

SOLUTION OF TWO-DIMENSIONAL HEAT AND MASS TRANSFER EQUATION WITH POWER-LAW TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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ABSTRACT. In this paper, we obtain the particular exact solutions of the two-dimensional heat and mass transfer equation with power-law temperature-dependent thermal conductivity using the Adomian's decomposition method. In comparison with existing techniques, the decomposition method is very effective in terms of accuracy and convergence. Also, it is an advantageous method for obtaining the solutions of non-linear differential equations without linearization and physically unrealistic assumptions. Numerical comparisons are presented in both tables and figures.

Keywords: Adomian's decomposition method, heat and mass transfer equation, thermal conductivity

AMS Subject Classification: 65M20, 34D20

1. INTRODUCTION

In this paper we consider the two-dimensional heat and mass transfer equation with power-law temperature-dependent thermal conductivity [10]

$$\frac{\partial \omega}{\partial t} = a \left[\frac{\partial}{\partial x} \left(\omega^m \frac{\partial \omega}{\partial x} \right) \right] + b \left[\frac{\partial}{\partial y} \left(\omega^m \frac{\partial \omega}{\partial y} \right) \right] + \beta \omega \quad (1)$$

where m can be an integer, fractional or negative number. Also, a , b and β are some parameters. Some very special cases of Eq.(1) in one dimension have been studied in [6,9]. In [9] the authors have obtained the particular exact solutions of the porous media equation (the case $a = 1$, $b = 0$, $\beta = 0$) that usually occurs in nonlinear problems of heat and mass transfer, and in biological systems. When $a = 1$, $b = 1$, $m = 1$ and $\beta = 0$, Eq. (1) is called *Boussinesq equation*. It arises in nonlinear heat conduction theory and the theory of unsteady flows through porous media with a free surface [10]. Also, when $a = b$, $m = -1$ and $\beta = 0$, the equation admits travelling-wave solutions [10].

To obtain mathematical models of physical or biological phenomena, one generally faces nonlinear partial differential equations, and to find exact solutions of such equations is generally not easy. There are some methods to obtain approximate solution of this kind of equations. Some of them are linearization of the equation, perturbation and numerical methods. In the beginning of the 1980's, Adomian aimed to find a new method which

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is called the Adomian Decomposition Method (ADM)[1]. This method is very useful to find series solution of ordinary differential equations, partial differential equations and integral equations. Also, this method avoids linearization of the problem and unnecessary assumptions.

In [2,3] the Authors apply ADM for fin efficiency of convective straight fins with temperature dependent thermal conductivity. Also, in [5] ADM has been applied to obtain the solution for the convective longitudinal fins with variable thermal conductivity.

In the next section we apply the decomposition method to Eq. (1), and in the last section we present some numerical examples and results that show how rapidly the series solution converges to the exact solution.

2. METHOD

If we let $f(\omega) = \omega^m \frac{\partial \omega}{\partial x}$, $g(\omega) = \omega^m \frac{\partial \omega}{\partial y}$ and $h(\omega) = \beta \omega$, Eq.(1) can be written in an operator form

$$L_t \omega = aL_x [f(\omega)] + bL_y [g(\omega)] + h(\omega) \quad (2)$$

where $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial}{\partial x}$ and $L_y = \frac{\partial}{\partial y}$ symbolize the linear differential operators. We assume integration inverse operators exist and they are defined as $L_t^{-1} = \int_0^t (\cdot) dt$, $L_x^{-1} = \int_0^x (\cdot) dx$ and $L_y^{-1} = \int_0^y (\cdot) dy$, respectively. Therefore, the solution of Eq.(1) in t direction can be written as [1,6,7,8,9]

$$\omega(x, y, t) = \omega(x, y, 0) + aL_t^{-1}(L_x f(\omega)) + bL_t^{-1}(L_y g(\omega)) + L_t^{-1}(h(\omega)). \quad (3)$$

By ADM [1] the solution of Eq.(1) can be written in series form as

$$\omega(x, y, t) = \sum_{n=0}^{\infty} \omega_n(x, y, t). \quad (4)$$

To find the solution, one has to solve the recursive relations

$$\omega_0 = \omega(x, y, 0), \quad \omega_{n+1} = L_t^{-1}(aL_x(A_n)) + L_t^{-1}(bL_y(B_n)) + L_t^{-1}(C_n) \quad n \geq 0, \quad (5)$$

where the Adomian polynomials A_n , B_n and C_n are [1]

