TWMS J. App. Eng. Math. V.4, N.2, 2014, pp. 216-225.

PARTIAL COMPLETE CONTROLLABILITY OF DETERMINISTIC SEMILINEAR SYSTEMS

AGAMIRZA E. BASHIROV[1], MAHER JNEID[2],§

Abstract. In this paper the concept of partial complete controllability for deterministic semilinear control systems in separable Hilbert spaces is investigated. Some important systems can be expressed as a first order differential equation only by enlarging the state space. Therefore, the ordinary controllability concepts for them are too strong. This motivates the partial controllability concepts, which are directed to the original state space. Based on generalized contraction mapping theorem, a sufficient condition for the partial complete controllability of a semilinear deterministic control system is obtained in this paper. The result is demonstrated through appropriate examples.

Keywords: Complete controllability, partial controllability, semilinear system.

AMS Subject Classification: 93B05

1. Introduction

The concepts of controllability for deterministic control systems, described by first order differential equations in infinite dimensional spaces, has been adequately examined for more than half of decade by many authors. The complete controllability describes a property of steering any initial state to any point in the state space and was defined by Kalman [16]. This property does not hold for many infinite dimensional systems (see Fattorini [15] and Russel [22]). Therefore, the complete controllability concept was weakened to the approximate controllability, which is a property of steering any initial state to arbitrarily small neighbourhood of any point in the state space. Necessary and sufficient conditions for complete and approximate controllability are almost completely studied and presented in, for example, Curtain and Zwart [14], Bensoussan [4], Bensoussan et al. [5], Zabczyk [26], Bashirov [3], Klamka [17] etc. for linear systems, Balachandran and Dauer [1, 2], Klamka [18], Mahmudov [20] etc. for nonlinear systems, Sakthivel at al [23, 24], Yan [25] etc. for fractional differential systems and Ren et al [21] for differential inclusions.

Recently, in Bashirov et al [6, 7] the partial controllability concepts were defined and they are studied for linear control systems. The idea of these concepts is that some control systems, including higher order differential equations, wave equations and delay equations, can be written as a first order differential equation only by enlarging the dimension of the state space. Therefore, the theorems on controllability, which are formulated for control

[1] Eastern Mediterranean University, Gazimagusa, North Cyprus; Institute of Cybernetics, ANAS, Baku, Azerbaijan.
e-mail: agamirza.bashirov@emu.edu.tr
[2] Eastern Mediterranean University, Gazimagusa, North Cyprus.
e-mail: maher.jneid@emu.edu.tr;
§ Manuscript received: March 3, 2014.

systems in the form of first order differential equations, are too strong for them because they involve the enlarged state space. In such cases the partial controllability concepts become preferable, which assume the original state space. The basic controllability conditions for linear systems, including resolvent conditions from Bashirov and Mahmudov [8] and Bashirov and Kerimov [9] (see also [10, 11, 12]), are extended to partial controllability concepts by just a replacement of the controllability operator by its partial version.

A sufficient condition for partial complete controllability for semilinear control systems is obtained in Bashirov and Jneid [13]. This condition covers many systems, but there are others which remain out of this sufficient condition. In this paper we prove an alternative sufficient condition for partial complete controllability for semilinear control systems.

2. Setting of the problem

Consider the basic semilinear control system

$$x_t' = Ax_t + Bu_t + \sigma_t f(t, x_t, u_t),\ 0 < t \le T,\ x_0 = \xi \in X, \tag{1}$$

with $T > 0$, where x is a state process and u is a control. We assume that the following conditions hold.

(A): X and U are separable Hilbert spaces, H is a closed subspace of X and L is a projection operator from X to H;

(B): A is a densely defined closed linear operator on X, generating a strongly continuous semigroup e^{At}, $t \ge 0$;

(C): B is a bounded linear operator from U to X;

(D): f is a nonlinear function from $[0,T] \times X \times U$ to X, satisfying

- f is continuous on $[0,T] \times X \times U$;
- f is bounded on $[0,T] \times X \times U$, that is, there is $M \ge 0$ such that $\|f(t,x,u)\| \le M$ for all $(t,x,u) \in [0,T] \times X \times U$;
- f is Lipschitz continuous with respect to x and u, that is, there is $K \ge 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \le K(\|x-y\| + \|u-v\|)$$

for all $t \in [0,T]$, $x, y \in X$ and $u, v \in U$;

(E): σ is a continuous nonnegative real-valued function on $[0,T]$.

