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# WEAK CONVERGENCE THEOREM FOR THE ERGODIC DISTRIBUTION OF A RANDOM WALK WITH NORMAL DISTRIBUTED INTERFERENCE OF CHANCE

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Abstract. In this study, a semi-Markovian random walk process $(X(t))$ with a discrete interference of chance is investigated. Here, it is assumed that the $\zeta_n$, $n = 1, 2, 3, ...$, which describe the discrete interference of chance are independent and identically distributed random variables having restricted normal distribution with parameters $(a, \sigma^2)$. Under this assumption, the ergodicity of the process $X(t)$ is proved. Moreover, the exact forms of the ergodic distribution and characteristic function are obtained. Then, weak convergence theorem for the ergodic distribution of the process $W_a(t) \equiv X(t)/a$ is proved under additional condition that $\sigma/a \to 0$ when $a \to \infty$.

Keywords: Random walk; discrete interference of chance; normal distribution; ergodic distribution; weak convergence.

AMS Subject Classification: 60G50; 60K15.

## 1. Introduction

Many applied problems of queueing, reliability, inventory, control, insurance and other theories are formulated in terms of random walks with various types of barrier or discrete interference of chance. Some important studies on this topic exist in the literature (see, for example, Afanasyeva and Bulinskaya [2]; Aliyev et. al. [3], [4]; Alsmeyer [5]; Anisimov and Artalejo [6]; Borovkov [7]; Brown and Solomon [8]; Chang [9]; Chang and Peres [10]; Jansen and Leeuwaarden [13], [14]; Khaniyev and Mammadova [15]; Khaniyev and Aksop [16]; Khorsunov [17]; Korolyuk and Borovskikh [18]; Lotov [19]; Nasirova [19]; Siegmund [22],[23]; Skorohod and Slobodenyuk [24]; Spitzer [25] etc.).

Note that, in the studies Aliyev et. al. [3], [4] and Khaniyev and Aksop [16], the random variables $\zeta_n$, $n = 1, 2, 3...$ which describe the discrete interference of chance has gamma, triangular and beta distributions, respectively. In this study, unlike Aliyev et. al. [3], [4] and Khaniyev and Aksop [16], we assume that the random variables $\zeta_n$, $n = 1, 2, 3...$

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are independent and identically distributed random variables having restricted normal distribution.

1.1. **The Model.** Consider a stochastic model which comes up in the insurance theory. This model can be described as follows.

Suppose that amount of initial capital of an insurance company is equal to $z \in (0, \infty)$. Assume that the premiums and claims arrive to the system at the times $T_n = \sum_{i=1}^{n} \xi_i$, $n \geq 1$. Here $\xi_i$, $i \geq 1$ are inter-arrival times of two successive customers. Level of the total capital passes from a state to another by jumping in accordance with $\{-\eta_n\}$, $n \geq 1$. The random variables $\eta_n, n = 1, 2, ...$ express difference of claims and premiums. Next, amount of the total capital keeps on its variation until a random time $\tau_1$ that is the first time at which the capital level falls below zero. At the epoch $\tau_1$, the amount of the capital is immediately increased to the level $\zeta_1$ having a restricted normal distribution in the interval $[0, \infty)$. Thus, the first period is completed. Then, the insurance company keeps working in a similar way. The amount of total capital of the insurance company at time t, denote by $X(t)$. The process $X(t)$ is known as "A semi-Markovian random walk with a Normal distributed interference of chance". Now, we proceed to mathematical construction of the process $X(t)$.

## 2. Mathematical Construction of The Process $X(t)$

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ be two independent sequences of random variables defined on any probability space $(\Omega, \Im, P)$, such that variables in each sequence are independent and identically distributed. Suppose that $\xi_n$'s take only positive values, $\eta_n$'s take both positive and negative values. Introduce a sequence of normal distributed random variables $\{Y_n\}$, $n \geq 1$ with parameters $(a, \sigma^2)$, $a > 0$, $\sigma > 0$, as well. In other words, probability density function of $Y_n$ can be written as follows:

$$f_Y(x) = \frac{1}{\sigma}\varphi(\frac{x-a}{\sigma}), \quad x \in R.$$

Here $\varphi(u)$ is the probability density function of standard normal distribution, i.e.,

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} exp(-u^2/2).$$

Moreover, we put $\zeta_n \equiv \max\{0, Y_n\}$, $n = 1, 2, 3, ...$ and denote the distribution function of $\zeta_n$ by $\pi(z)$. In this case, it is hold that $\pi(z) \equiv P\{\zeta_n \leq z\} = \Phi((z-a)/\sigma)$ when $z \geq 0; and \quad \pi(z) \equiv P\{\zeta_n \leq z\} = 0$ when $z < 0$. Here, $\Phi(u)$ is the standard normal distribution function. Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0, \quad n = 1, 2, ...$$

and a sequence of integer - valued random variables $\{N_n\}$ as:

$$N_0 = 0, N_1 \equiv N(z) = \inf\{n \geq 1 : z - S_n < 0\}, z \geq 0;$$

$$N_{n+1} = \inf\{k \geq 1 : \zeta_n - (S_{N_1+N_2+...+N_n+k} - S_{N_1+N_2+...+N_n} < 0)\}, n = 1, 2, ...$$

and $\inf\{\emptyset\} = +\infty$ is stipulated.