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f \left(\sum_{n=0}^{\infty} \lambda^n \omega_n \right) \right]_{\lambda=0}, \quad n \geq 0. \quad (6)$$

$$B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[g \left(\sum_{n=0}^{\infty} \lambda^n \omega_n \right) \right]_{\lambda=0}, \quad n \geq 0. \quad (7)$$

$$C_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[h \left(\sum_{n=0}^{\infty} \lambda^n \omega_n \right) \right]_{\lambda=0}, \quad n \geq 0. \quad (8)$$

We obtain the first few Adomian polynomials for the functions $f(\omega)$, $g(\omega)$ and $h(\omega)$ defined above

$$\begin{aligned}
A_0 &= \omega_0^m \frac{\partial \omega_0}{\partial x}, \\
A_1 &= m\omega_0^{m-1}\omega_1 \frac{\partial \omega_0}{\partial x} + \omega_0^m \frac{\partial \omega_1}{\partial x}, \\
A_2 &= m\omega_0^{m-1}\omega_2 \frac{\partial \omega_0}{\partial x} + m\omega_0^{m-1}\omega_1 \frac{\partial \omega_1}{\partial x} + \omega_0^m \frac{\partial \omega_2}{\partial x} + \frac{m}{2}(m-1)\omega_0^{m-2}\omega_1^2 \frac{\partial \omega_0}{\partial x}, \\
&\vdots \\
B_0 &= \omega_0^m \frac{\partial \omega_0}{\partial y}, \\
B_1 &= m\omega_0^{m-1}\omega_1 \frac{\partial \omega_0}{\partial y} + \omega_0^m \frac{\partial \omega_1}{\partial y}, \\
B_2 &= m\omega_0^{m-1}\omega_2 \frac{\partial \omega_0}{\partial y} + m\omega_0^{m-1}\omega_1 \frac{\partial \omega_1}{\partial y} + \omega_0^m \frac{\partial \omega_2}{\partial y} + \frac{m}{2}(m-1)\omega_0^{m-2}\omega_1^2 \frac{\partial \omega_0}{\partial y}, \\
&\vdots \\
C_0 &= \beta\omega_0, \\
C_1 &= \beta\omega_1, \\
C_2 &= \beta\omega_2, \\
&\vdots
\end{aligned}$$

The convergence of the series given by (4) is studied in [4].

We let $\phi_n(x, y, t)$ represent the partial sum

$$\phi_n(x, y, t) = \sum_{k=0}^n \omega_k(x, y, t). \quad (9)$$

Therefore,

$$\omega(x, y, t) = \lim_{n \rightarrow \infty} \phi_n(x, y, t) \quad (10)$$

as it is clear from (4). In the following section, we consider some examples and compute the absolute errors $|\omega(x, y, t) - \phi_n(x, y, t)|$ in tables where $\omega(x, y, t)$ is the exact solution and $\phi_n(x, y, t)$ is the n th partial sum as defined above.

3. APPLICATIONS

Example 1: We take $m = 1$ and $\beta = 0$ in Eq.(1), then Eq.(1) becomes

$$\frac{\partial \omega}{\partial t} = a \left[\frac{\partial}{\partial x} \left(\omega \frac{\partial \omega}{\partial x} \right) \right] + b \left[\frac{\partial}{\partial y} \left(\omega \frac{\partial \omega}{\partial y} \right) \right]. \quad (11)$$

In [10] the authors give the exact solution to this equation as $\omega(x, y, t) = x + y + (a + b)t$. Taking $\omega_0 = x + y$ and using ADM to obtain recursive relation, we can find the Adomian

polynomials as

$$\begin{aligned} A_0 &= B_0 = x + y, \\ \omega_1 &= aL_t^{-1}(L_x(A_0)) + bL_t^{-1}(L_y(B_0)) = (a + b)t, \\ A_1 &= B_1 = (a + b)t, \\ \omega_2 &= aL_t^{-1}(L_x(A_1)) + bL_t^{-1}(L_y(B_1)) = 0, \\ &\vdots \end{aligned}$$