Here, σ is a some sort adjusting function. Below we will put additional condition on σ so that to get a controllability property of the system (1).

Under these conditions the semilinear system (1) has a unique continuous solution in the mild sense for every $u \in C(0,T;U)$ (the space of all U-valued continuous functions on $[0,T]$, equipped with maximum-norm) and $\xi \in X$ (see, Li and Yong [19]), that is, there is a unique continuous function x from $[0,T]$ to X such that

$$x_t = e^{At}\xi + \int_0^t e^{A(t-s)}(Bu_s + \sigma_s f(s, x_s, u_s))\,ds. \tag{2}$$

Denote $U_{\mathrm{ad}} = C(0,T;U)$, regarding this space as a set of admissible controls. Let

$$D_{\xi,T} = \{h \in H : x_0 = \xi \text{ and } \exists u \in U_{\mathrm{ad}} \text{ such that } h = Lx_T\}.$$

In accordance with the definitions from Bashirov et al [6, 7], the semilinear control system (1) is said to be L-partially complete controllable on U_{ad} for the time T if $D_{\xi,T} = H$ for all $\xi \in X$. Similarly, it is said to be L-partially approximate controllable on U_{ad} for the time T if $\overline{D_{\xi,T}} = H$ for all $\xi \in X$, where $\overline{D_{\xi,T}}$ is the closure of $D_{\xi,T}$. In the case $H = X$, we have $L = I$ (the identity operator). Respectively, these definitions reduce to the well known definitions of complete and approximate controllability, respectively. In this paper we deal with L-partial complete controllability.

The reason for studying L-partial controllability concepts is that many systems can be written in the form of (1) if the original state space is enlarged. Therefore, suitable controllability concepts for such systems are the L-partial controllability concepts with the operator L projecting the enlarged state space to the original one.

Introduce the controllability operator

$$Q_t = \int_0^t e^{As} BB^* e^{A^* s} ds, \ 0 \le t \le T,$$

and the L-partial controllability operator

$$\tilde{Q}_t = LQ_tL^*, \ 0 \le t \le T,$$

where L^* is the adjoint of L. The L-partial controllability operator $\tilde{Q}_t$ becomes the controllability operator Q_t if $L = I$ (the identity operator). The operator-valued function $\tilde{Q}$ have the following properties, which will be used without reference:

(i) $\tilde{Q}_0 = 0$;
(ii) $\tilde{Q}_t$ is a nonnegative for all $0 \le t \le T$, that is $\langle \tilde{Q}_t h, h\rangle \ge 0$ for all $h \in H$;
(iii) $\tilde{Q}_t = \tilde{Q}_s + \tilde{Q}_{t-s}$ for all $0 \le s \le t \le T$;
(iv) $\tilde{Q}$ is increasing, that is $\langle (\tilde{Q}_t - \tilde{Q}_s)h, h\rangle \ge 0$ for all $h \in H$ and $0 \le s \le t \le T$;
(v) $\|\tilde{Q}_s\| \le \|\tilde{Q}_t\|$ for $0 \le s \le t \le T$;
(vi) $\tilde{Q}$ is continuous (in the uniform operator topology) function on $[0, T]$.

Here, (i) and (ii) are trivial, (iii) follows from

$$\tilde{Q}_t = \tilde{Q}_s + \int_s^t Le^{Ar} BB^* e^{A^* r} L^* \, dr = \tilde{Q}_s + \tilde{Q}_{t-s},$$

and (iv)–(vi) follow (iii). Since $Q = \tilde{Q}$ for $L = I$, (i)–(vi) hold for Q as well.