Let $\tau_0 = 0, \quad \tau_1 \equiv \tau(z) = T_{N(z)} = \sum_{i=1}^{N(z)} \xi_i, \quad z \geq 0; \quad \tau_n = T_{N_1+...+N_n}, \quad n \geq 2$ and define $\nu(t)$ as:

$$\nu(t) = \max\{n \geq 0 : T_n \leq t\}.$$

Now, we can construct the desired stochastic process $X(t)$ as follows:

$$X(t) = \zeta_n - (S_{\nu(t)} - S_{N_0+N_1+...+N_n}), \tag{1}$$

if $\tau_n \le t < \tau_{n+1}$, $n = 0, 1, 2, ...$, $\zeta_0 = z \in [0, \infty)$.

The main purpose of this study is to prove weak convergence theorem for the ergodic distribution of the process $X(t)$, as $a \to \infty$. Therefore, we first prove the ergodicity of the process $X(t)$.

## 3. The Ergodicity of The Process $X(t)$

Firstly, we state the following theorem on the ergodicity of the process $X(t)$.

**Theorem 3.1** (The Ergodic Theorem). *Let the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$ satisfy the following supplementary conditions:*

*(i)$E(\xi_1) < \infty$; (ii) $E(\eta_1) > 0$; (iii) $E(\eta_1^2) < \infty$;*

*(iv) $\eta_1$ is non-arithmetic random variable.*

*Then, the process $X(t)$ is ergodic and for any bounded measurable function $f(x)(f : [0, \infty) \to R)$ the following relation holds, with probability 1:*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(u))du = \frac{1}{E(N(\zeta_1))}\int_0^\infty f(x)d_x(E(A(x,\zeta_1))), \tag{2}$$

*where $E(N(\zeta_1)) = \int_0^\infty E(N(z))d\pi(z)$; $E(A(x,\zeta_1)) = \int_0^\infty A(x,z)d\pi(z)$.*

$$A(x,z) = \sum_{n=0}^{\infty} a_n(x,z); a_n(x,z) = P\{z - S_i > 0; i = \overline{1,n}; z - S_n \le x\}; \quad x, z \ge 0.$$

*Proof.* The process $X(t)$ belongs to a wide class of processes which is known in the literature as the class of semi-Markov processes with a discrete interference of chance. General ergodic theorem for this class exists in literature (see, Gihman and Skorohod [12], p.243). According to this theorem, it is sufficient to verify the following assumptions for proving the ergodicity of $X(t)$:

**Assumption 1.** It is required to choose a sequence of ascending random times, such that the values of the process $X(t)$ at these times form an embedded Markov chain which is ergodic. For this reason, it is sufficient to consider the sequence of the random times $\{\tau_n\}$, $n \ge 1$ which is defined in Section 2. On the other hand, the values of the process $X(t)$ at these times $X(\tau_n) = \zeta_n \equiv \max\{0, Y_n\}$, $n = 1, 2, 3, ...$ form a sequence of the independent and identically distributed random variables. Accordingly, the embedded Markov chain $\{X(\tau_n)\}$, $n \ge 1$ is ergodic with stationary distribution function
$\pi(z) \equiv P\{\zeta_n \le z\} = \Phi((z - a)/\sigma)$, $z \ge 0$. Hence, the first assumption of the general ergodic theorem is satisfied.

**Assumption 2.** The mathematical expectation of the time between successive Markov moments $\{\tau_n\}$, $n = 1, 2, 3, ...$ must be finite, i.e., for all $n = 1, 2, 3, ...$

$$E(\tau_n - \tau_{n-1}) < \infty. \tag{3}$$

Since the random variables $\tau_n - \tau_{n-1}$, $n = 1, 2, 3, ...$ are independent and the random variables $\tau_n - \tau_{n-1}$, $n = 2, 3, ...$ are identically distributed random variables, then for holding the condition (3), it is sufficient to show that $E(\tau_1) = E(\tau(z)) < \infty$ and

$$E(\tau_n - \tau_{n-1}) \equiv \int_0^\infty E(\tau(z))d\pi(z) < \infty, \quad n = 2, 3, ... \tag{4}$$

On the other hand, by using Wald's identity (see, Feller [11], p.601), we have:

$$E(\tau(z)) = E(\sum_{i=1}^{N(z)} \xi_i) = E(\xi_1)E(N(z)). \tag{5}$$

Therefore, we have

$$E(\tau_n - \tau_{n-1}) = E(\xi_1) \int_0^{\infty} E(N(z)) d\pi(z), \quad n = 2, 3, ... \tag{6}$$

