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DISTANCE MAJORIZATION SETS IN GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is said to be a distance majorization set (or dm - set) if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that $d(u, v) \geq deg(u) + deg(v)$. The minimum cardinality of a dm - set is called the distance majorization number of G (or dm - number of G) and is denoted by $dm(G)$, Since the vertex set of G is a dm - set, the existence of a dm - set in any graph is guaranteed. In this paper, we find the dm - number of standard graphs like $K_n, K_{1,n}, K_{m,n}, C_n, P_n$, compute bounds on $dm-$ number and dm- number of self complementary graphs and mycielskian of graphs.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [2]. The degree of a vertex v, denoted by $deg(v)$, is the cardinality of its adjacent vertices. Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of a vertex of G. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $rad(G)$ and the maximum eccentricity is its diameter, $diam(G)$ of G.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ is dominated by at least one vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A subset D of $V(G)$ is said to be a distance majorization set (or dm - set) if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that $d(u, v) \geq deg(u) + deg(v)$. The minimum cardinality of a dm - set is called the distance majorization number of G (or dm - number of G) and is denoted by $dm(G)$. A dominating set need not be a dm - set. For example, in $K_{1,n}$, the set consisting of the central vertex is a dominating set but it is not a dm - set if $n \geq 3$. A dm - set may not be a dominating set. For example, in P_5, the

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set containing the pendent vertices is a dm - set but it is not a dominating set. Thus, the concept of dm - sets is different from dominating sets.

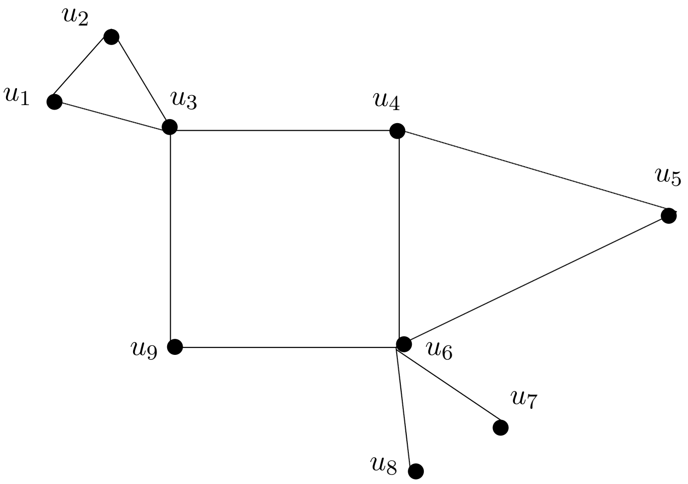
2. Main Results

Definition 2.1. *Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is said to be distance majorization set (dm - set) if for every $u \in V - S$, there exists a vertex $v \in S$ such that $d(u, v) \geq d(u) + d(v)$. The minimum cardinality of a dm - set is called the distance majorization number (dm - number) and is denoted by $dm(G)$.*

Remark 2.1. *For any graph G, $V(G)$ is always a dm - set of G. Then the existence of a dm-set is guaranteed.*

Example 2.1.

G :



$S = \{u_3, u_4, u_5, u_6, u_7, u_9\}$ is a dm - set of G and hence it is easily seen that $dm(G) = 6$.

Remark 2.2. *Let $u, v \in V(G)$. Then u is dm - dominated by v if $d(u, v) \geq deg(u)+deg(v)$.*

Theorem 2.1. *$dm(G) = 1$ if and only if G has an isolate.*

Proof. If G has an isolate say u, then $\{u\}$is a dm - set of G and hence $dm(G) = 1$. Suppose $dm(G) = 1$. Let $\{u\}$ be a dm - set of G. Suppose u is not an isolate. Then there exists $v \in V(G)$ such that u and v are adjacent. Therefore, $d(u, v) = 1$ and $deg(u) + deg(v) \geq 2$, a contradiction. Hence u is an isolate of G. □

Theorem 2.2. *For a star graph $K_{1,n}$, $dm(K_{1,n}) = 2$.*

Proof. Let S be a dm - set of $K_{1,n}$. Let $V(K_{1,n}) = \{u, v_1, v_2, \cdots, v_n\}$. Let u be the central vertex of $K_{1,n}$. Thus $u \in S$. u can not dm-dominate any $v_i, 1 \leq i \leq n$, Since $d(v_i, v_j) = 2$, for all $i, j, 1 \leq i, j, \leq n$, $v_i \in S$ for some i. v_i dm-dominates v_j for all $j, j \neq i, i \neq j, 1 \leq j \leq n$, Therefore, $dm(K_{1,n}) = 2$.

□

Observation 2.1.

