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LOWER AND UPPER SOLUTIONS FOR GENERAL TWO-POINT FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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Abstract. This paper establishes the existence of a positive solution of fractional order two-point boundary value problem,

$$D_{a+}^{q_1} y(t) + f(t, y(t)) = 0, \ t \in [a, b],$$
$$y(a) = 0, \ y'(a) = 0, \ \alpha D_{a+}^{q_2} y(b) - \beta D_{a+}^{q_3} y(a) = 0,$$

where $D_{a+}^{q_i}, i = 1, 2, 3$ are the standard Riemann-Liouville fractional order derivatives, $2 < q_1 \leq 3, \ 0 < q_2, q_3 < q_1, \ \alpha, \beta$ are positive real numbers and $b > a \geq 0$, by an application of lower and upper solution method and fixed-point theorems.

Keywords: Fractional derivative, Boundary value problem, Two-point, Green's function, Positive solution.

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1. Introduction

Fractional order differential equations have a wide range of applications in various fields of science and engineering such as physics, fluid flows, flow in porous media, electrical networks and viscoelasticity. The existence of a solution via lower and upper solutions, coupled with a monotone iterative technique, provides an effective and flexible mechanism that offers theoretical as well as constructive results for nonlinear problems on a closed set. The lower and upper solutions for two-point fractional order boundary value problems, an improvement by a monotone iterative process, serve as bounds for solution. The idea imbedded in this technique has proved to be of immense value and has played an important role in unifying a variety of nonlinear problems.

Recently, much interest has been achieved in establishing the existence of solutions via lower and upper solutions for boundary value problems (BVPs) associated with integer and fractional order differential equations. To mention the related papers along these lines, we refer to Habets and Zanolin [4], Lee [8], Li, Sun and Jia [9] for integer order differential equations, Shi and Zhang [14] given sufficient conditions for the existence of at least one solution for fractional order boundary value problem,

$$D^{\delta} u(t) + g(t, u) = 0, \ t \in (0, 1),$$
$$u(0) = a, \ u(1) = b,$$

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where $1 < \delta \leq 2$, $g : [0,1] \times R \to R$ and D^δ is Caputo fractional order derivative, using upper and lower solutions method. In consequence, this method allows us to ensure the existence of a solution of the considered problem lying between the lower and the upper solution, that is, we have information about the existence and location of the solutions. So the problem of finding a solution of the considered problem is replaced by that of finding two well-ordered functions that satisfy some suitable inequalities.

This paper is concerned with establishing the existence of a positive solution for fractional order two-point boundary value problem,

$$D^{q_1}_{a^+} y(t) + f(t, y(t)) = 0, \; t \in [a, b], \tag{1}$$

$$y(a) = 0, \; y'(a) = 0, \; \alpha D^{q_2}_{a^+} y(b) - \beta D^{q_3}_{a^+} y(a) = 0, \tag{2}$$

where $D^{q_i}_{a^+}, i = 1, 2, 3$ are the standard Riemann-Liouville fractional order derivatives, $2 < q_1 \leq 3, \; 0 < q_2, q_3 < q_1, \; \alpha, \beta$ are positive real numbers and $b > a \geq 0$.

We assume that the following conditions hold throughout the paper:

(P1) $f : [a, b] \times R^+ \to R^+$ is continuous,

(P2) $\alpha > \min\left\{ \frac{\beta\Gamma(q_1-q_3-2)a^{q_1-q_3-3}}{\Gamma(q_1-q_2-2)b^{q_1-q_2-3}}, \frac{[k_1-2ak_2]}{k_3\Gamma(q_1)} \frac{\Gamma(q_1-q_3)}{\Gamma(q_1-q_2)} \right\}$.

If $q_1 \in (2, 3]$, then define

$$q_1^* = \begin{cases} [q_1] + 1, & \text{if } q_1 \in (2, 3), \\ q_1, & \text{if } q_1 = 3. \end{cases}$$

Let $y(t) \in C^{q_1^*}[a, b]$ be a solution of fractional order boundary value problem (1)-(2).

The rest of the paper is organized as follows. In Section 2, the Green's function for the homogeneous boundary value problem corresponding to (1)-(2) is constructed and certain lemmas are proved. In Section 3, sufficient conditions for the existence of a positive solution of fractional order boundary value problem (1)-(2) are established using lower and upper solution method, and the Schauder fixed point theorem. Finally as an application, the result is demonstrated with an example.