Recall that, in this case, $0 < E(\xi_1) < \infty$ is hold. To provide the condition (3), the following inequalities should be satisfied, i.e., $E(N(z)) < \infty$ and

$$\int_0^{\infty} E(N(z)) d(\pi(z)) < \infty. \tag{7}$$

For this purpose, we introduce the ladder epoch ($\nu_1^+$) and ladder height ($\chi_1^+$) of the random walk $\{S_n\}$, $n \geq 0$:

$$\nu_1^+ = \min\{n \geq 1 : S_n > 0\}; \quad \chi_1^+ = S_{\nu_1^+} = \sum_{i=1}^{\nu_1^+} \eta_i.$$

Let the random variables $(\nu_i^+, \chi_i^+)$; $i = 1, 2, ..$ be mutually independent and have the same distribution as a pair $(\nu_1^+, \chi_1^+)$ (see, Feller [11]). According to E. Dynkin's principle, $N(z)$ and $S_{N(z)}$ can be presented as follows:

$$N(z) \equiv \sum_{i=1}^{H(z)} \nu_i^+ \quad ; \quad S_{N(z)} = \sum_{i=1}^{H(z)} \chi_i^+, \tag{8}$$

where

$$H(z) = \min\{n \geq 1 : \sum_{i=1}^{n} \chi_i^+ > z\}. \tag{9}$$

By using Wald's identity, we have:

$$E(N(z)) = E(H(z)) E(\nu_1^+). \tag{10}$$

$H(z)$ is a renewal process generated by the ladder heights $\{\chi_n^+\}$, $n \geq 1$. For each $0 < z < +\infty$, the condition $E(H(z)) \equiv U_+(z) < \infty$ is satisfied (see, Feller [11]). On the other hand, since $E(\eta_1) > 0$ then $E(\nu_1^+) < +\infty$ (see, Feller [11], p.396-397). Therefore, the inequality $E(N(z)) < +\infty$ is true. Additionally, we should show that

$$EU_+(\zeta_1) \equiv \int_0^{\infty} U_+(z) d\pi(z) < +\infty \tag{11}$$

is hold. By the sharper form of the renewal theorem (see, Feller [11], p. 366)

$$U_+(z) = \frac{z}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + g(z) \tag{12}$$

can be written, as $z \to \infty$. Here, $\mu_k \equiv E((\chi_1^+)^k), \quad k = 1, 2, ..$ and the function $g(z)$ tends to zero, as $z \to \infty$, i.e., $lim_{z \to \infty} g(z) = 0$. For this reason, for each $\varepsilon > 0$ it is possible to find the number $b \equiv b(\varepsilon)$ such that $0 < b(\varepsilon) < +\infty$, and for each $z \geq b(\varepsilon)$

$$|g(z)| < \frac{\varepsilon}{2}. \tag{13}$$

The expression (11) can be rewritten as follows:

$$EU_+(\zeta_1) \equiv \int_0^{b(\varepsilon)} U_+(z) d\pi(z) + \int_{b(\varepsilon)}^{\infty} U_+(z) d\pi(z) \equiv J_1(\varepsilon) + J_2(\varepsilon). \tag{14}$$

Since the function $U_+(z)$ is a monotone non-decreasing function, then for all $0 \leq z \leq b(\varepsilon)$ is $U_+(z) \leq U_+(b(\varepsilon)) < +\infty$. Therefore,

$$J_1(\varepsilon) \equiv \int_0^{b(\varepsilon)} U_+(z)d\pi(z) \leq U_+(b(\varepsilon)) \int_0^{b(\varepsilon)} d\pi(z) \leq U_+(b(\varepsilon)). \tag{15}$$

On the other hand, because of the definition of the number $b(\varepsilon)$ according to (12), we have:

$$U_+(b(\varepsilon)) \leq \frac{b(\varepsilon)}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + \frac{\varepsilon}{2}. \tag{16}$$

Hence, from (15) and (16), we obtain the following inequality:

$$J_1(\varepsilon) \leq \frac{b(\varepsilon)}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + \frac{\varepsilon}{2}. \tag{17}$$

We now estimate the second term in (14):

$$J_2(\varepsilon) \equiv \int_{b(\varepsilon)}^{\infty} U_+(z)d\pi(z) \leq \frac{1}{\mu_1} \int_{b(\varepsilon)}^{\infty} zd\pi(z) + (\frac{\mu_2}{2\mu_1^2} + \frac{\varepsilon}{2}) \int_{b(\varepsilon)}^{\infty} d\pi(z)$$

$$\leq \frac{E(\zeta_1)}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + \frac{\varepsilon}{2}, \tag{18}$$

where

$$E(\zeta_1) = \int_0^{\infty} zd\pi(z) = \int_0^{\infty} z\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})dz = a\Phi(a/\sigma) + \sigma\varphi(a/\sigma) \leq a + \frac{\sigma}{\sqrt{2\pi}}.$$

Using (17) and (18), from (14) we have:

$$EU_+(\zeta_1) \equiv J_1(\varepsilon) + J_2(\varepsilon) \leq \frac{a + \sigma/\sqrt{2\pi}}{\mu_1} + \frac{b(\varepsilon)}{\mu_1} + \frac{\mu_2}{\mu_1^2} + \varepsilon. \tag{19}$$