(1) Let $u \in V(G)$ be a full degree vertex of G. Then, clearly $d(u, v) = 1$, for all $v \in V(G)$. Thus any dm - set of G contain u.
(2) Every vertex of K_n is a full degree vertex. Therefore, $dm(K_n) = n$

(3) $dm(\overline{K_n}) = 1$, since each vertex is an isolate.
(4) For any connected graph G, $2 \leq dm(G) \leq n$.
(5) If for any vertex $u \in V(G)$ of degree greater than or equal to $diam(G)$, then u belongs to a dm - set of G.
(6) For any graph G, $1 \leq dm(G) \leq n$.

Theorem 2.3. *For a double star $D_{r,s}$, $dm(D_{r,s}) = 3$.*

Proof. Let S be a $dm-$ set of $D_{r,s}$. Let $V(D_{r,s}) = \{u, v, x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_s\}$. Let u, v be the central vertices of $D_{r,s}$. Clearly, $d(u, x_i) = 1, d(u, y_j) = 2$ for all $i, j, 1 \leq i \leq r; 1 \leq j \leq s$ and $d(u, v) = 1$. Thus , $u, v \in S$. Since $d(x_i, y_j) = 3$, for all $i, j; 1 \leq i \leq r; 1 \leq j \leq s, x_1 \in S$. Hence , $dm(K_{1,n}) \geq 3$. $deg(x_i)+deg(x_j) < d(x_i, y_j), i, j, 1 \leq i \leq r; 1 \leq j \leq s$, $x_1 \in S$, and hence no other $x_i, y_j \in S$. Therefore, $dm(K_{1,n}) \leq 3$. Hence, $dm(K_{1,n}) = 3$. □

Theorem 2.4. *For a complete bipartite graph $K_{m,n}$, $dm(K_{m,n}) = m + n$.*

Proof. Let S be a dm - set of $K_{m,n}$. Let $V(K_{m,n}) = \{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\}$. Since $diam(K_{m,n}) = 2$ and $d(x_i) = n \geq 1, d(y_j) = m \geq 1$, for all $i, j; 1 \leq i \leq m; 1 \leq j \leq n$ Hence , $dm(K_{m,n}) \geq m + n$. $deg(x_i) + deg(x_j) < d(x_i, X_j)$, $i, j, 1 \leq i \leq m; 1 \leq j \leq n$, $dm(K_{m,n}) \leq m + n$. □

Theorem 2.5. *For a path P_n , $dm(P_n) = \begin{cases} 2 & n = 3 \text{ and } n \geq 7 \\ 3 & n = 4, 5, 6 \end{cases}$*

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let S be a dm - set of P_n. Let $n = 3$. Then $P_3 \cong K_{1,2}$, $dm(P_3) = 2$. Let $n = 4$. $diam(P_4) = 3$. Since $d(v_1, v_4) \geq deg(v_1) + deg(v_4)$ either v_1 or v_4 belongs to S. Thus $dm(P_4) = 3$. Let $n = 5$. Then,clearly S contains v_1, v_5, v_3 since v_2 and v_4 are $dm-$dominated by v_1 and v_5 respectively. Let $n = 6$. Then,clearly S contains v_1, v_6, v_3, since v_4, v_5 and v_2 are $dm-$dominated by v_1 and v_6 respectively.

Let $n \geq 7$. Then clearly, v_1 $dm-$ dominates the vertices $v_i, 4 \leq i \leq n - 1$ and v_n $dm-$ dominates the vertices $v_i, 2 \leq i \leq n - 3$. Therefore, $dm(P_n) = 2, n \geq 7$. □

Theorem 2.6. *For a cycle C_n, $dm(C_n) = \begin{cases} n & n \leq 7 \\ 4 & n = 8 \\ 3 & 9 \leq n \leq 13 \\ 2 & n \geq 14 \end{cases}$*

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$. Let S be a dm - set of C_n. $diam(C_n) \leq 3, n \leq 7$. $deg(v) = 2$, for all $v \in V(C_n)$. Therefore, $d(v_i, v_j) < deg(v_i)+deg(v_j)$, for all $i, j, 1 \leq i, j \leq n$. Hence $dm(C_n) = n, n \leq 7$.

Let $n = 8$. Then,clearly S contains v_1, v_2, v_3, v_4. Since v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$,v_2 dm - dominates the vertices $v_i, 6 \leq i \leq n - 2$, v_3 dm - dominates the vertices $v_i, 7 \leq i \leq n - 1$ and v_4 dm - dominates the vertices $v_i, 8 \leq i \leq n$. Therefore, $dm(C_n) = 4$.

Let $9 \leq n \leq 13$. Then $S = \{v_1, v_4, v_7\}$, since v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$, v_4 dm - dominates the vertices $v_i, 8 \leq i \leq n$, v_7 dm - dominates the remaining vertices of C_n. Therefore, $dm(C_n) = 3$.

