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SOME NEW CLASSES OF GRACEFUL DIAMETER SIX TREES

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ABSTRACT. Here we denote a *diameter six tree* by $(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; $a_i,\ i = 1, 2, \ldots, m$, $b_j,\ j =\ 1, 2, \ldots, n$, and $c_k,\ k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to a_0; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. Here we give graceful labelings to some new classes of diameter six trees $(a_0; a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ in which the branches of a diameter four tree incident on a_0 are of same type, i.e. either they are all odd branches or even branches. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree and an even branch if its center has an even degree.

Keywords: graceful labeling, diameter six tree, component moving transformation, transfers of the first and second types, BD8TF

AMS Subject Classification: 05C78

1. Introduction

Definition 1.1. A *diameter six tree* is a tree which has a representation of the form $(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n;\ c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; $a_i,\ i = 1, 2, \ldots, m$, $b_j,\ j =\ 1, 2, \ldots, n$, and $c_k,\ k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to a_0; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. We observe that in a diameter six tree with above representation $m \geq 2$, i.e. there should be at least two (vertices) a_i s adjacent to c which are the centers of diameter four trees. Here we use the notation D_6 to denote a diameter six tree.

Graceful Tree Conjecture [5] of Ringel and Kotzig was published in 1964. Numerous efforts of last five decades have failed to resolve this conjecture. One may refer to the latest survey paper of Gallian [1] to have an idea regarding the progress made so far in resolving graceful tree conjecture. All trees up to diameter five [2] are known to be graceful. Among the diameter six trees only *banana trees* [6] are known to be graceful. By a *banana tree* we mean a tree which is obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two. Here we give graceful labelings to some new classes of diameter six trees $(a_0; a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ in which all the branches of a diameter four tree incident on a_0 are of same type, i.e. either they are all odd branches or even branches. Here by a branch we mean a star, i.e. we call a star

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an odd branch if its center has an odd degree and an even branch if its center has an even degree. We use the concept of component moving and inverse transformation concepts of graceful labelings discussed in [2], [3], and [4] to give graceful labelings to diameter six trees in this paper. Next we give some fundamental prerequisite tools to deduce our results.

Definition 1.2. [3], [4] Let T be a tree and a and b be two vertices of T. By $a \rightarrow b$ *transfer* we mean that some components from a have been moved to b. The successive transfers $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_4, \ldots$ is simply written as the transfer $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \ldots$. In the transfer $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{n-1} \rightarrow a_n$, each vertex a_i, $i = 1, 2, \ldots, n-1$ is called a vertex of transfer. Let T be a labelled tree with a labeling f. We consider the vertices of T whose labels form the sequence $(a, b, a-1, b+1, a-2, b+2)$ (respectively, $(a, b, a+1, b-1, a+2, b-2)$). Let a be adjacent to some vertices having labels different from the above labels. The $a \longrightarrow b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers. The $a \longrightarrow b$ transfer is called *a transfer of the second type* if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type $a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow a-2$ (respectively, $a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a+2$), is called a *backward double 8 transfer of the first type* or $BD8TF$ a to $a-2$ (respectively, a to $a+2$).

Theorem 1.1. [3], [4] In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels), which satisfy $(n+1+i)+(n+p-i) = a+b$, $i \geq 0$ (respectively, $(n+i)+(n+p-1-i) = a+b$, $i \geq 0$). Then the following hold.
(a) By making a transfer $a \rightarrow b$ of first type we can keep an odd number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.
(b) If A contains an even number of elements, then by making a sequence of transfers of the second type $a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow a-2 \rightarrow b+2 \rightarrow \ldots$ (respectively, $a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a+2 \rightarrow b-2 \rightarrow \ldots$), an even number of elements from A can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.
(c) By a BD8TF a to $b+1$ (respectively, $b-1$), we can keep an even number of elements from A at $a, b, a-1$, and $b+1$ (respectively, $a, b, a+1$, and $b-1$), and move the rest to $a-2$ (respectively, $a+2$). The resultant tree formed in each of the above cases is graceful.
(d) Consider the transfer $R' : a \rightarrow b \rightarrow a-1 \rightarrow b+1 \rightarrow \ldots \rightarrow \ldots$ (respectively, $a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow \ldots \rightarrow \ldots$), such that R' is partitioned as $R' : T'_1 \rightarrow T'_2$, where T'_1 is sequence of transfers consisting of the transfers of the first type and BD8TF and T'_2 is a sequence of transfer of the second type. The tree T^{**} obtained from T by making the transfer R' is graceful.