Under the conditions of the Theorem 3.1, the conditions $\mu_1 > 0$ and $\mu_2 \equiv E(\chi_1^+)^2 < +\infty$ are hold. On the other hand, $a < \infty$; $\sigma < \infty$ and for any $\varepsilon > 0$ the condition $b(\varepsilon) < \infty$ is true. Therefore, from (19) we have:

$$EU_+(\zeta_1) < \infty. \tag{20}$$

Hence, $E(\tau_1) \equiv E(\tau(z)) < \infty$ and $E(\tau_n - \tau_{n-1}) < \infty$, $n = 2, 3, ...$ are proved. This shows that the Assumption 2 is also satisfied. It means that under the conditions of Theorem 3.1, the conditions of the general ergodic theorem are satisfied. Thereby, the process $X(t)$ is ergodic. In this case, for any bounded measurable function f(x) the relation (2) holds with probability 1 (see, Gihman, Skorohod [12], p.243).

This completes the proof of Theorem 3.1.

□

**Corollary 3.1.** *The ergodic distribution function* $(Q(x))$ *of the process* $X(t)$ *can be presented as follows:*

$$Q_X(x) = \lim_{t\to\infty} P\{X(t) \leq x\} = \frac{E(A(x,\zeta_1))}{E(N(\zeta_1))}. \tag{21}$$

*Proof.* Substituting the indicator function instead of the $f(x)$ in (2), we can obtain the equation (21).

□

Now, we define the characteristic function of the ergodic distribution of the process $X(t)$ as follows: $\varphi_X(\theta) \equiv \lim_{t\to\infty} E\{exp(i\theta X(t))\}$, $\theta \in R$.

**Corollary 3.2.** *The characteristic function* $(\varphi_X(\theta))$ *of the ergodic distribution of the process* $X(t)$ *can be expressed as follows:*

$$\varphi_X(\theta) = \lim_{t\to\infty} E\{exp(i\theta X(t))\} = \frac{1}{E(N(\zeta_1))}\int_0^\infty e^{i\theta z} d_z E(A(z,\zeta_1)). \tag{22}$$

Using the basic identity for the random walks (see, Feller [11], p.514), from (22), we obtain the following lemma.

**Lemma 3.1.** *Let the conditions of Theorem 3.1 be satisfied. Then, for each* $\theta \in R/\{0\}$, *the characteristic function* $\varphi_X(\theta)$ *of the ergodic distribution of the process* $X(t)$ *can be expressed by means of the characteristics of the pair* $(N(z), S_{N(z)})$ *and the random variable* $\eta_1$ *as follows:*

$$\varphi_X(\theta) = \frac{1}{EN(\zeta_1)}\int_0^\infty e^{i\theta z}\frac{\varphi_{S_{N(z)}}(-\theta)-1}{\varphi_\eta(-\theta)-1} d\pi(z), \tag{23}$$

*where*

$$EN(\zeta_1) \equiv \int_0^\infty EN(z)d\pi(z); \quad \varphi_{S_{N(z)}}(-\theta) = Eexp(-i\theta S_{N(z)}); \quad \varphi_\eta(-\theta) = Eexp(-i\theta\eta_1).$$

## 4. Weak Convergence Theorem for The Ergodic Distribution of the Process $W_a(t)$

The main aim of this section is to prove the weak convergence theorem for the ergodic distribution of the process $W_a(t) \equiv X(t)/a$, as $a \to \infty$. Before that, we need to prove the following lemma.

**Lemma 4.1.** *Under the conditions* $E(\eta_1^2) < +\infty$, $E(\eta_1) > 0$ *and* $\sigma/a \to 0$, *as* $a \to \infty$, *the following asymptotic relation can be written:*

$$E(S_{N(\zeta_1)}) = a + \tilde{\mu}_{21} + o(1). \tag{24}$$

*Here,* $\tilde{\mu}_{21} \equiv \mu_2/2\mu_1, \quad \mu_k \equiv E(\chi_1^{+k}), \quad k = 1,2,...$

*Proof.* In the study [21], Rogozin proved that if $\mu_2 \equiv E(\chi_1^{+2}) < \infty$, then the following asymptotic relation is true when $z \to \infty$:

$$E(S_{N(z)}) = z + \tilde{\mu}_{21} + o(1). \tag{25}$$

On other hand,

$$\begin{aligned} E(S_{N(\zeta_1)}) &= \int_0^\infty E(S_{N(z)})d\pi(z) \\ &= E(S_{N(0)})\bar{\Phi}(T) + \int_0^\infty E(S_{N(z)})\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})dz \\ &= E(\chi_1^+)\bar{\Phi}(T) + \int_0^\infty E(S_{N(z)})\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})dz. \end{aligned} \tag{26}$$

Here, $T \equiv a/\sigma$, $\bar{\Phi}(T) = 1 - \Phi(T)$. By the asymptotic expansion for the normal distribution function (see, Abramowitz and Stegun [1], p.298), the following asymptotic relation can be written, as $T \to \infty$:

$$\bar{\Phi}(T) = \int_t^\infty \varphi(u)du = \frac{\varphi(T)}{T}(1 + o(1)).$$

Therefore, as $a \to \infty$

$$E(\chi_1^+)\bar{\Phi}(T) = \mu_1 \frac{\varphi(T)}{T}(1 + o(1)) = \frac{\sigma\mu_1}{a}\varphi(\frac{a}{\sigma})(1 + o(1)). \tag{27}$$

To shorten of the notations, we put: $\varphi_\sigma(u) \equiv (1/\sigma)\varphi(u/\sigma)$; $M_1(z) \equiv E(S_{N(z)})$.