We will additionally assume that

(E): For every $0 < t \le T$, $\tilde{Q}_t$ is coercive.

The coercivity of $\tilde{Q}_t$ means that there is a number $\gamma_t > 0$ such that

$$\langle \tilde{Q}_t h, h\rangle \ge \gamma_t \|h\|^2 \text{ for all } h \in H.$$

This implies the existence of the bounded inverse $\tilde{Q}_t^{-1}$ with

$$\|\tilde{Q}_t^{-1}\| \le \frac{1}{\gamma_t}.$$

Clearly γ_t is not unique. We will let

$$\gamma_t = \max\{\alpha > 0 : \langle \tilde{Q}_t h, h\rangle \ge \alpha \|h\|^2 \text{ for all } h \in H\}.$$

Here, γ_t is the maximum of a nonempty (by condition (E)) closed and bounded above set and, therefore, γ_t exists. One can easily deduce the following properties of the function $\gamma : (0, T] \to (0, \infty)$:

(vii) γ is increasing, that is $s < t$ implies $\gamma_s \le \gamma_t$;
(viii) γ is continuous;
(ix) $\lim_{t\to 0^+} \gamma_t = 0$.

By (ix), the integral $\int_0^T \frac{dt}{\gamma_t}$ is an improper integral. Therefore, we additionally set the rate of convergence in (ix) as follows.

(F): $\int_0^T (\sigma_t/\gamma_{T-t})\, dt < \infty$.

In particular, if there exists $\alpha > 0$ such that $\gamma_{T-t} \ge (T-t)^{1-\alpha}\sigma_t$ for every $0 \le t \le T$, then

$$\int_0^T \frac{\sigma_t\,dt}{\gamma_{T-t}} = \int_0^T (T-t)^{\alpha-1}\,dt = \frac{T^\alpha}{\alpha} < \infty.$$

Therefore, in this case the condition (F) holds.

3. Partially Complete Controllability

Choose $0 < \tau \le T$ and let $\tilde{U}_\tau = C(0,\tau;U)$ and $\tilde{X}_\tau = C(0,\tau;X)$. Then $\tilde{X}_\tau \times \tilde{U}_\tau$ is a Banach space with the norm

$$\|(\cdot,\cdot)\|_{\tilde{X}_\tau\times\tilde{U}_\tau} = \|\cdot\|_{\tilde{X}_\tau} + \|\cdot\|_{\tilde{U}_\tau}.$$

Define the operator $G_\tau : \tilde{X}_\tau \times \tilde{U}_\tau \to \tilde{X}_\tau \times \tilde{U}_\tau$ by

$$G_\tau(y,v)_t = (Y_t, V_t),\ 0 \le t \le \tau, \tag{3}$$

where

$$Y_t = e^{At}\xi + Q_t e^{A^*(T-t)}L^*\tilde{Q}_T^{-1}(h - Le^{AT}\xi) + \int_0^t e^{A(t-s)}\sigma_s f(s,y_s,v_s)\,ds$$
$$- \int_0^t Q_{t-s}e^{A^*(T-t)}L^*\tilde{Q}_{T-s}^{-1}e^{A(T-s)}\sigma_s f(s,y_s,v_s)\,ds \tag{4}$$

and

$$V_t = B^*e^{A^*(T-t)}L^*\left(\tilde{Q}_T^{-1}(h - Le^{AT}\xi) - \int_0^t \tilde{Q}_{T-s}^{-1}e^{A(T-s)}\sigma_s f(s,y_s,v_s)\,ds\right), \tag{5}$$

where $\xi \in X$ and $h \in H$.