Lemma 1.1. [2] If g is a graceful labeling of a tree T with n edges then the labeling g_n defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the *inverse transformation* of g is also a graceful labeling of T.

2. RESULTS

Notations: Let $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ be diameter six tree. We may have one of or both $n = 0$ and $r = 0$. For next couple of results we will consistently use the following notations.
$m_e =$ Number of diameter four trees adjacent to a_0 with centers having odd degree.
$m_o =$ Number of diameter four trees adjacent to a_0 with centers having even degree, i.e. $m = m_e + m_o$.
$n_e =$ Number of stars adjacent to a_0 with center having odd degree.
$n_o =$ Number of stars adjacent to a_0 with center having even degree, i.e. $n = n_e + n_o$.

Theorem 2.1. If $m+n$ is odd, $m_e \cong 0 \; mod \; 4$, $n_e \cong 0 \; mod \; 4$, and the branches incident on the center a_i of the diameter four tree are all odd branches or all even branches then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.
(c) If m is odd then $D_6 = \{a_0; a_1, a_2, \ldots, a_m\}$ has a graceful labeling.
(d)If m is odd then $D_6 = \{a_0; a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

Proof: We prove part (a) first. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Let us remove the pendant vertices adjacent to a_0 and represent the new graceful tree by $D_6^{(1)}$. Consider the graceful tree G as represented in Figure 1.

Let $A = \{k+1, k+2, \ldots, q-k-r-1\}$. Observe that $(k+i)+(q-r-k-i) = q-r$. Consider the sequence of transfer $T_1 : q-r \to 1 \to q-r-1 \to 2 \to q-r-2 \to \ldots \to k \to q-r-k \to k+1$ of the vertex levels in the set A. Observe that the transfer T_1 and the set A satisfy the properties of Theorem 1.1.

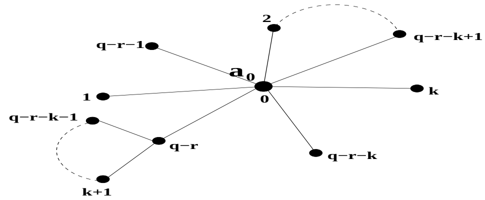


FIGURE 1. The graceful tree G.

Suppose that the centers $a_1, a_2, \ldots, a_{m_o}$ are the centers of diameter four trees adjacent to a_0 with even degree and $a_{m_o+1}, a_{m_o+2}, \ldots, a_m$ are the centers of diameter four trees adjacent to a_0 with odd degree. Suppose that the centers $b_1, b_2, \ldots, b_{n_e}$ are the centers of stars adjacent to a_0 with odd degree and $b_{n_e+1}, b_{n_e+2}, \ldots, b_n$ are the centers of stars adjacent to a_0 with even degree. Here T_1 consists of m_o successive transfers of the first type, followed by $\frac{m_e+n_e}{4}$ successive BD8TF, and finally n_o successive transfers of the first type. We carry out the transfer T_1 by keeping desired number of elements of A at each vertex of the transfer. Observe that

$$a_i = \begin{cases} q-r-\frac{i-1}{2} \text{ if } i \text{ is odd} \\ \frac{i}{2} \text{ if } i \text{ is even} \end{cases} \text{ and } b_j = \begin{cases} \begin{cases} q-r-\frac{m+j-1}{2} \text{ if } j \text{ is odd} \\ \frac{m+j}{2} \text{ if } i \text{ is even} \end{cases} & \text{if } m \text{ is even} \\ \\ \begin{cases} \frac{m+j}{2} \text{ if } j \text{ is odd} \\ q-r-\frac{m+j-1}{2} \text{ if } j \text{ is even} \end{cases} & \text{if } m \text{ is odd} \end{cases}$$

LetA_1 be the set of vertex labels of A which have come to the vertex $k+1$ after the transfer T_1. Since each transfer in T_1 is either a transfer of 1st type or a BD8TF, the elements of A_1 are the consecutive integers. Next consider the transfer $T_2 : k+1 \to q-r-k-1 \to k+2 \to q-r-k-2 \to \ldots, \to t$, where $t = \begin{cases} k+k_1+1; \text{ if } m \text{ is odd} \\ q-r-k-k_1; \text{ if } m \text{ is even} \end{cases}$, $k_1 = \sum_{i=1}^{m} deg(a_i) - m$

