq \sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2. \]

Next we prove upper inequality in (2) by contradiction method. Assume that \(\sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2 > B\). Then, there exist \(m' \in \mathbb{Z}^N\) such that

\[ \sup_{m \in \mathbb{Z}^N} (|\hat{\phi}(m)|^2) = |\hat{\phi}(m')|^2 \text{ and } N|\hat{\phi}(m')|^2 > B. \]

Choose \(x \in \mathbb{C}^N\) such that \(\hat{x}(m) = 0\) for \(m \neq m'\) and \(\hat{x}(m) = \hat{\phi}(m')\) for \(m = m'\).

We compute

\[ \sum_{k \in \mathbb{Z}^N} |\langle T_k \phi, x \rangle|^2 = \sum_{k \in \mathbb{Z}^N} \langle T_k \phi, x \rangle \langle T_k \phi, x \rangle \]

\[ = \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \overline{\langle T_k \phi, x \rangle} \langle T_k \phi, x \rangle \]

\[ = \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \langle M_{-k} \hat{x}, \hat{x} \rangle \]

\[ = \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} \left[ \sum_{n \in \mathbb{Z}^N} \hat{\phi}(n)e^{-2\pi i nk/N} \overline{\hat{x}(n)} \sum_{m \in \mathbb{Z}^N} \hat{\phi}(m)e^{2\pi i nk/N} \hat{x}(m) \right] \]

\[ = \frac{1}{N^2} \sum_{k \in \mathbb{Z}^N} N \left| \left\langle \hat{\phi}, \frac{1}{\sqrt{N}} e^{2\pi i (\bullet) k/N} x \right\rangle \right|^2 \]

\[ = \frac{1}{N} \sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2 |\hat{x}(m)|^2 \]

\[ = \frac{1}{N} |\hat{\phi}(m')|^2 \hat{\phi}(m')|^2 + \frac{1}{N} |\hat{\phi}(m')|^2 |\hat{\phi}(m')|^2 \]

\[ > \frac{B}{N} |\hat{\phi}(m')|^2 \]

\[ = B \|x\|_2^2 \]

\[ = N B \|x\|_2^2 \geq B \|x\|_2^2. \]
This shows that $B$ is not an upper bound for $\{T_k\phi\}_{k \in \mathbb{Z}^N}$, a contradiction. Hence we must have $\sum_{m \in \mathbb{Z}^N}|\hat{\phi}(m)|^2 \leq B$. This completes the proof. \hfill \Box

We now demonstrate by a concrete example that condition given in Theorem 2.2 is not sufficient.

**Example 2.1.** Let $N > 1$. Choose $\phi = (1, 1, ..., 1)^T \in \mathbb{C}^N$. Then, by definition of pointwise Fourier transform, we have

$$\hat{\phi}(0) = 1\phi(0) + 1\phi(1) + ... + 1\phi(N-1) = N.$$ 

This gives $\sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2 = |\phi(0)|^2 + \sum_{m \in \mathbb{Z}^N \setminus \{0\}} |\hat{\phi}(m)|^2 > 0$. Therefore, there exist $A, B > 0$ such that

$$A \leq \sum_{m \in \mathbb{Z}^N} |\hat{\phi}(m)|^2 \leq B.$$ 

Hence condition (2) given in Theorem 2.2 is satisfied. On the other hand, the family of vectors $\{T_k\phi\}_{k \in \mathbb{Z}^N} = \{(1, 1, ..., 1)^T\}$ is not a frame for $\mathbb{C}^N$ (see Theorem 1.1).

Let $\{f_k\}_{k \in I}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k \in I}$ for $\mathcal{H}$ satisfying

$$f = \sum_{k \in I} \langle f, g_k \rangle g_k \text{ for all } f \in \mathcal{H} \tag{4}$$

is called a *dual frame* of $\{f_k\}_{k \in I}$. Let $S$ be the frame operator for $\{f_k\}_{k \in I}$. Then, the family of vectors $\{S^{-1}f_k\}_{k \in I}$ is a frame for $\mathcal{H}$ and satisfies (4) (see equation (1)). The frame $\{S^{-1}f_k\}_{k \in I}$ is called the *canonical dual* frame of $\{f_k\}_{k \in I}$. The following theorem shows that the canonical dual of $DFT$ in $\mathbb{C}^N$ have the same structure.