2. Green's function and Lemmas

In this section, the Green's function for the homogeneous boundary value problem corresponding to (1)-(2) is constructed and certain lemmas are proved, which are essential to establish the main results.

Let $G(t, s)$ be the Green's function for the homogeneous fractional order differential equation

$$-D^{q_1}_{a^+} y(t) = 0, \; t \in [a, b], \tag{3}$$

satisfying the boundary conditions (2).

Lemma 2.1. *Let* $d = \Gamma(q_1 - q_2)(k_1 + a^2 k_3 - 2ak_2) \neq 0$. *If* $h(t) \in C[a, b]$ *and* $h(t) \geq 0$, *then the fractional order differential equation,*

$$D^{q_1}_{a^+} y(t) + h(t) = 0, \; t \in [a, b], \tag{4}$$

satisfying (2) *has a unique solution,*

$$y(t) = \int_a^b G(t, s) h(s) ds,$$

where

$$G(t, s) = \begin{cases} G_1(t, s), & a \leq t \leq s \leq b, \\ G_2(t, s), & a \leq s \leq t \leq b, \end{cases} \tag{5}$$

$$G_1(t,s) = \frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}](t-a)^2 t^{q_1-3},$$
$$G_2(t,s) = \frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}](t-a)^2 t^{q_1-3} - \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}, \ and$$
$$k_{i+1} = \Big[\frac{\alpha\Gamma(q_1-i)b^{q_1-q_2-i-1}}{\Gamma(q_1-q_2-i)} - \frac{\beta\Gamma(q_1-i)a^{q_1-q_3-i-1}}{\Gamma(q_1-q_3-i)}\Big], \ for\ i = 0, 1, 2.$$

Proof. Let $y(t) \in C^{q_1^*}[a,b]$ be a solution of fractional order boundary value problem (4),(2). Then

$$I^{q_1}_{a^+} D^{q_1}_{a^+} y(t) = -I^{q_1}_{a^+} h(t),$$

and hence

$$y(t) = \frac{-1}{\Gamma(q_1)}\int_a^t (t-s)^{q_1-1} h(s)ds + c_1 t^{q_1-1} + c_2 t^{q_1-2} + c_3 t^{q_1-3}.$$

Using the boundary conditions (2), c_1, c_2 and c_3 are determined as

$$c_1 = \int_a^b \Big[\frac{k_3\alpha(b-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)(k_1+a^2k_3-2ak_2)}\Big]h(s)ds,$$
$$c_2 = -\int_a^b \Big[\frac{2ak_3\alpha(b-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)(k_1+a^2k_3-2ak_2)}\Big]h(s)ds,$$
$$c_3 = \int_a^b \Big[\frac{a^2k_3\alpha(b-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)(k_1+a^2k_3-2ak_2)}\Big]h(s)ds.$$

Hence, the unique solution of (4), (2) is

$$\begin{aligned} y(t) &= \int_a^b \frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}][t^{q_1-1} - 2at^{q_1-2} + a^2t^{q_1-3}]h(s)ds \\ &- \int_a^t \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}h(s)ds \\ &= \int_a^b G(t,s)h(s)ds, \end{aligned}$$

where $G(t,s)$ is given in (5). □

Lemma 2.2. *Assume that the condition* $(P2)$ *is satisfied. Then the Green's function* $G(t,s)$ *of* (3),(2) *is nonnegative, for all* $t, s \in [a,b]$.

Proof. The Green's function $G(t,s)$ is given in (5). From the condition $(P2)$, for $a \le t \le s \le b$,

$$\begin{aligned} G_1(t,s) =& \frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}](t-a)^2 t^{q_1-3} \\ >& \frac{[k_1-2ak_2]\Gamma(q_1-q_3)(b-s)^{q_1-q_2-1}(t-a)^2 t^{q_1-3}}{d\Gamma(q_1-q_2)\Gamma(q_1)} \\ \ge& 0. \end{aligned}$$

By using the condition $(P2)$, we can establish the nonnegativity of the Green's function $G(t,s)$ for $a \le s \le t \le b$,

$$\begin{aligned}
G_2(t,s) =&\frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}](t-a)^2 t^{q_1-3} - \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}\\
\ge&\frac{1}{d}[k_3\alpha(b-s)^{q_1-q_2-1}](t-a)^2 t^{q_1-3} - \frac{(b-s)^{q_1-1}}{\Gamma(q_1)}\\
\ge&\frac{k_3\Gamma(q_1)\alpha(b-s)^{q_1-q_2-1}(t-a)^2(t-a)^{q_1-3} - d(b-s)^{q_1-1}}{d\Gamma(q_1)}\\
>&\frac{\Big[(k_1-2ak_2)\Gamma(q_1-q_3)(t-a)^{q_1-1} - d\Gamma(q_1-q_2)(b-s)^{q_2}\Big](b-s)^{q_1-q_2-1}}{d\Gamma(q_1)\Gamma(q_1-q_2)}\\
\ge&0.
\end{aligned}$$