Therefore, one gets $\omega_n = 0$ for $n \geq 2$. These polynomials are enough to get the exact solution of the Eq.(11)

$$\omega(x, y, t) = \omega_0(x, y, t) + \omega_1(x, y, t) = x + y + (a + b)t.$$

Example 2: We take $m = 2$ and $\beta = 0$ in Eq.(1), then Eq.(1) becomes

$$\frac{\partial \omega}{\partial t} = a \left[\frac{\partial}{\partial x} \left(\omega^2 \frac{\partial \omega}{\partial x} \right) \right] + b \left[\frac{\partial}{\partial y} \left(\omega^2 \frac{\partial \omega}{\partial y} \right) \right]. \quad (12)$$

In [10] the exact solution of this equation is given as

$$\omega(x, y, t) = \left[\frac{2(x + y + t)}{(a + b)} \right]^{1/2}. \quad (13)$$

We take $\omega_0 = \omega(x, y, 0) = \left[\frac{2(x + y)}{(a + b)} \right]^{1/2}$. To find the series solution of this equation by ADM, we find Adomian polynomials and the terms of the series as

$$\begin{aligned} A_0 &= 2^{1/2}(a + b)^{-3/2}(x + y)^{1/2}, \\ B_0 &= 2^{1/2}(a + b)^{-3/2}(x + y)^{1/2}, \\ A_1 &= 2^{-1/2}(a + b)^{-3/2}(x + y)^{-1/2}t, \\ B_1 &= 2^{-1/2}(a + b)^{-3/2}(x + y)^{-1/2}t, \\ A_2 &= -2^{-5/2}(a + b)^{-3/2}(x + y)^{-3/2}t^2, \\ B_2 &= 2^{-5/2}(a + b)^{-3/2}(x + y)^{-3/2}t^2, \\ \omega_0 &= 2^{1/2}(a + b)^{-1/2}(x + y)^{1/2}, \\ \omega_1 &= 2^{-1/2}(a + b)^{-1/2}(x + y)^{-1/2}t, \\ \omega_2 &= -2^{-5/2}(a + b)^{-1/2}(x + y)^{-3/2}t^2, \\ \omega_3 &= 2^{-7/2}(a + b)^{-1/2}(x + y)^{-5/2}t^3, \end{aligned}$$

and so on. The other terms of the series solution are found by using *Matcad7*. Therefore, the series solution of Eq.(12) is

$$\begin{aligned} \omega(x, y, t) &= \omega_0(x, y, t) + \omega_1(x, y, t) + \omega_2(x, y, t) + \omega_3(x, y, t) + \dots \\ &= 2^{1/2}(a + b)^{-1/2}(x + y)^{1/2} + 2^{-1/2}(a + b)^{-1/2}(x + y)^{-1/2}t \\ &\quad - 2^{-5/2}(a + b)^{-1/2}(x + y)^{-3/2}t^2 + 2^{-7/2}(a + b)^{-1/2}(x + y)^{-5/2}t^3 + \dots \end{aligned}$$

This gives the exact solution given by (13) in the closed form which can be verified through substitution.

Example 3: If we plug $m = -1$, $a = b = \alpha$ and $\beta = 0$ in the Eq.(1), then Eq.(1) becomes

$$\frac{\partial \omega}{\partial t} = \alpha \left[\frac{\partial}{\partial x} \left(\frac{1}{\omega} \frac{\partial \omega}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\omega} \frac{\partial \omega}{\partial y} \right) \right]. \quad (14)$$

We will obtain the series solution of the problem with the initial condition

$$\omega_0 = (\sin y + \exp x)^{-2}.$$

Therefore, one gets

$$\begin{aligned} A_0 &= -2 \exp x (\sin y + \exp x)^{-1}, \\ B_0 &= -2 \cos y (\sin y + \exp x)^{-1}, \\ \omega_1 &= 2\alpha t (\sin y + \exp x)^{-2}, \\ A_n &= B_n = 0, \quad n = 1, 2, 3 \dots, \\ \omega_n &= 0, \quad n = 2, 3, 4 \dots. \end{aligned}$$

This gives the exact solution to Eq.(14)

$$\begin{aligned} \omega(x, y, t) &= \omega_0(x, y, t) + \omega_1(x, y, t), \\ &= (\sin y + \exp x)^{-2} + 2\alpha t (\sin y + \exp x)^{-2}, \\ &= \frac{2\alpha t + 1}{(\sin y + \exp x)^2}. \end{aligned} \quad (15)$$