Lemma 3.1. *Under the conditions* (A)–(E), *let* $0 < \tau < T$, $\xi \in X$, *and* $h \in H$. *Then the operator* G_τ, *defined by* (3)–(5), *has a unique fixed point.*

Proof. Clearly, G_τ transforms $\tilde{X}_\tau \times \tilde{U}_\tau$ into $\tilde{X}_\tau \times \tilde{U}_\tau$. Take any $(y,v),(z,w) \in \tilde{X}_\tau \times \tilde{U}_\tau$. Let

$$N = \sup_{0\le t\le T}\|e^{At}\| \text{ and } \|\sigma\| = \max_{0\le t\le T}\sigma_t.$$

Denote $G_\tau(y,v) = (Y,V)$ and $G_\tau(z,w) = (Z,W)$. Then

$$G_\tau(y,v)_t - G_\tau(z,w)_t = (Y_t - Z_t, V_t - W_t).$$

Here, $\|Y_t - Z_t\|_X$ can be estimated as follows:

$$\|Y_t - Z_t\| \le \int_0^t \|Q_{t-s}e^{A^*(T-t)}L^*\tilde{Q}_{T-s}^{-1}e^{A(T-s)}\sigma_s\|\|f(s,y_s,v_s) - f(s,z_s,w_s)\|\,ds$$
$$+ \int_0^t \|e^{A(t-s)}\sigma_s\|\|f(s,y_s,v_s) - f(s,z_s,w_s)\|\,ds$$
$$\le N^2\|\sigma\|\int_0^t \frac{\|Q_{t-s}\|}{\gamma_{T-s}}\|f(s,y_s,v_s) - f(s,z_s,w_s)\|\,ds$$
$$+ N\|\sigma\|\int_0^t \|f(s,y_s,v_s) - f(s,z_s,w_s)\|\,ds$$
$$\le \left(1 + \frac{\|Q_T\|N}{\gamma_{T-\tau}}\right)NK\|\sigma\|\int_0^t (\|y_s - z_s\| + \|v_s - w_s\|)\,ds. \tag{6}$$

Similarly, for $\|V_t - W_t\|_U$ we have

$$\begin{aligned}\|V_t - W_t\| &\le \|B^* e^{A^*(T-t)} L^*\| \int_0^t \|\tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s\| \|f(s, y_s, v_s) - f(s, z_s, w_s)\| \, ds \\ &\le N^2 \|B\| \|\sigma\| \int_0^t \frac{1}{\gamma_{T-s}} \|f(s, y_s, v_s) - f(s, z_s, w_s)\| \, ds \\ &\le \frac{\|B\| N}{\gamma_{T-\tau}} N K \|\sigma\| \int_0^t (\|y_s - z_s\| + \|v_s - w_s\|) \, ds. \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain the inequality

$$\|G_\tau(y, v)_t - G_\tau(z, w)_t\| \le k \int_0^t \|(y_s, v_s) - (z_s, w_s)\| ds, \tag{8}$$

where

$$k = \left(1 + \frac{\|Q_T\| N}{\gamma_{T-\tau}} + \frac{\|B\| N}{\gamma_{T-\tau}}\right) N K \|\sigma\|.$$

Let $G_\tau^0(y, v) = (y, v)$ and define $G_\tau^n(y, v) = G_\tau(G_\tau^{n-1}(y, v))$. Then (8) implies

$$\begin{aligned}\|G_\tau^2(y, v)_t - G_\tau^2(z, w)_t\| &\le k \int_0^t \|G_\tau^1(y, v)_s - G_\tau^1(z, w)_s\| \, ds \\ &\le k^2 \int_0^t \int_0^s \|(y_r, v_r) - (z_r, w_r)\| \, dr ds.\end{aligned}$$

Repeating this procedure n times, we obtain

$$\begin{aligned}\|G_\tau^n(y, v)_t - G_\tau^n(z, w)_t\| &\le k^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|(y_{t_n}, v_{t_n}) - (z_{t_n}, w_{t_n})\| \, dt_n \cdots dt_2 dt_1 \\ &\le k^n \|(y, v) - (z, w)\|_{\tilde{X}_\tau \times \tilde{U}_\tau} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_2 dt_1 \\ &\le \frac{k^n t^n}{n!} \|(y, v) - (z, w)\|_{\tilde{X}_\tau \times \tilde{U}_\tau}.\end{aligned}$$