In this case, we can write:

$$\int_0^\infty E(S_{N(z)})\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})dz = \int_0^\infty M_1(z)\varphi_\sigma(z-a)dz$$

$$= \int_0^a M_1(z)\varphi_\sigma(z-a)dz + \int_a^\infty M_1(z)\varphi_\sigma(z-a)dz. \tag{28}$$

Denote the summands of the equation (28) as $I_1(a)$ and $I_2(a)$, respectively, i.e.,

$$I_1(a) \equiv \int_0^a M_1(z)\varphi_\sigma(a-z)dz \equiv M_1(a) * \varphi_\sigma(a),$$

$$I_2(a) \equiv \int_a^\infty M_1(z)\varphi_\sigma(z-a)dz.$$

Firstly, investigate the asymptotic behavior of $I_1(a)$, as $a \to \infty$. For this purpose, apply the Laplace transform to $I_1(a)$:

$$\tilde{I}_1(\lambda) \equiv \int_0^\infty I_1(a)e^{-\lambda a}da = \tilde{M}_1(\lambda)\tilde{\varphi}_\sigma(\lambda). \tag{29}$$

From the definition, $\tilde{\varphi}_\sigma(\lambda)$ can be presented as follows:

$$\tilde{\varphi}_\sigma(\lambda) = \int_0^\infty e^{-\lambda a}\varphi_\sigma(a)da = exp(\frac{\lambda^2\sigma^2}{2})(1 - \Phi(\lambda\sigma)). \tag{30}$$

Writing the Taylor expansion of $\Phi(\lambda\sigma)$ as $\lambda \to 0$, we have:

$$(1 - \Phi(\lambda\sigma)) = \frac{1}{2} - \frac{\sigma}{\sqrt{2\pi}}\lambda + o(\lambda^2). \tag{31}$$

On other hand, when $\lambda \to 0$, the following asymptotic relation can be written:

$$exp(\frac{\lambda^2\sigma^2}{2}) = 1 + \frac{\sigma^2}{2}\lambda^2 + o(\lambda^2). \tag{32}$$

Taking account the expansions (31) and (32) into (30), we have, as $\lambda \to 0$:

$$\tilde{\varphi}_\sigma(\lambda) = \frac{1}{2} - \frac{\sigma}{\sqrt{2\pi}}\lambda - \frac{\sigma^2}{4}\lambda^2 + o(\lambda^2). \tag{33}$$

It is known that, when $\lambda \to 0$ (Rogozin [21])

$$\tilde{M}_1(\lambda) = \frac{1}{\lambda^2} + \frac{\tilde{\mu}_{21}}{\lambda} + o(1). \tag{34}$$

Taking account the expansions (33) and (34) into (29), we obtain

$$\tilde{I}_1(\lambda) \equiv \tilde{M}_1(\lambda)\tilde{\varphi}_\sigma(\lambda) = \frac{1}{2\lambda^2} + \frac{\tilde{\mu}_{21}}{2\lambda} - \frac{\sigma}{\sqrt{2\pi}}\frac{1}{\lambda} + o(1). \tag{35}$$

According to Tauber - Abel theorem, from (35), we have:

$$I_1(a) \equiv M_1(a)\varphi_\sigma(a) = \frac{a}{2} + [\frac{\tilde{\mu}_{21}}{2} - \frac{\sigma}{\sqrt{2\pi}}] + o(1). \tag{36}$$

Now, investigate the asymptotic behavior of $I_2(a)$, as $a \to \infty$. It is known that, when $z \to \infty$ (see, Rogozin [21]).

$$M_1(z) = z + \tilde{\mu}_{21} + g(z). \tag{37}$$

where the function $g(z)$ is bounded function and it goes to zero when $z \to \infty$. Because of that $I_2(a)$ can be presented as follows:

$$I_2(a) = \int_a^\infty M_1(z)\varphi_\sigma(z-a)dz = \int_a^\infty [z + \tilde{\mu_{21}} + g(z)]\varphi_\sigma(z-a)dz$$

$$= \int_a^\infty z\varphi_\sigma(z-a)dz + \tilde{\mu}_{21}\int_a^\infty \varphi_\sigma(z-a)dz + \int_a^\infty g(z)\varphi_\sigma(z-a)dz. \tag{38}$$