Example 4: In this example, we obtain a series solution for two-dimensional heat and mass transfer equation with power-law temperature-dependent thermal conductivity with a source term for some cases. We take $m = -1$, $a = b = \alpha$ in the Eq.(1), therefore it becomes

$$\frac{\partial \omega}{\partial t} = \alpha \left[\frac{\partial}{\partial x} \left(\frac{1}{\omega} \frac{\partial \omega}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\omega} \frac{\partial \omega}{\partial y} \right) \right] + \beta \omega, \quad (16)$$

where α and β are parameters. In [10] a partial exact solution of this equation is given by

$$\omega(x, y, t) = \exp(\beta t) u(x, y, \eta) \quad (17)$$

where

$$\eta = \frac{1}{\beta} [1 - \exp(-\beta t)] \quad , \quad u(x, y, \eta) = \frac{2\alpha\eta + 1}{(\sin y + \exp(x))^2}$$

with the initial condition $\omega_0 = \omega(x, y, 0) = (\sin y + \exp x)^{-2}$. If we compute the Adomian polynomials and the terms of the series we obtain

$$\begin{aligned} A_0 &= -2 \exp x (\sin y + \exp x)^{-1}, \\ B_0 &= -2 \cos y (\sin y + \exp x)^{-1}, \\ C_0 &= \beta (\sin y + \exp x)^{-2}, \\ \omega_1 &= (2\alpha + \beta)t (\sin y + \exp x)^{-2}, \\ A_n &= B_n = 0, \quad n = 1, 2, 3 \dots, \\ C_1 &= \beta t (2\alpha + \beta) (\sin y + \exp x)^{-2}, \\ \omega_2 &= \frac{\beta t^2}{2} (2\alpha + \beta) (\sin y + \exp x)^{-2}, \\ C_2 &= \frac{\beta^2 t^2}{2} (2\alpha + \beta) (\sin y + \exp x)^{-2}, \\ \omega_3 &= \frac{\beta^2 t^3}{6} (2\alpha + \beta) (\sin y + \exp x)^{-2}, \end{aligned}$$

TABLE 1. Numerical results for $|\omega(x, y, t) - \phi_8(x, y, t)|$ where $\omega(x, y, t) = (x + y + t)^{1/2}$ for Eq. (12) in the case $a = b = 1$

$x_i, y_i t_i$	0.1	0.3	0.5	0.6	0.8
$x = 0.2, y = 0.3$	$3.377E - 9$	$5.159E - 5$	$4.185E - 3$	$0.02E + 00$	$0.226E + 00$
$x = 1.5, y = 0.2$	$1.144E - 13$	$2.054E - 9$	$1.876E - 7$	$9.307E - 7$	$1.152E - 5$
$x = 2, y = 0.3$	$9.104E - 15$	$1.628E - 10$	$1.515E - 8$	$7.582E - 8$	$9.524E - 7$
$x = 2.5, y = 1$	$0.000E + 00$	$4.752E - 12$	$4.511E - 10$	$2.278E - 9$	$2.912E - 8$
$x = 2.5, y = 0.2$	$2.22E - 15$	$4.228E - 11$	$3.968E - 9$	$1.994E - 8$	$2.522E - 7$

TABLE 2. Numerical results for $|\omega(x, y, t) - \phi_8(x, y, t)|$ where $\omega(x, y, t) = \frac{3 \exp(t) - 2}{(\sin y + \exp(x))^2}$ for Eq. (16) in the case $\alpha = \beta = 1$

$x_i, y_i t_i$	0.1	0.5	0.8	1	1.5
$x = 0.1, y = 0.1$	$5.184E - 13$	$2.119E - 7$	$9.427E - 6$	$5.756E - 5$	$1.57E - 3$
$x = 0.2, y = 1.5$	$1.528E - 13$	$6.248E - 8$	$2.78E - 6$	$1.698E - 5$	$4.63E - 4$
$x = 2, y = 0.6$	$1.19E - 14$	$4.863E - 9$	$2.164E - 7$	$1.321E - 6$	$3.604E - 5$
$x = 3.5, y = 3$	$0.00E + 00$	$2.782E - 10$	$1.238E - 8$	$7.557E - 8$	$2.061E - 6$
$x = 3.5, y = 0.2$	$0.000E + 00$	$2.772E - 10$	$1.233E - 8$	$7.531E - 8$	$2.054E - 6$