Hence,

$$\|G_\tau^n(y, v) - G_\tau^n(z, w)\|_{\tilde{X}_\tau \times \tilde{U}_\tau} \le \frac{(kT)^n}{n!} \|(y, v) - (z, w)\|_{\tilde{X}_\tau \times \tilde{U}_\tau}.$$

Since

$$\lim_{n \to \infty} \frac{(kT)^n}{n!} = 0,$$

G_τ^n is a contraction mapping for sufficiently large n. Then by generalised contraction mapping theorem, the operator G_τ has a unique fixed point. □

Lemma 3.2. *Under the conditions* (A)–(E)*, let* $0 < \sigma < \tau < T$*,* $\xi \in X$*, and* $h \in H$*. Define the operator* G_τ *by* (3)–(5)*. Let* (x, u) *be a fixed points of* G_τ*. Then the restriction* $(x|_{[0,\sigma]}, u|_{[0,\sigma]})$ *of* (x, u) *from the interval* $[0, \tau]$ *to the interval* $[0, \sigma]$ *is a fixed point of* G_σ*.*

Proof. From (5)–(7), one can observe that if $(Y, V) = G_\tau(y, v)$, then

$$(Y|_{[0,\sigma]}, V|_{[0,\sigma]}) = G_\tau(y, v)|_{[0,\sigma]} = G_\sigma(y|_{[0,\sigma]}, v|_{[0,\sigma]}).$$

Therefore, if (x, u) is a fixed point of G_τ, then $(x|_{[0,\sigma]}, u|_{[0,\sigma]})$ is a fixed point of G_σ. □

Lemma 3.3. *Under the conditions* (A)–(F)*, let* $\xi \in X$*, and* $h \in H$*. Then the operator* G_T *defined by* (3)–(5) *for* $\tau = T$*, has a unique fixed point.*

Proof. By Lemma 3.2, there exists a unique pair (x, u) of X- and U-valued continuous functions on $[0, T)$ such that

$$x_t = e^{At}\xi + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\xi) + \int_0^t e^{A(t-s)}\sigma_s f(s, x_s, u_s)\, ds$$
$$- \int_0^t Q_{t-s} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s f(s, x_s, u_s)\, ds \tag{9}$$

and

$$u_t = B^* e^{A^*(T-t)} L^* \left(\tilde{Q}_T^{-1}(h - Le^{AT}\xi) - \int_0^t \tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s f(s, x_s, u_s)\, ds \right). \tag{10}$$

Since

$$\|Q_{t-s} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s f(s, x_s, u_s)\| \le \frac{\|Q_T\| N^2 M \sigma_s}{\gamma_{T-s}},\ 0 \le s \le T,$$

and the improper integral

$$\int_0^T \frac{\sigma_s}{\gamma_{T-s}}\, ds$$

converges (see condition (F)), x_T is well-defined by equation (9) with $\lim_{t\to T} x_t = x_T$. The same arguments work for (10) as well. So, the pair (x, u) is well-defined in $\tilde{X}_T \times \tilde{U}_T = C(0, T; X) \times U_{\text{ad}}$. Furthermore, $(x|_{[0,\tau]}, u|_{[0,\tau]}) = G_\tau(x|_{[0,\tau]}, u|_{[0,\tau]})$ for all $0 < \tau < T$ and continuity of (x, u) implies $G_T(x, u) = (x, u)$. Therefore, the pair (x, u), defined by (9)–(10) on $[0, T]$, is a fixed point of G_T. If G_T has two fixed points, then G_τ should have two fixed points for some $0 < \tau < T$ that contradicts Lemma 3.1. Thus the pair (x, u) from (9)–(10) is a unique fixed point of G_T. □

Theorem 3.1. *Under the conditions* (A)–(F)*, the semilinear system* (1) *is L-partially complete controllable on* $[0, T]$.