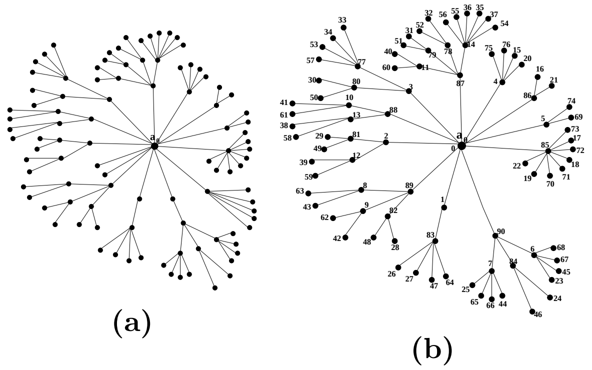
Observe that the vertices of transfer T_2 and the elements of A_1 satisfy the hypothesis of Theorem 1.1. Let the sum of all odd (or even) branches of the diameter four trees with centers $a_1, a_2, \ldots, a_m$ be s. So T_2 consists of $s-1$ successive transfers. Each transfer of T_2 is a transfer of the first or second type according as the branches of the diameter four trees are all odd or even. By executing the transfer T_2 we get back the tree $D_6^{(1)}$ and by Theorem 1.1 it is graceful. Attach the vertices $c_1, c_2, \ldots, c_r$ to a_0 and assign them the labels $q-r+1, q-r+2, q-r+3, \ldots, q$ so as to get back D_6 with a graceful labeling.

Proofs of the remaining cases follow from that of part (a). For part (b) we set $r = 0$, for part (c) we set $n = 0$ and $r = 0$, and for part (d) we set $n = 0$.

Theorem 2.2. If $r \geq 0$, $m+n$ is even, $m_e \cong 0 \; mod \; 4$, either $n_e \cong 0 \; mod \; 4$ and $n_o \geq 1$ or $n_e \cong 1 \; mod \; 4$, and the branches incident on the center a_i of the diameter four tree are all odd branches or all even branches then $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

Proof: Designate the vertex b_n as the center of a star with even degree if $n_e \cong 0 \; mod \; 4$ and $n_o \geq 1$ and the center of a star with odd degree if $n_e \cong 1 \; mod \; 4$. Construct a tree G_6 from D_6 by removing the vertices $b_n, c_1, c_2, \ldots, c_r$. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6)| = q_1$. Repeat the procedure involving the proof of Theorem 2.1(a) by replacing n with $n-1$ and $q-r$ with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach $b_n, c_1, c_2, \ldots, c_r$ to a_0 and assign the labels $q_1+1, q_1+2, \ldots, q_1+r, q_1+r+1$ to them. Obviously, the tree $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+r+1} to $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ so that the label of the vertex b_n becomes 0. By Lemma 1.1, g_{q_1+r+1} is a graceful labeling of $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign the labels $q_1+r+2, q_1+r+3, \ldots, q_1+r+p+1$ to them. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

Example 2.1. The diameter six tree in Figure 2 (a) is a diameter six of the type in Theorem 2.2. Here $q = 98$, $m = 7$, and $n = 5$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing all the pendant vertices and one star adjacent to a_0. Subsequently we get the graceful labelings of the given tree through graceful trees in figures (c), (d), and (e).