□

Lemma 2.3. *If $y(t) \in C^{q_1^*}[a,b]$ and is a positive solution of fractional order boundary value problem* (1)-(2)*, then*

$$m\phi(t) \le y(t) \le \mathcal{M}\phi(t),$$

where

$$\phi(t) = \frac{k_3\alpha(b-a)^{q_1-q_2}(t-a)^2 t^{q_1-3}}{d(q_1-q_2)} - \Big[\frac{(t-a)^{q_1}-(t-b)^{q_1}}{\Gamma(q_1+1)}\Big],$$

and $m, \mathcal{M}$ are two constants.

Proof. Since $y(t) \in C^{q_1^*}[a,b]$, there exists $\mathcal{M}' > 0$ such that $|y(t)| \le \mathcal{M}'$ for $t \in [a,b]$. Choosing

$$m = \min_{(t,y)\in[a,b]\times[0,\mathcal{M}']} f(t,y(t)),$$
$$\mathcal{M} = \max_{(t,y)\in[a,b]\times[0,\mathcal{M}']} f(t,y(t)).$$

From Lemma 2.1, we have

$$m\int_a^b G(t,s)ds \le \int_a^b G(t,s)f(s,y(s))ds \le \mathcal{M}\int_a^b G(t,s)ds.$$

By simple algebraic calculation, we have

$$\begin{aligned}
\phi(t) =&\int_a^b G(t,s)ds\\
=&\int_a^b \frac{k_3\alpha(t-a)^2 t^{q_1-3}(b-s)^{q_1-q_2-1}}{d}ds - \int_a^b \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}ds\\
=&\frac{k_3\alpha(b-a)^{q_1-q_2}(t-a)^2 t^{q_1-3}}{d(q_1-q_2)} - \Big[\frac{(t-a)^{q_1}-(t-b)^{q_1}}{\Gamma(q_1+1)}\Big].
\end{aligned}$$

Hence the proof of Lemma 2.3 is verified. □

Now we introduce the following two definitions about the lower and upper solutions of fractional order boundary value problem (1)-(2).

Definition 2.1. *A function $v(t)$ is called a lower solution of fractional order boundary value problem* (1)-(2) *if $v(t) \in C^{q_1^*}[a,b]$ and $v(t)$ satisfies*

$$-D_{a^+}^{q_1}v(t) \le f(t,v(t)),\ t\in[a,b],\ 2<q_1\le 3,$$
$$v(a)\le 0,\ v'(a)\le 0,\ \alpha D_{a^+}^{q_2}v(b) - \beta D_{a^+}^{q_3}v(a) \le 0.$$

Definition 2.2. *A function $w(t)$ is called an upper solution of fractional order boundary value problem* (1)-(2) *if $w(t) \in C^{q_1^*}[a,b]$ and $w(t)$ satisfies*

$$-D_{a^+}^{q_1} w(t) \geq f(t, w(t)),\ t \in [a,b],\ 2 < q_1 \leq 3,$$
$$w(a) \geq 0,\ w'(a) \geq 0,\ \alpha D_{a^+}^{q_2} w(b) - \beta D_{a^+}^{q_3} w(a) \geq 0.$$

3. Existence of a positive solution

In this section, the existence of a positive solution of fractional order boundary value problem (1)-(2) is established, using lower and upper solution method and the Schauder fixed point theorem. And as an application, the result is demonstrated with an example.

Theorem 3.1. *The fractional order BVP* (1)-(2) *has a positive solution $y(t)$ if the following conditions are satisfied:*

$(A1)$ $f(t,y) \in C([a,b] \times R^+ \to R^+)$ *is nondecreasing relative to* y,

$(A2)$ $f(t, \phi(t)) \neq 0,\ for\ t \in (a,b)$,

$(A3)$ *there exists a positive constant* $\eta < 1$ *such that*

$k^\eta f(t,y) \leq f(t,ky),\ for\ all\ 0 \leq k \leq 1,$

and using the Schauder fixed point theorem.