Proof. Take any $\xi \in X$ and $h \in H$. We have to show that there exists $u \in U_{\text{ad}}$ such that $Lx_T = h$, where x is a solution of (2) corresponding to u. Let (x, u) be a fixed point of G_T, defined by (3)–(5) for $\tau = T$. Then (x, u) satisfies (9)–(10). We can find another representation for x as follows:

$$\begin{aligned}
x_t &= e^{At}\xi + \int_0^t e^{A(t-s)}\sigma_s f(s, x_s, u_s)\, ds \\
&\quad + \int_0^t e^{A(t-s)} BB^* e^{A^*(t-s)} e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\xi)\, ds \\
&\quad - \int_0^t \int_s^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s f(s, x_s, u_s)\, drds \\
&= e^{At}\xi + \int_0^t e^{A(t-r)}\sigma_r f(r, x_r, u_r)\, dr \\
&\quad + \int_0^t e^{A(t-r)} BB^* e^{A^*(T-r)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\xi)\, dr \\
&\quad - \int_0^t \int_0^r e^{A(t-r)} BB^* e^{A^*(T-r)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} \sigma_s f(s, x_s, u_s)\, dsdr \\
&= e^{At}\xi + \int_0^t e^{A(t-r)}\sigma_r f(r, x_r, u_r)\, dr + \int_0^t e^{A(t-r)} Bu_r\, dr.
\end{aligned}$$

Hence, the fixed point (x, u) of G_T is so that $u \in U_{\text{ad}}$ and x is a mild solution of the equation (1), corresponding to u. Next, we calculate

$$Lx_T = Le^{AT}\xi + LQ_T L^* \tilde{Q}_T^{-1}(h - Le^{AT}\xi) + \int_0^T e^{A(T-s)}\sigma_s f(s, x_s, u_s)\, ds$$
$$- \int_0^T LQ_{T-s}L^* \tilde{Q}_{T-s}^{-1}\sigma_s f(s, x_s, u_s)\, ds = h.$$

Thus, for every $\xi \in X$ and $h \in H$, there is $u \in U_{\text{ad}}$ such that $Lx_T = h$. This means that the system (1) is L-partially complete controllable. □

4. Examples

We demonstrate Theorem 3.1 in the following examples of control systems.

Example 4.1. The condition (F) in Theorem 3.1 is rather hard condition. If we set $\sigma_t = 1$ in (F), then it becomes

$$\int_0^T \frac{dt}{\gamma_t} = \int_0^T \frac{dt}{\gamma_{T-t}} < \infty.$$

This importer integral is divergent even for simple examples. The one-dimensional linear system

$$x'_t = ax_t + bu_t,$$

where $a \neq 0$ and $b \neq 0$, is obviously completely controllable on $[0, T]$ since

$$Q_T = \int_0^T e^{at} bbe^{at}\, dt = \frac{b^2(e^{2aT} - 1)}{2a} > 0.$$

Here $\gamma_t = b^2(e^{2at} - 1)/2a$. Hence

$$\int_0^T \frac{dt}{\gamma_t} = \frac{2a}{b^2}\int_0^T \frac{1}{e^{2at} - 1}\, dt = -\frac{2a}{b^2}\int_0^T \left(1 + \frac{e^{2at}}{1 - e^{2at}}\right) dt$$
$$= -\frac{2aT + \ln|1 - e^{2aT}| - \lim_{t\to 0^+} \ln|1 - e^{2at}|}{b^2} = \infty.$$

Therefore, the function σ should be selected so that to adjust the convergence of the improper integral in (F).

Example 4.2. Consider the system of differential equations

$$\begin{cases} x'_t = y_t + bu_t, \ x_0 \in \mathbb{R}, \\ y'_t = \sigma_t f(t, x_t, y_t, u_t), \ y_0 \in \mathbb{R}, \end{cases} \tag{11}$$

on $[0, T]$, where $\mathbb{R}$ is the space of real numbers, $u \in U_{\text{ad}} = C(0, T; \mathbb{R})$. Write this system in $\mathbb{R}^2$ as the following semilinear system

$$z'_t = Az_t + \sigma_t F(t, z_t, u_t) + Bu_t, \tag{12}$$

where

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \ F(t, z, u) = \begin{bmatrix} 0 \\ f(t, x, y, u) \end{bmatrix},$$

assuming that

$$z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

One can calculate that

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

The controllability operator is

$$Q_t = \int_0^t e^{As} BB^* e^{A^* s}\, ds = b^2 t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ 0 < t \le T.$$

Hence, Q_t is not coercive and the conditions for complete controllability, based on coercivity of Q_t, fail for this example. We can investigate the partial complete controllability for this system being interested in just the first component x_t of z_t.