and so on. The other terms of the series solution are obtained by using *Matcad7*. Therefore, the series solution of Eq.(16) is

$$\begin{aligned}
 \omega(x, y, t) &= \omega_0(x, y, t) + \omega_1(x, y, t) + \omega_2(x, y, t) + \dots \\
 &= (\sin y + \exp x)^{-2} + (2\alpha + \beta)t(\sin y + \exp x)^{-2} \\
 &+ \frac{\beta t^2}{2}(2\alpha + \beta)(\sin y + \exp x)^{-2} \\
 &+ \frac{\beta^2 t^3}{6}(2\alpha + \beta)(\sin y + \exp x)^{-2} + \dots .
 \end{aligned} \tag{18}$$

This is the exact solution given by (17) in the closed form which can be verified through substitution.

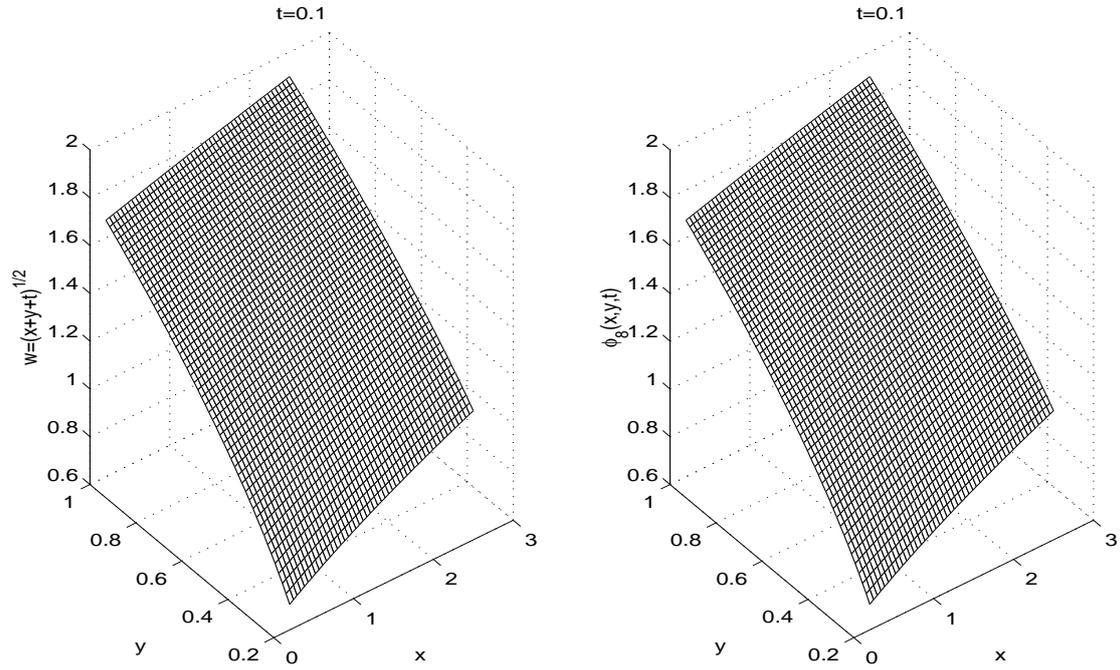


FIGURE 1. Numerical comparison for the solution to Eq. (12) at $t = 0.1$ in the case $a = b = 1$.