Moreover,

$$\int_a^\infty \varphi_\sigma(z-a)dz = \int_0^\infty \varphi_\sigma(v)dv = \int_0^\infty \frac{1}{\sigma}\varphi(\frac{v}{\sigma})dv$$

$$= \int_0^\infty \varphi(u)du = \frac{1}{2}; \quad \int_a^\infty z\varphi_\sigma(z-a)dz = \frac{a}{2} + \frac{\sigma}{\sqrt{2\pi}}.$$

When $z \to \infty$, $g(z) \to 0$ is satisfied. Then, the following asymptotic relation is hold: $\int_a^\infty g(z)\varphi_\sigma(z-a)dz \to 0$. Therefore,

$$I_2(a) = \frac{a}{2} + \frac{\sigma}{\sqrt{2\pi}} + \frac{\tilde{\mu}_{21}}{\sigma} + o(1). \tag{39}$$

Taking account the expansions (36) and (39) into (28), we have, as $a \to \infty$:

$$\int_0^\infty E(S_{N(z)})\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})dz = \int_0^\infty M_1(z)\varphi_\sigma(z-a)dz = I_1(a) + I_2(a)$$

$$= \frac{a}{2} + \frac{\tilde{\mu}_{21}}{\sigma} - \frac{\sigma}{\sqrt{2\pi}} + o(1) + \frac{a}{2} + \frac{\sigma}{\sqrt{2\pi}} + \frac{\tilde{\mu}_{21}}{\sigma} + o(1) = a + \tilde{\mu}_{21} + o(1). \tag{40}$$

Substituting the expansions (40) and (27) in (26), we finally obtain, as $a \to \infty$:

$$E(S_{N(\zeta_1)}) = \frac{\sigma\mu_1}{a}\varphi(\frac{a}{\sigma})(1 + o(1)) + a + \tilde{\mu}_{21} + o(1) = a + \tilde{\mu}_{21} + o(1). \tag{41}$$

This completes the proof of Lemma 4.1.

□

Now, let us investigate the asymptotic behavior of the characteristic function of the ergodic distribution of the process $W_a(t) \equiv X(t)/a$, as $a \to \infty$. For this purpose, we put:

$$\varphi_W(\theta) \equiv \lim_{t\to\infty} E\{exp(i\theta W_a(t))\}.$$

**Theorem 4.1.** *Under the conditions of the Lemma 4.1, for the characteristic function $(\varphi_W(\theta))$ of the ergodic distribution of the process $W_a(t)$, the following asymptotic expansion can be written, as $a \to \infty$:*

$$\varphi_W(\theta) = \frac{e^{i\theta} - 1}{i\theta} + \frac{1}{a}C(\theta) + o(\frac{1}{a}),$$

*where* $C(\theta) \equiv [e^{i\theta} - 1]\tilde{m}_{21} - [(e^{i\theta} - 1 - i\theta)/(i\theta)]\tilde{\mu}_{21}$, $\tilde{\mu}_{21} = \mu_2/2\mu_1$,
$\tilde{m}_{21} = m_2/2m_1$, $\mu_k = E(\chi_1^{+k})$, $m_k = E(\eta_k)$, $k = 1, 2, ..$

*Proof.* $W_a(t) \equiv X(t)/a$ is a linear transform of the process $X(t)$. Hence, under the conditions of Theorem 3.1, the process $W_a(t)$ is ergodic. In this case, the characteristic function ($\varphi_W(\theta)$) of the ergodic distribution of the process $W_a(t)$ is expressed in terms of the characteristic function ($\varphi_X(\theta)$) of the ergodic distribution of the process $X(t)$ as follows:

$$\varphi_W(\theta) = \varphi_X(\frac{\theta}{a}). \tag{42}$$

On the other hand, according to Lemma 3.2, the characteristic function $\varphi_X(\theta)$ is presented as follows:

$$\varphi_X(\theta) = \frac{1}{E(N(\zeta_1))}\int_0^\infty e^{i\theta z}\frac{E\{exp(-i\theta S_{N(z)})\}-1}{E\{exp(-i\theta\eta_1)\}-1}d\pi(z). \tag{43}$$

Using (42) and (43), the function $\varphi_W(\theta)$ can be rewritten in the following form:

$$\varphi_W(\theta) = \frac{1}{E(N(\zeta_1))}\int_0^\infty e^{i\frac{\theta}{a}z}\frac{E\{exp(-i\frac{\theta}{a}S_{N(z)})\}-1}{E\{exp(-i\frac{\theta}{a}\eta_1)\}-1}d\pi(z). \tag{44}$$

According to Wald identity,

$$M_1(z) \equiv E(S_{N(z)}) = E(\eta_1)E(N(z)) = m_1E(N(z)).$$

Therefore, we have

$$E(N(z)) = \frac{E(S_{N(z)})}{m_1}$$

and

$$E(N(\zeta_1)) \equiv \int_0^\infty E(N(z))d\pi(z) = \frac{1}{m_1}\int_0^\infty M_1(z)d\pi(z) = \frac{E(M_1(\zeta_1))}{m_1}.$$