For $n \geq 14$. $diam(C_n) \geq \lceil \frac{n}{2} \rceil$, v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$,$v_{\lceil \frac{n}{2} \rceil}$ dm - dominates the vertices $\{v_1, v_2, v_3, v_4\} \cup \{v_i, 12 \leq i \leq n\}$. Therefore, $dm(C_n) = 2$. □

Theorem 2.7. *If $d(u) + d(v) > diam(G)$ for every $u, v \in V(G)$, then $dm(G) = n$.*

Proof. Suppose $d(u) + d(v) > diam(G)$, for every $u, v \in V(G)$. Suppose $dm(G) < n$. Let S be a minimum dm - set of G. Let $u \in V - S$. Then , there exists $v \in S$ such that $d(u, v) \geq d(u) + d(v) > diam(G)$, a contradiction. Hence $dm(G) = n$.

□

Theorem 2.8. *If $2\delta(G) > d(u, v)$, for every $u, v \in V(G)$, then $dm(G) = n$.*

Proof. Suppose $2\delta(G) > d(u, v)$, for every $u, v \in V(G)$. Let S be a dm - set of G. By the definition of dm - set, each $u \in V - S$ there exists a vertex $v \in S$ such that $d(u, v) \geq d(u) + d(v) \geq \delta(G) + \delta(G) \geq 2\delta(G)$, a contradiction. Therefore, $dm(G) = n$. □

Theorem 2.9. *For any subgraph H of G , $dm(H) \leq dm(G)$.*

Proof. Let G be a graph. Let H be a subgraph of G. Let $u, v \in V(G)$. Suppose H contains an isolate. Then $dm(H) = 1 \leq dm(G)$. Suppose G does not contain an isolate. Let S be a dm - set of G. Clearly, $d_G(u, v) \geq d_H(u, v)$ and $deg_G(u) \geq deg_H(u), deg_G(v) \geq deg_H(v)$. Hence $dm(H) \leq dm(G)$.

□

Theorem 2.10. *For any spanning tree T of G , $dm(T) \leq dm(G)$.*

Proof. Let T be a spanning tree of G. Then clearly, $d_T(u, v) \geq d_G(u, v) \geq deg_G(u) + deg_G(v) \geq deg_T(u) + deg_T(v)$. Therefore,$dm(T) \leq dm(G)$.

□

Theorem 2.11. *For any tree T, $dm(T) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let T be a tree. Let S be a dm - set of T. Let u and v be diametrically opposite vertices of T. Let $u \in S$. Consider $T_1 = T - \{u, v\}$. Let $x, y \in T_1$ and x and y are diametrically opposite vertices of T_1. Then $S \cup \{x\}$. Consider $T_2 = T_1 - \{x, y\}$.Continuing this process until we get either K_1 or K_2, since any tree has exactly either one or two centers. Clearly, $dm(T) \leq \lceil \frac{n}{2} \rceil$.

□

Observation 2.2. *$dm(G) = n - 1$ if and only if $d(u) + d(v) > diam(G)$, for exactly one pair of vertices $u, v \in V(G)$.*

Theorem 2.12. *Let G be a graph. $diam(G) = 2$ and $dm(G) = 2$ if and only if G is a star.*

Proof. Let G be a graph. Let S be a dm - set of G. Suppose G is a star. Then clearly, $diam(G) = 2$ and $dm(G) = 2$.
Conversely, if $diam(G) = 2$ and $dm(G) = 2$. Since $diam(G) = 2$, G is non complete. Therefore, $deg(u) \leq n - 1, \forall u \in V(G)$. Let $u, v \in V(G)$. If $deg(x) \geq 2$, for every $x \in V - S, x \neq u, v$, then $x \in S$, a contradiction, $dm(G) = 2$. Since $diam(G) = 2$, G is connected. Therefore, u and v are adjacent with at least one vertex $x \in V - S$. As $deg(x) = 1, \forall x \in V - S$, either u or v adjacent with x. Without loss of generality, u is adjacent with x. Suppose v is not adjacent with u. Then v is an isolate, a contradiction. Therefore, u and v are adjacent. Hence G is a star. □

Lemma 2.1. *Let G be a self complementary graph. Then G contains exactly two pendent vertices.*

Proof. Suppose G contains a pendent vertex say u and its support v. If v is not adjacent with $n-2$ vertices of G, then $d(u, x) \geq 3, x \in V(G)-\{u, v\}$. Therefore, $deg_G(v) = n-1$. But $deg_{\overline{G}}(v) = 0$, a contradiction since $G \cong \overline{G}$. Therefore, G contains more than one pendent vertex. Suppose G contains more than 3 pendent vertices say u, v, w. Let u', v', w' be its support. Moreover, if x, y, z be three pendent vertices in $\overline{G}$, then $deg(x), deg(y), deg(z)$ is $n-2$ in G, a contradiction. Therefore, G contains exactly two pendent vertices. □

Theorem 2.13. *Let G be a self complementary graph. Then $dm(G) = n$ or $n-1$.*

Proof. Every nontrivial self-complementary graph G has diameter 2 or 3 [5].