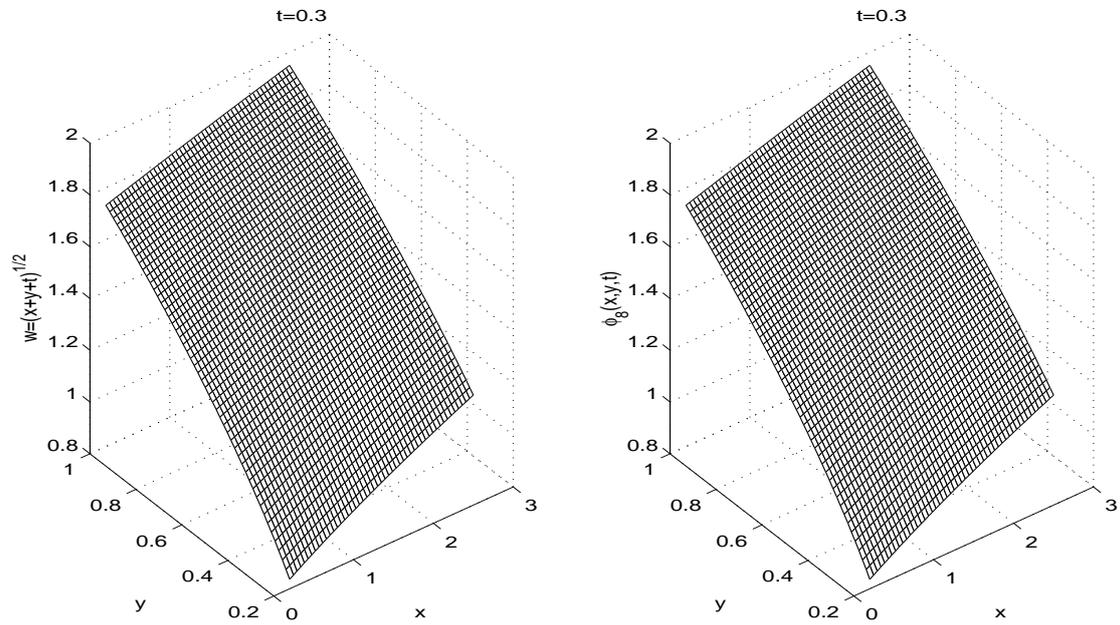


FIGURE 2. Numerical comparison for the solution to Eq. (12) at $t = 0.3$ in the case $a = b = 1$.

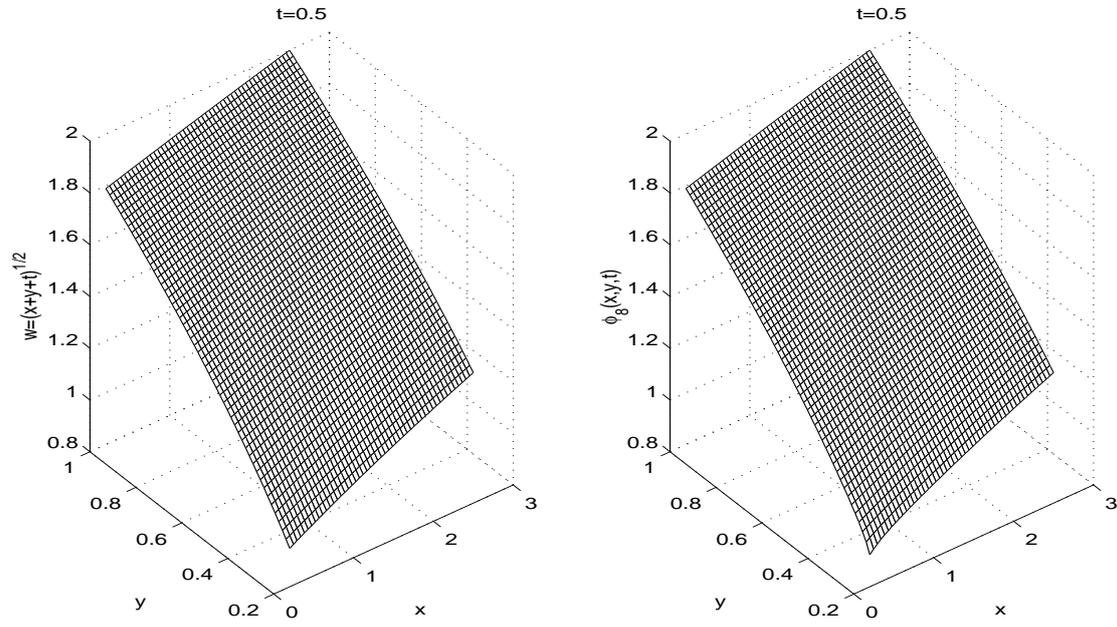


FIGURE 3. Numerical comparison for the solution to Eq. (12) at $t = 0.5$ in the case $a = b = 1$