On the other hand, $E(M_1(\zeta_1)) = a + \tilde{\mu}_{21} + o(1)$ (see, Lemma 4.1).
Therefore,

$$E(N(\zeta_1)) = \frac{1}{m_1}[a + \tilde{\mu}_{21} + o(1)] = \frac{a}{m_1}[1 + \frac{\tilde{\mu}_{21}}{a} + o(\frac{1}{a})]. \tag{45}$$

Since $m_2 < \infty$, the following expansion can be written, as $a \to \infty$ (see, Feller [11], p.514):

$$E(exp(-i\frac{\theta}{a}\eta_1)) = 1 - i\frac{\theta}{a}m_1 + \frac{(i\theta)^2}{2a^2}m_2 + o(\frac{1}{a^2}). \tag{46}$$

Then,

$$\begin{aligned} E(exp(-i\frac{\theta}{a}\eta_1)) - 1 &= -i\frac{\theta}{a}m_1 + \frac{(i\theta)^2}{2a^2}m_2 + o(\frac{1}{a^2}) \\ &= -i\frac{\theta}{a}m_1\{1 - \frac{i\theta}{a}\tilde{m}_{21} + o(\frac{1}{a})\}. \end{aligned} \tag{47}$$

Taking (45) and (47) into consideration, we obtain:

$$\begin{aligned} I(a) \equiv E(N(\zeta_1))[E(exp(-i\frac{\theta}{a}\eta_1)) - 1] &= -i\theta[1 + \frac{1}{a}\tilde{\mu}_{21} + o(\frac{1}{a})] \\ = [1 - \frac{i\theta}{a}\tilde{m}_{21} + o(\frac{1}{a})] &= -i\theta\{1 + \frac{1}{a}[\tilde{\mu}_{21} - i\theta\tilde{m}_{21}] + o(\frac{1}{a})\}. \end{aligned} \tag{48}$$

Now, we denote the numerator of fraction (44) by J(a) and rewrite it as follows:

$$\begin{aligned} J(a) &= \int_0^\infty e^{i\frac{\theta}{a}z}[E(exp(-i\frac{\theta}{a}S_{N(z)})) - 1]d\pi(z) \\ &= \int_0^\infty \{E(exp(-i\frac{\theta}{a}[S_{N(z)} - z])) - exp(i\theta)exp(-i\frac{\theta}{a}(z-a))\}d\pi(z) \\ &= E(exp(-i\frac{\theta}{a}\bar{S}_{N(\zeta_1)})) - e^{i\theta}E(exp(i\frac{\theta}{a}(\zeta_1 - a))), \end{aligned} \tag{49}$$

where $\bar{S}_{N(\zeta_1)} \equiv S_{N(\zeta_1)} - \zeta_1$. Besides,

$$E(\zeta_1) = \int_0^\infty z d\pi(z) = \int_0^\infty z\frac{1}{\sigma}\varphi(\frac{z-a}{\sigma})d\pi(z)$$

$$= \int_0^\infty (a+\sigma v)\frac{1}{\sigma}\varphi(v)\sigma dv = \int_{-\infty}^{+\infty}(a+\sigma v)\varphi(v)dv - \int_{-\infty}^{-T}(a+\sigma v)\varphi(v)dv$$

$$= a - a\sigma\frac{\varphi(T)}{a}(1+o(1)) + \sigma\varphi(T)$$

$$= a + o(\varphi(\frac{a}{\sigma})), \tag{50}$$

where $T \equiv a/\sigma$.
Therefore, the following expansion can be written, as $a \to \infty$ (see, Feller [12], p.514):

$$E(exp(i\frac{\theta}{a}(\zeta_1 - a))) = 1 - \frac{i\theta}{a}E(\zeta_1 - a) + o(\frac{1}{a}E(\zeta_1 - a))$$

$$= 1 + o(\frac{1}{a}exp(-\frac{a^2}{2\sigma^2})). \tag{51}$$

On the other hand, taking Lemma 4.1 into consideration, we have, as $a \to \infty$ (see, Feller [11], p.514):

$$E(exp(-i\frac{\theta}{a}\bar{S}_{N(\zeta_1)})) = 1 - \frac{i\theta}{a}[E(S_{N(\zeta_1)}) - E(\zeta_1)] + o(\frac{1}{a})$$

$$= 1 - \frac{i\theta}{a}[a + \tilde{\mu}_{21} + o(1) - (a + o(\varphi(\frac{a}{\sigma})))] + o(\frac{1}{a})$$

$$= 1 - \frac{i\theta}{a}\tilde{\mu}_{21} + o(\frac{1}{a}). \tag{52}$$

Taking expansions (51) and (52) into consideration, we obtain the following expansion for the $J(a)$:

$$J(a) = 1 - \frac{i\theta}{a}\tilde{\mu}_{21} + +o(\frac{1}{a}) - e^{i\theta}(1 + o(\frac{1}{a}exp(\frac{-a^2}{2\sigma^2})))$$

$$= 1 - e^{i\theta} - \frac{i\theta}{a}\tilde{\mu}_{21} + o(\frac{1}{a}). \tag{53}$$