Proof. Let

$$k_1 \in \Big(0, \min\Big\{\frac{1}{\xi_2}, (\xi_1)^{\frac{\eta}{1-\eta}}\Big\}\Big],\ k_2 \geq \max\Big\{\frac{1}{\xi_1}, (\xi_2)^{\frac{\eta}{1-\eta}}\Big\},$$
$$\xi_1 = \min\Big\{b, \inf_{t\in[a,b]} f(t,\phi(t))\Big\} > 0,\ \xi_2 = \max\Big\{b, \sup_{t\in[a,b]} f(t,\phi(t))\Big\},\ \text{and}$$
$$g(t) = \int_a^b G(t,s) f(s,\phi(s)) ds.$$

We can prove that $v(t) = k_1 g(t),\ w(t) = k_2 g(t)$ are lower and upper solutions of fractional order boundary value problem (1)-(2) respectively.
From Lemma 2.2, we know that $g(t)$ is a positive solution of the following fractional order boundary value problem

$$\left.\begin{aligned} &D_{a^+}^{q_1} y(t) = f(t,\phi(t)),\ t \in [a,b],\ 2 < q_1 \leq 3,\\ &y(a) = 0,\ y'(a) = 0,\ \alpha D_{a^+}^{q_2} y(b) - \beta D_{a^+}^{q_3} y(a) = 0. \end{aligned}\right\} \tag{6}$$

From Lemma 2.3, we have

$$\xi_1 \phi(t) \leq g(t) \leq \xi_2 \phi(t),\ t \in [a,b]. \tag{7}$$

Using the conditions $(A1)$-$(A2)$, it shows that

$$k_1\xi_1 \leq \frac{v(t)}{\phi(t)} \leq k_1\xi_2 \leq 1,\ \frac{1}{k_2\xi_2} \leq \frac{\phi(t)}{w(t)} \leq \frac{1}{k_2\xi_1} \leq 1,$$
$$(k_1\xi_1)^\eta \geq k_1 \text{ and } (k_2\xi_2)^\eta \leq k_2.$$

From the condition $(A3)$ and $(k_1\xi_1)^\eta \geq k_1$, then the following relation satisfy

$$\left.\begin{aligned} f(t,v(t)) =& f\Big(t, \frac{v(t)}{\phi(t)}\phi(t)\Big) \geq \Big(\frac{v(t)}{\phi(t)}\Big)^\eta f(t,\phi(t))\\ \geq& (k_1\xi_1)^\eta f(t,\phi(t)) \geq k_1 f(t,\phi(t)), \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} k_2 f(t,\phi(t)) =& k_2 f\Big(t, \frac{\phi(t)}{w(t)} w(t)\Big) \geq k_2 \Big(\frac{\phi(t)}{w(t)}\Big)^{\eta} f(t, w(t)) \\ \geq& \Big(\frac{1}{k_2 \xi_2}\Big)^{\eta} f(t, w(t)) \geq f(t, w(t)). \end{aligned}\right\} \tag{9}$$

It implies that

$$\left.\begin{aligned} -D^{q_1}_{a^+} v(t) = k_1 f(t,\phi(t)) \leq f(t, v(t)),\ t \in [a,b],\ 2 < q_1 \leq 3, \\ -D^{q_1}_{a^+} w(t) = k_2 f(t,\phi(t)) \geq f(t, w(t)),\ t \in [a,b],\ 2 < q_1 \leq 3. \end{aligned}\right\} \tag{10}$$

Since $v(t) = k_1 g(t)$ and $w(t) = k_2 g(t)$ satisfy the boundary conditions (2). Therefore, $v(t) = k_1 g(t)$ and $w(t) = k_2 g(t)$ are lower and upper solutions of (1)-(2) respectively.

Now we suppose that

$$g(t, y(t)) = \begin{cases} f(t, v(t)), & if\ y(t) \leq v(t), \\ f(t, y(t)), & if\ v(t) \leq y(t) \leq w(t), \\ f(t, w(t)), & if\ w(t) \leq y(t), \end{cases} \tag{11}$$

and prove that fractional order boundary value problem

$$-D^{q_1}_{a+} y(t) = g(t, y(t)),\ t \in [a,b], \tag{12}$$

satisfying the boundary condition (2) has a solution.
Consider the operator $T : C^{q_1^*}[a,b] \to C^{q_1^*}[a,b]$ and is defined as