**Theorem 2.3.** Suppose that $\{T_k\phi\}_{k \in \mathbb{Z}^N}$ is a DFT for $\mathbb{C}^N$ with frame operator $S$. Then, the canonical dual frame of $\{T_k\phi\}_{k \in \mathbb{Z}^N}$ is $\{T_kS^{-1}\phi\}_{k \in \mathbb{Z}^N}$.

**Proof.** First we show that frame operator $S$ commutes with translation operator. For any $k' \in \mathbb{Z}^N$ and $\psi \in \mathbb{C}^N$, we compute

$$T_{k'}S\psi = T_{k'} \sum_{k \in \mathbb{Z}^N} \langle \psi, T_k\phi \rangle T_k \phi = \sum_{k \in \mathbb{Z}^N} \langle \psi, T_k\phi \rangle T_{k'} T_k \phi = \sum_{k \in \mathbb{Z}^N} \langle \psi, T_k\phi \rangle T_{(k' + k)} \phi = \sum_{k \in \mathbb{Z}^N} \langle \psi, T_{(k' - k)} \phi \rangle T_k \phi = \sum_{k \in \mathbb{Z}^N} \langle \psi, T_{-k'} T_k \phi \rangle T_k \phi = \sum_{k \in \mathbb{Z}^N} \langle T_{k' \psi}, T_k \phi \rangle T_k \phi = ST_{k'} \psi.$$ 

Therefore, the frame operator $S$ commutes with translation operator. This gives

$$S^{-1}T_k\phi = (T_{-k}^{-1}S)^{-1}\phi = (T_{-k}S)^{-1}\phi = (ST_{-k})^{-1}\phi = T_{-k}S^{-1}\phi = T_kS^{-1}\phi$$

Hence the canonical dual frame of $\{T_k\phi\}_{k \in \mathbb{Z}^N}$ is $\{T_kS^{-1}\phi\}_{k \in \mathbb{Z}^N}$. The theorem is proved. \hfill \Box
To conclude the paper, we characterize generator functions for DFT in $\mathbb{C}^2$.

**Theorem 2.4.** For $\phi = (x(0), x(1))^T \in \mathbb{C}^2$, a family of vectors $\{T_k \phi\}_{k \in \mathbb{Z}^2}$ is a DFT for $\mathbb{C}^2$ if and only if $(x(0))^2 \neq (x(1))^2$.

**Proof.** First suppose that $\{T_k \phi\}_{k \in \mathbb{Z}^2}$ is a DFT for $\mathbb{C}^2$. Then, $\phi \neq 0$. Let us write $\phi = (a, b)^T = (\frac{a}{b}, \frac{b}{a}),$ where $a = x(0)$ and $b = x(1)$. Without loss of generality, let $a \neq 0$. Let, if possible, $a^2 = b^2$. Then, for $c_1 = \frac{-b}{a}, c_2 = 1 \neq 0$, we have

$$c_1(a, b)^T + c_2(b, a)^T = \left(\frac{-b}{a} a + b, \frac{-b}{a} b + a\right)^T = (0, 0)^T.$$

which contradicts the linear independence of $\{T_k \phi\}_{k \in \mathbb{Z}^2}$. Hence $a^2 \neq b^2$.

For the converse part, assume that $a^2 \neq b^2$, where $a$ and $b$ are same as in forward part. Then, both $a$ and $b$ can not be zero. Without loss of generality, let $a \neq 0$. Let $c_1, c_2 \in \mathbb{C}$ be such that $c_1(a, b)^T + c_2(b, a)^T = 0$. Then, $c_1 a + c_2 b = 0$ and $c_1 b + c_2 a = 0$. This gives $c_1 = \frac{-c_2 b}{a}$ and $\frac{b^2 + a^2 c_2}{a} = 0$. By using that $a^2 \neq b^2$, we obtain $c_2 = c_1 = 0$. Therefore, $\{T_k \phi\}_{k \in \mathbb{Z}^2} = \{(a, b)^T, (b, a)^T\}$ is linearly independent and hence (by using Theorem 1.1) form a $DFT$ for $\mathbb{C}^2$. □

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**References**


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