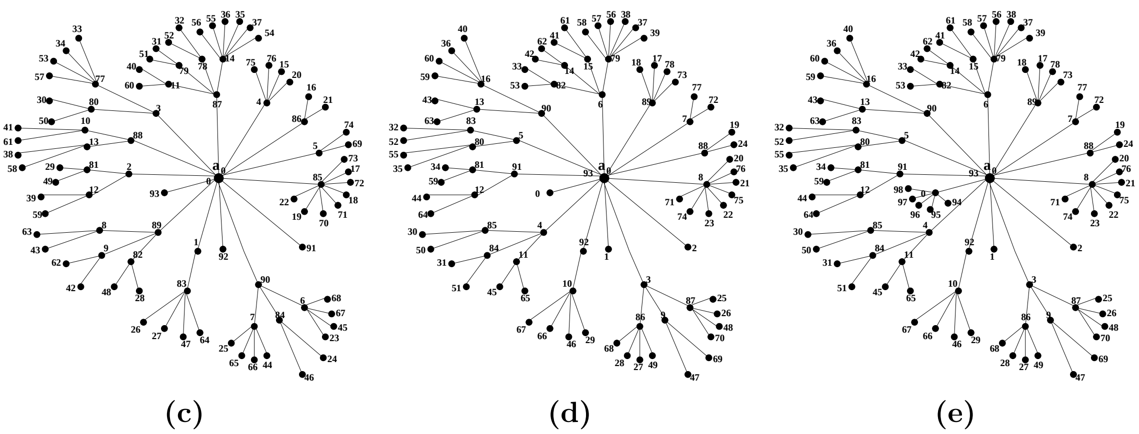


FIGURE 2. A diameter six tree of the type in Theorem 2.2(b) with a graceful labeling.

Theorem 2.3. Let $r \geq 0$, $m_e \cong 1\ mod\ 4$, the degree of at least one a_i, $1 \leq i \leq m$ is ≥ 4, and one of the following conditions hold.
(a) $n_e \cong 0\ mod\ 4$, $m+n$ is even.
(b) $m_e \cong 1\ mod\ 4$ $m+n$ is odd, either $n_e \cong 0$ and $n_o \geq 1$ or $n_e \cong 1$.
If the branches incident on the center a_i of the diameter four tree are all odd branches or all even branches then $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

Proof:(a) Let us designate the vertices a_1, a_2, ..., a_m in such a way that $deg(a_2)$ is maximum.
Case - I: If $deg(a_2)$ is even then we designate a_i, $1 \leq i \leq m_o$ as the centers of diameter four trees with even degree and a_i, $m_o + 1 \leq i \leq m$ as the centers of diameter four trees with odd degree. Excluding a_0 let there be $2p_i + 1$ neighbours of a_i, $i = 1, 2, \ldots, m_o$ and there be $2p_i$ neighbours of a_i, $i = m_o + 1, m_o + 2, \ldots, m$ in D_6. Construct a new diameter six tree, say G_6 by removing the vertices c_1, c_2, ..., c_r, and a_m and making $2p_m$ neighbours of a_m adjacent to the vertex a_2. Let $|E(G_6)| = q_1$. Repeat the procedure for giving labeling to $D_6^{(1)}$ in the proof of Theorem 2.1 (a) by replacing m_e with $m_e - 1$ and $q - r$ with q_1 and give a graceful labeling to G_6. Observe that the vertex a_2 gets label 1, and the $2(p_2 + p_m) + 1$ neighbours of a_2 get the labels $q_1 - x$, $x + 1 + i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_m$. While labeling G_6 we allot labels $x + i + 2$, $q_1 - x - i$, $i = 1, 2, \ldots, p_m$ to $2p_m$ neighbours of a_m that were shifted to a_2 while constructing G_6. Next attach the vertex a_m to a_0, assign it the label $q_1 + 1$ and move the vertices $x + i + 2, q_1 - x - i, i = 1, 2, \ldots, p_m$, to a_m. Since $(x + i + 2) + (q_1 - x - i) = q_1 + 2 = 1 + (q_1 + 1)$, for $i = 1, 2, \ldots, p_m$, by Theorem 1.1 the resultant tree, say G_1 thus formed is graceful. Finally, attach the pendant vertices c_1, $c_2, \ldots, c_r$ to a_0 and assign the labels $q_1 + 2, q_1 + 3, \ldots, q_1 + r + 1$ to them Observe that we finally get the tree D_6 with a graceful labeling.
Case - II: If $deg(a_2)$ is odd then we designate a_i, $1 \leq i \leq m_e - 1$ as the centers of diameter four trees with odd degree and a_i, $m_e \leq i \leq m - 1$ as the centers of diameter four trees with even degree, and a_m as the center of a diameter four tree with an odd degree. Excluding a_0 let there be $2p_i + 2$ neighbours of a_i, $i = 1, 2, \ldots, m_e - 1$, there be $2p_i + 1$ neighbours of a_i, $i = m_e, m_e + 1, \ldots, m - 1$, and there be $2p_m$ neighbours of a_m in D_6. Construct a new diameter six tree from D_6, say G_6 by removing the vertices c_1, c_2, ..., c_r, and a_m and making $2p_m$ neighbours of a_m adjacent to the vertex a_2. Repeat the procedure for giving graceful labeling to $D_6^{(1)}$ in the proof of Theorem 2.1 (a) by carrying the transfer $T_1 : q - r \to 1 \to q - r - 1 \to 2 \to q - r - 2 \to \ldots \to k \to q - r - k \to k + 1$ consisting of $\frac{m_e - 1}{4}$ successive BD8TF, followed by $m_o + n_o$ successive transfers of the first type, followed by $\frac{n_e}{4}$ successive BD8TF keeping desired number of elements of A at each vertex of the transfer and give a graceful labeling to G_6. Observe that the vertex a_2 gets label 1, and the $2(p_2 + p_m)$ neighbours of a_2 get the labels $q_1 - x$, $x + 1 + i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_m - 1$ and one more vertex . While labeling G_6 we allot labels $x + i + 2$, $q_1 - x - i$, $i = 1, 2, \ldots, p_m$ to $2p_m$ neighbours of a_m that were shifted to a_2 while constructing G_6. The remaining proof is same as that in Case - I.
(b): Designate the vertex a_2 as the center of diameter four tree whose degree ≥ 4. Designate the vertex a_m as the center of a diameter four tree with odd degree, say $deg(a_m) = 2p_m + 1$. Construct a tree G_6 from D_6 by removing the vertices c_1, c_2, ..., c_r, one star with center b_n, where $deg(b_n)$ is odd if $n_e \cong 0\ mod\ 4$ and $deg(b_n)$ is even if $n_e \cong 1\ mod\ 4$, and one diameter four tree with center a_m, and attaching $2p_m$ neighbours of a_m to a_2. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 2.1 (a) by replacing m with $m - 1$, n with $n - 1$ and $q - r$ with q_1 and give a graceful labeling to