Let $L = [1\ \ 0]$. Then

$$\tilde{Q}_t = LQ_T L^* = b^2 t > 0.$$

Therefore, $\tilde{Q}_t$ is coercive for all $0 < t \le T$. Here $\gamma_t = b^2 t$. So, if $\sigma_t \le (T-t)^\alpha$ for some $\alpha > 0$, then

$$\int_0^T \frac{\sigma_t}{\gamma_{T-t}}\, dt \le \int_0^T \frac{(T-t)^\alpha}{b^2 (T-t)}\, dt = \frac{1}{b^2} \int_0^T t^{\alpha-1}\, dt = \frac{T^\alpha}{b^2 \alpha} < \infty.$$

Thus if, additionally, f is continuous bounded and also satisfies Lipschitz condition in x, y and u, then the system (12) is L-partially complete controllable on $[0,T]$.

Example 4.3. Delay equations are typical for application of partial controllability concepts. Consider the nonlinear delay equation

$$\begin{cases} x_t' = a x_t + b u_t + \sigma_t f\big(t, x_t, \int_{-\varepsilon}^0 x_{t+\theta}\, d\theta, u_t\big), \\ x_0 = \xi,\ x_\theta = \eta_\theta,\ -\varepsilon \le \theta \le 0, \end{cases} \tag{13}$$

on $[0,T]$, where $a \ne 0$, $b \ne 0$, $0 < \varepsilon < T$, $\xi \in \mathbb{R}$, $\eta \in L_2(-\varepsilon, 0; \mathbb{R})$ (the space of square integrable functions) and $u \in U_{\mathrm{ad}} = C(0,T;\mathbb{R})$.

Introduce the function $\bar{x} : [0,T] \to L_2(-\varepsilon, 0; \mathbb{R})$ by

$$[\bar{x}_t]_\theta = x_{t+\theta},\ 0 \le t \le T,\ -\varepsilon \le \theta \le 0.$$

This function is a solution of

$$\bar{x}_t' = (d/d\theta)\bar{x}_t,\ \bar{x}_0 = \eta,\ 0 < t \le T.$$

Denote by $\mathcal{T}_t$, $t \ge 0$, the semigroup generated by the differential operator $d/d\theta$ and let Γ be the integral operator from $L_2(-\varepsilon, 0; \mathbb{R})$ to $\mathbb{R}$, defined by

$$\Gamma h = \int_{-\varepsilon}^0 h_\theta\, d\theta,\ h \in L_2(-\varepsilon, 0; \mathbb{R}).$$

Then for

$$y_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix},\ \zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}),$$

the system (13) can be written as

$$y_t' = A y_t + \sigma_t F(t, y_t, u_t) + B u_t,\ y_0 = \zeta, \tag{14}$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & d/d\theta \end{bmatrix},\ F(t,y,u) = \begin{bmatrix} f(t, x, \Gamma\bar{x}, u) \\ 0 \end{bmatrix},\ B = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

where the variable y consists of two components:

$$y = \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}).$$

The semigroup, e^{At} has the form

$$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & \mathcal{T}_t \end{bmatrix},\ t \ge 0.$$

Therefore, the controllability operator Q_t for the system (14) equals to

$$Q_t = \int_0^t e^{As} B^* B e^{A^* s}\, ds = \int_0^t \begin{bmatrix} b^2 e^{2as} & 0 \\ 0 & 0 \end{bmatrix} dt = \begin{bmatrix} b^2(e^{2at}-1)/2a & 0 \\ 0 & 0 \end{bmatrix}.$$

This is not a coercive operator.