By lemma 2.1, G has exactly two pendent vertices and degree of the remaining $n-2$ vertices is greater than or equal to 2. Therefore, $dm(G) = n$ or $n-1$. □

Definition 2.2. *Mycielski construction to create triangle-free graphs with large chromatic numbers. For a graph G, on n vertices $V(G) = \{v_1, v_2, \cdots, v_n\}$, let $\mu(G)$ be the graph on vertices $X \cup Y \cup \{z\} = \{x_1, x_2, \cdots, y_1, y_2, \cdots, y_n, z\}$ with edges zy_i for all i and edges x_ix_j, y_ix_j for all edges v_iv_j in G. For example, $\mu(K_2) = C_5$.*

Theorem 2.14. *For a graph G without isolated vertices, $dm(\mu(G)) = \max\{|V(\mu(G))|, 2n+2-l\}$,where l is the total number of pendent vertices in G.*

Proof. For a graph G without isolated vertices, $diam(\mu(G)) = min(max(2, diam(G)), 4)$ [6].

Clearly, $deg(z) = n, deg(x_i) = 2deg(v_i)$, $1 \leq i \leq n$ and $deg(y_j) = deg(v_j), 1 \leq j \leq n$.In $\mu(G)$, we have $d(z, x_i) = 2, d(x_i, y_i) = 2, d(y_i, y_j) = 2, d(x_i, y_j) \leq 3$ and $d(x_i, x_j) \leq 4$, for all $i \neq j$.

case(i): $diam(\mu(G)) = 2$.

Since $deg(v) \geq 2$, $v \in V(\mu(G))$, $dm(\mu(G)) = |V(\mu(G)|$.

case(ii): $diam(\mu(G)) = 4$.

Let S be a dm - set of G. z is not dm-dominated by x_i, y_j, x_i is not dm-dominated by y_i, y_i is not dm-dominated by y_j and x_i is not dm-dominated by y_j. $diam(G) \geq 4$.

Suppose $d(x_i, x_j) = 4$. In this case, $diam(\mu(G)) = 4$. $deg(x_i) = 2$ (Suppose $deg(x_i) = 3$. Then the degree of the vertex dm-dominates x_i is 1,a contradiction). $deg(x_i) = 2$ then $deg(v_i) = 1$. If x_i is dominated by x_j, then $deg(v_j) = 1$. Hence $dm(\mu(G)) = 2n+2-l$,where l is the total number of pendent vertices in G.

case(iii): $diam(\mu(G)) = diam(G)$.

In this case, if $diam(G) \geq 4$, $diam(\mu(G)) = min(diam(G), 4) = 4$. As the same lines in case(ii), we get the result. □

Lemma 2.2. *If G be a graph with $\alpha(G) = 1$, then $dm(L(G)) = n$, where $\alpha(G)$ is the vertex covering number of G.*

Proof. If $\alpha(G) = 1$, then G contains a spanning subgraph, that is star, then $L(G)$ is a complete graph. Hence $dm(L(G)) = n$. □

Lemma 2.3. *If $diam(L(G)) = 1$, then $dm(G) = 2$ or 3.*

Proof. $diam(L(G)) = 1$ if and only if G is either K_3 or $K_{1,n-1}$. Hence $dm(G) = 2$ or 3. □

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References

[1] Chartrand, G. and Lesniak, L., (2004), Graphs and Digraphs (4th ed.), CRC Press, ISBN 978-1-58488-390-6.

[2] Buckley, F. and Harary, F.. (1990), Distance in Graphs, Addision-Wesley, Redwood City, CA.

[3] West, D. W., (2001), Introduction to Graph Theory - Second edition, Prentice Hall.
[4] Farrugia, A., (1999), Self-complementary graphs and generalisations: A comprehensive reference manual, University of Malta.
[5] Harary, F. and Robinson, R. W., (1985), The Diameter of a Graph and its Complement, The American Mathematical Monthly, Vol. 92, No. 3, pp. 211-212.
[6] Fisher, D. C., McKenna and Boyer, E. D., (1998), Hamiltonicity, diameter, domination, packing and biclique partitions of Mycielski's graphs, Discrete Applied Mathematics, 84, pp. 93-105.

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