G_6. Next attach the vertex a_m to a_0 and assign the label $q_1 + 1$ and shift the $2p_m$ vertices from a_2 to a_m as discussed in the proof of part (a). Let the new graceful tree thus formed be G_1. Next attach vertices c_1, c_2, ..., c_r, and b_n to a_0 and assign the labels $q_1 + 2$, $q_1 + 3$, ..., $q_1 + r + 1$, and $q_1 + r + 2$, respectively. Obviously, the tree $G_1 \cup \{c_1,\ c_2,\ \ldots,\ c_r,\ b_n\}$ is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+r+2} to G_1 so that the label of the vertex b_n becomes 0. By Lemma 1.1, g_{q_1+r+2} is a graceful labeling of G_1. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign labels $q_1 + r + 3,\ q_1 + r + 4,\ \ldots,\ q_1 + r + p + 2$ to them. Observe that we get the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

Theorem 2.4. If $m_e \cong 0 \ mod\, 4$ and $n = n_e \cong 0 \ mod\, 4$, m is even, $m_o \geq 2$, the degree of the center of at least one diameter four tree adjacent to $a_0 \geq 4$, and the branches incident on the center a_i of the diameter four tree are all odd branches or all even branches then $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Proof: Designate the vertex a_2 as the center of diameter four tree whose degree $\geq$ 4. Designate the vertex a_m as the center of a diameter four tree with even degree, say $deg(a_m) = 2p_m + 2$. Construct a tree G_6 from D_6 by removing the vertex a_m and attaching any $2p_m$(out of $2p_m + 1$) neighbours of a_m to a_2. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 2.1 (a) by replacing m with $m - 1$, n with $n - 1$ and $q - r$ with q_1 and give a graceful labeling to G_6. Next attach the vertex a_m to a_0, assign the label $q_1 + 1$, and shift the $2p_m$ vertices from a_2 to a_m as discussed in the proof of part (a) of Theorem 2.3 to a get a graceful tree, say G_1 with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to G_1 so that the label of the vertex a_{m_o} becomes 0. By Lemma 1.1, g_{q_1+1} is a graceful labeling of G_1. Now attach one remaining vertex to a_{m_o} and assign the label $q_1 + 2$ to it. Let this graceful labeling of the new tree, say G_2 thus formed be g_1. Let there be p neighbours of $q_1 + 2$ in D_6. Apply inverse transformation $g_{1_{q_1+2}}$ to G_2 so that the label of the vertex $q_1 + 2$ of G_2 becomes 0. By Lemma 1.1, g_{q_1+2} is a graceful labeling of G_2. Now attach the p pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1 + 3, q_1 + 4, \ldots, q_1 + p + 2$. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

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