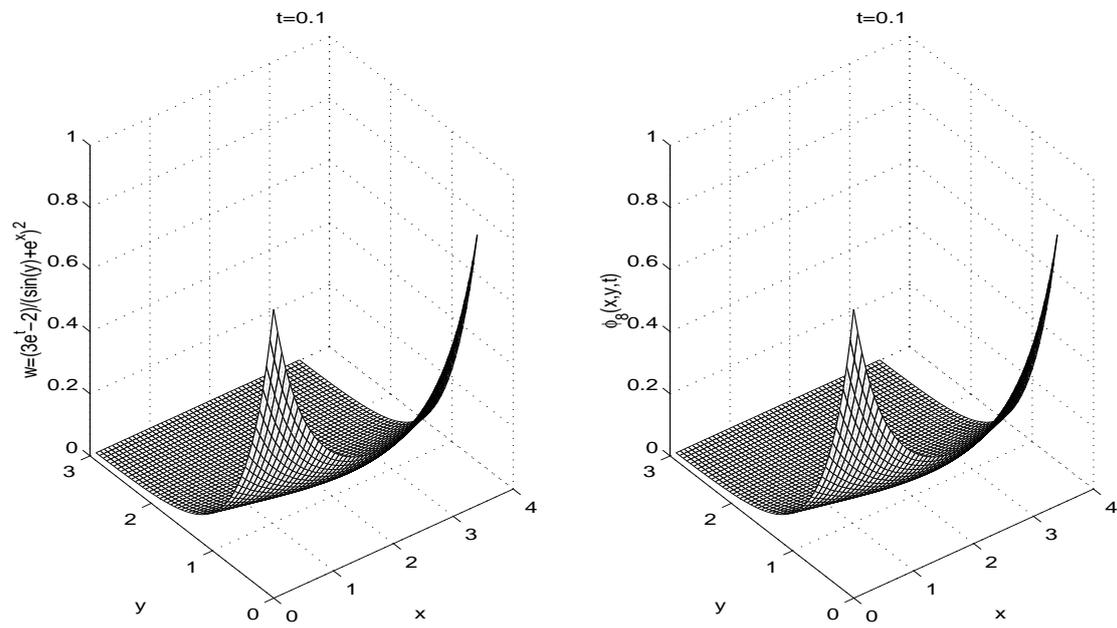


FIGURE 4. Numerical comparison for the solution to Eq. (16) at $t = 0.1$ in the case $\alpha = \beta = 1$

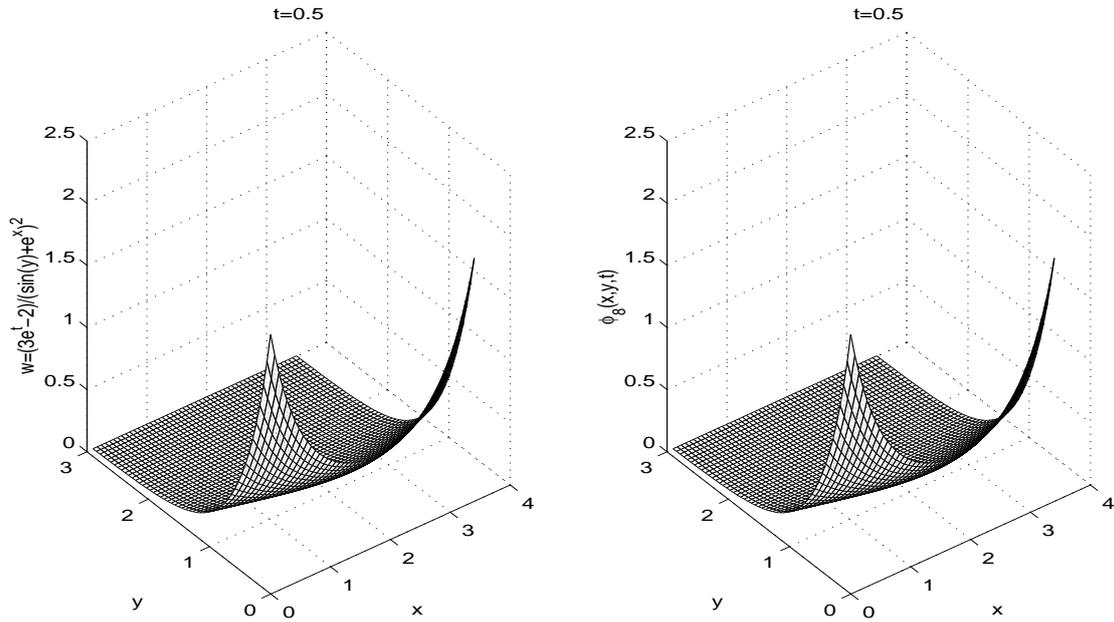


FIGURE 5. Numerical comparison for the solution to Eq. (16) at $t = 0.5$ in the case $\alpha = \beta = 1$