$$Ty(t) = \int_a^b G(t,s) g(s, y(s)) ds.$$

From Lemma 2.2, $y \in C^{q_1^*}[a,b]$, we have $Ty(t) \geq 0$, for all $t \in [a,b]$ and it is easy to observe that T is continuous in $C^{q_1^*}[a,b]$. Now, the condition $(A1)$ is used to obtain

$$f(t, v(t) \leq g(t, y(t)) \leq f(t, w(t)),\ t \in [a,b]. \tag{13}$$

Thus, there exists a positive constant $\mathcal{M}$ such that $|g(t, y(t))| \leq \mathcal{M}$, which implies that the operator T is uniformly bounded, for all $y(t) \in C^{q_1^*}[a,b]$ and $a \leq t_1 \leq t_2 \leq b$, it follows that

$$\begin{aligned} |Ty(t_2) - Ty(t_1)| \leq& \int_a^b |G(t_2,s) - G(t_1,s)| g(s, y(s)) ds \\ =& \frac{\mathcal{M} k_3 \alpha (b-a)^{q_1 - q_2}}{d(q_1 - q_2)} \Big[(t_2 - a)^2 t_2^{q_1 - 3} - (t_1 - a)^2 t_1^{q_1 - 3}\Big] - \\ & \mathcal{M}\Big[\frac{(t_2 - a)^{q_1} - (t_2 - b)^{q_1} - (t_1 - a)^{q_1} + (t_1 - b)^{q_1}}{\Gamma(q_1 + 1)}\Big], \end{aligned}$$

which shows that the operator T is equicontinuous. Thus, from the Arzela–Ascoli theorem, T is a completely continuous operator. Therefore, from the Schauder fixed point theorem [3], the operator T has a fixed point, i.e., fractional order boundary value problem (12),(2) has a solution.

Finally, we prove that fractional order boundary value problem (1)-(2) has a positive solution. Suppose that $y^*(t)$ is a solution of fractional order boundary value problem (12),(2). Since the function $f(t, y)$ is nondecreasing in y, we know that

$$f(t, v(t)) \leq g(t, y(t)) \leq f(t, w(t)),\ t \in [a,b].$$

Assuming $z(t) = w(t) - y^*(t)$,

$$-D^{q_1}_{a^+} z(t) \geq f(t, w(t)) - g(t, y^*(t)) \geq 0, \tag{14}$$

$$z(a) = 0,\ z'(a) = 0,\ \alpha D^{q_2}_{a^+} z(b) - \beta D^{q_3}_{a^+} z(a) = 0. \tag{15}$$

Obviously, from Lemma 2.2 and Lemma 2.3, $z(t) \geq 0$, i.e., $y^*(t) \leq w(t)$ for $t \in [a, b]$. Similarly, $v(t) \leq y^*(t)$ for $t \in [a, b]$. Therefore $y^*(t)$ is a positive solution of fractional order boundary value problem (1)-(2). This completes the proof of theorem. □

Example 3.1. *Consider the two-point fractional order boundary value problem*

$$D_{0^+}^{2.9} y(t) + f(t, y(t)) = 0, \ 0 < t < 1, \tag{16}$$

$$y(0) = 0, \ y'(0) = 0, \ 5D_{0^+}^{0.8} y(1) - 3D_{0^+}^{0.5} y(0) = 0, \tag{17}$$

where

$$f(t, y) = t + \sin t + y^{0.6}.$$

The Green's function $G(t, s)$ of corresponding homogeneous boundary value problem is given by

$$G(t,s) = \begin{cases} \frac{2.8080 t^{1.9}(1-s)^{1.1}}{9.1364}, & t \leq s, \\ \frac{2.8080 t^{1.9}(1-s)^{1.1}}{9.1364} - \frac{(t-s)^{1.9}}{\Gamma(2.9)}, & s \leq t. \end{cases}$$

Since $k^{0.6} \leq 1$ and $0 \leq k \leq 1$. *It is easy to verify that*

$$\begin{aligned} k^{0.6} f(t, y) =& k^{0.6} t + k^{0.6} \sin t + k^{0.6} y^{0.6} \\ \leq& t + \sin t + (ky)^{0.6} \\ =& f(t, ky). \end{aligned}$$

Then all the conditions of Theorem 3.1 are satisfied. Therefore, it follows from Theorem 3.1, the fractional order boundary value problem (16)-(17) *has a positive solution* $y(t)$.

4. Conlusion

We derived sufficient conditions for the existence of a positive solution for the considered fractional order differential equations satisfying the general two-point boundary conditions. We established the existence of a positive solution via lower and upper solution method, and the Schauder fixed point theorem.

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