From (48) and (53), we get the following asymptotic expansion for the characteristic function $\varphi_W(\theta)$, as $a \to \infty$:

$$\varphi_W(\theta) = \frac{J(a)}{I(a)} = \frac{1 - e^{i\theta} - \frac{i\theta}{a}\tilde{\mu}_{21} + o(\frac{1}{a})}{-i\theta\{1 + \frac{1}{a}[\tilde{\mu}_{21} - i\theta\tilde{m}_{21}] + o(\frac{1}{a})\}}$$

$$= \frac{e^{i\theta}-1}{-i\theta}\{1 + \frac{i\theta\tilde{\mu}_{21}}{(e^{i\theta}-1)a} + o(\frac{1}{a})\}.\{1 - \frac{1}{a}[\tilde{\mu}_{21} + i\theta\tilde{m}_{21}] + o(\frac{1}{a})\}$$

$$= \frac{e^{i\theta}-1}{i\theta} + \frac{\tilde{\mu}_{21}}{a} - \frac{e^{i\theta}-1}{i\theta a}(\tilde{\mu}_{21} - i\theta\tilde{m}_{21}) + o(\frac{1}{a})$$

$$= \frac{e^{i\theta}-1}{i\theta} + \frac{1}{a}\{(e^{i\theta}-1)\tilde{m}_{21} - [\frac{e^{i\theta}-1-i\theta}{i\theta}]\tilde{\mu}_{21}\} + o(\frac{1}{a})$$

$$= \frac{e^{i\theta}-1}{i\theta} + \frac{C(\theta)}{a} + o(\frac{1}{a}), \tag{54}$$

where $C(\theta) = (e^{i\theta}-1)\tilde{m}_{21} - [(e^{i\theta}-1-i\theta)/(i\theta)]\tilde{\mu}_{21}$.

Thus, we obtained the asymptotic expansion (54) for the characteristic function $\varphi_W(\theta)$. This completes the proof of Theorem 4.1.

□

**Theorem 4.2** (Weak Convergence Theorem)**.** *Under the conditions of Theorem 4.1, the ergodic distribution* $(Q_W(x))$ *of the process* $W_a(t)$ *weakly converges to Uniform distribution over the interval* $[0,1]$*, i.e., for each* $x \in [0,1]$*, as* $a \to \infty$*,*

$$Q_W(x) \to G(x) \equiv x,$$

*where* $Q_W(x) \equiv \lim_{t\to\infty} P\{W_a(t) \le x\}$.

*Proof.* In Theorem 4.1, the expansion (54) is obtained for the characteristic function $(\varphi_W(\theta))$ of the ergodic distribution of the process $W_a(t)$. First term of this expansion is $\varphi_0(\theta) \equiv (e^{i\theta}-1)/(i\theta)$. This is the explicit expression of the characteristic function of the uniform distribution over the interval $[0,1]$. Second term of the expansion (54) is equal to $C(\theta)/a$ where

$$C(\theta) = (e^{i\theta}-1)\tilde{m}_{21} - [\frac{e^{i\theta}-1-i\theta}{i\theta}]\tilde{\mu}_{21}.$$

For each $\theta \in R$, the following inequalities are well known in literature:

$$|e^{i\theta}-1| \le |\theta|, \quad |e^{i\theta}-1-i\theta| \le \frac{|\theta^2|}{2}.$$

Using these inequalities, we can evaluate $C(\theta)$:

$$|C(\theta)| \le \tilde{m}_{21}|e^{i\theta}-1| + \tilde{\mu}_{21}|\frac{e^{i\theta}-1-i\theta}{i\theta}|$$

$$\le \tilde{m}_{21}|\theta| + \tilde{\mu}_{21}\frac{|\theta|^2}{2|\theta|} = \tilde{m}_{21}|\theta| + \tilde{\mu}_{21}\frac{|\theta|}{2} = (\tilde{m}_{21} + \frac{\tilde{\mu}_{21}}{2})|\theta|.$$

According to conditions of the Theorem 4.2, $E(\eta_1) > 0$ and $E(\eta_1^2) < \infty$. Because of that $\tilde{m}_{21}$ and $\tilde{\mu}_{21}$ are finite. Therefore, for each finite $\theta$, $C(\theta)$ is finite. Hence, $C(\theta)/a \to 0$, as $a \to \infty$. Finally, we have:

$$\lim_{a\to\infty} \varphi_W(\theta) = \varphi_0(\theta) = \frac{e^{i\theta}-1}{i\theta}.$$

According to the Continuity Theorem for characteristic functions (see, Feller [11], p.508), the ergodic distribution function of the process $W_a(t)$ weakly converges to the limit distribution function $G(x) \equiv x$, as $a \to \infty$, uniformly for $x \in [0,1]$; i.e., for each $x \in [0,1]$, as $a \to \infty$,

$$\lim_{t\to\infty} P\{W_a(t) \le x\} \equiv Q_W(x) \to G_0(x) \equiv x.$$

This completes the proof of Theorem 4.2. □

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