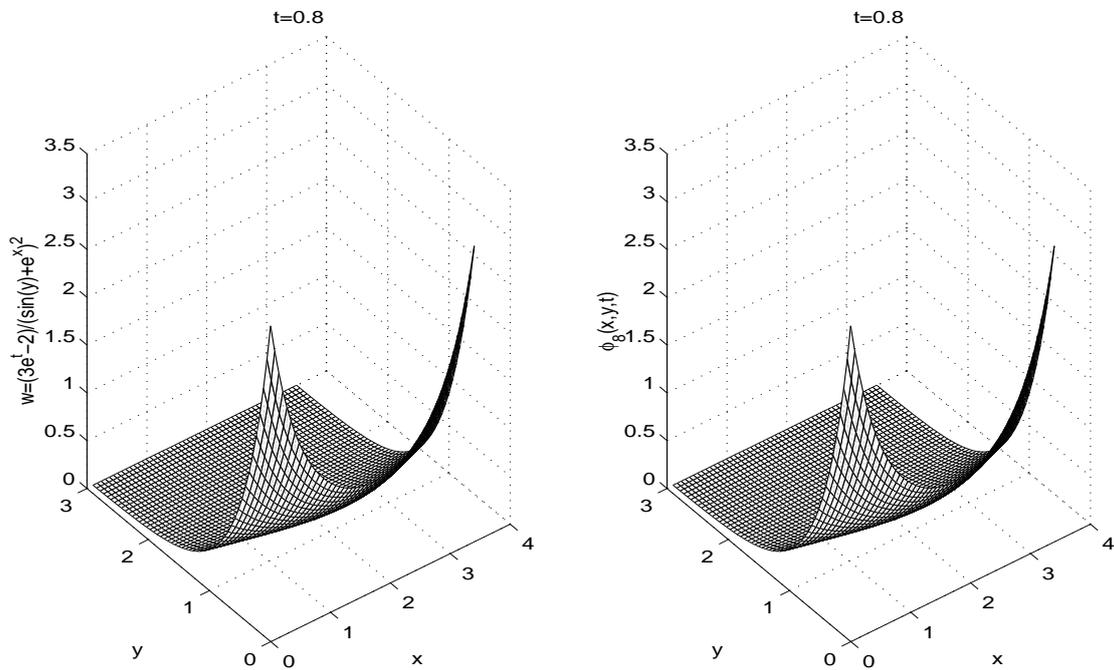


FIGURE 6. Numerical comparison for the solution to Eq. (16) at $t = 0.8$ in the case $\alpha = \beta = 1$

4. RESULTS

In this paper, we have shown that the approximate solutions obtained by using ADM are very close to the exact solutions of nonlinear partial differential equations of the form Eq. (1). For equations (12) and (16) the absolute errors $|\omega(x, y, t) - \phi_n(x, y, t)|$, where $\omega(x, y, t)$ is the exact solution and $\phi_n(x, y, t)$ is the n th partial sum given by Eq. (9), are shown in Tables 1 and 2, respectively. For numerical purposes, we take $n = 8$. As seen from Tables 1 and 2, the absolute errors are very small. For this type of nonlinear problems, we achieve a very good approximation to the partial exact solution by using only 8 terms of the decomposition series, which shows that the speed of convergence of this method is very fast, and the overall errors can be made very small by adding new terms to the series (9).

In Figures 1-3 and Figures 4-6 we compare the exact solutions with the 8-term series expansions for Eq. (12) and Eq. (16), respectively. As seen from the six figures the exact solutions are almost identical to those we have obtained by using only 8 terms of the decomposition series.

As a result, when we use ADM there is no need to make linearization or unnecessary assumptions to solve the non-linear differential equations.

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Serdal Pamuk, for the photography and short autobiography, see TWMS J. of Appl. and Eng. Math., V.3, No.2.



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