Taking into account that the original system is given by (13), and (14) is just representation of (13) in the standard form, enlarging the original state space $\mathbb{R}$ to $\mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R})$, one can see that the complete controllability for the system (13) is in fact L-partial complete controllability for the system (14) if

$$L = [1 \;\; 0] : \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}) \to \mathbb{R}.$$

Calculating partial controllability operator $\tilde{Q}_t$, we have

$$\tilde{Q}_t = L Q_t L^* = \frac{b^2(e^{2at}-1)}{2a} > 0,$$

So, $\tilde{Q}_t$ is coercive for $0 < t \le T$. Furthermore,

$$\gamma_t = \frac{b^2(e^{2at}-1)}{2a}.$$

Hence, if $\sigma_t \le a(e^{2a(T-t)} - 1)(T-t)^{1-\alpha}$ for some $\alpha > 0$, then

$$\int_0^T \frac{\sigma_t}{\gamma_{T-t}}\, dt \le \frac{2a^2}{b^2} \int_0^T (T-t)^{1-\alpha}\, dt = \frac{2a^2 T^\alpha}{b^2 \alpha} < \infty.$$

Thus if, additionally, f is continuous bounded and also satisfies Lipschitz condition in x, y and u, then the system (13) is L-partially complete controllable on $[0, T]$.

5. Conclusion

Generally, it is difficult to obtain complete controllability conditions. In this paper one such sufficient condition for partial complete controllability of a semilinear control system is proved. This condition requires convergence of the improper integral in (F). Given examples demonstrate how to adjust the function σ for fulfilling the convergence of this improper integral. Another feature of the obtained result is that it allows to get complete controllability of a component of the state process. There are other kinds of systems which besides semilinearity include impulsiveness, fractional derivatives, randomness etc. The result of this paper can be extended to these systems as well.

References

[1] Balachandran, K., and Dauer, J., (2002), Controllability of nonlinear systems in Banach spaces: a survey, *Journal of Optimization Theory and Applications*, 115, 7–28.

[2] Balachandran, K., and Dauer, J., (1998), Local controllability of semilinear evolution systems in Banach spaces, *Indian Journal of Pure and Applied Mathematics*, 29, 311-320.

[3] Bashirov, A. E., (2003), *Partially Observable Linear Systems under Dependent Noises; Systems & Control: Foundations & Applications*, Birkhäuser, Basel.

[4] Bensoussan, A., (1992), *Stochastic Control of Partially Observable Systems*, Cambridge University Press, London.

[5] Bensoussan, A., Da Prato, G., Delfour, M. C., and Mitter, S. K., (1993), *Representation and Control of Infinite Dimensional Systems, Volume 2; Systems & Control: Foundations & Applications*, Birkhäuser, Boston.

[6] Bashirov, A. E., Etikan, H., and Şemi, N., (2010), Partial controllability of stochastic linear systems, *International Journal of Control*, 83, 2564–2572.

[7] Bashirov, A. E., Mahmudov, N. I., Şemi, N., and Etikan, H., (2007), Partial controllability concepts, *International Journal of Control*, 80, 1–7.

[8] Bashirov, A. E., and Mahmudov N. I., (1999), On concepts of controllability for deterministic and stochastic systems, *SIAM Journal of Control and Optimization*, 37, 1808–1821.
[9] Bashirov, A. E., and Kerimov, K. R., (1997), On controllability conception for stochastic systems, *SIAM Journal of Control and Optimization*, 35, 384–398.
[10] Bashirov, A. E., (1996), On weakening of the controllability concepts, *Proceedings of the 35th Conference on Decission and Control, Kobe, Japan, December, 1996*, 640–645.
[11] Bashirov, A. E., and Mahmudov, B. I., (1999), Controllability of linear deterministic and stochastic systems, *Proceedings of the 38th Conference on Decision and Control, Phoenix, Arisona, USA, December, 1999*, 3196–3201.
[12] Bashirov, A. E., and Mahmudov, N. I., (1999), Some new results in theory of controllability, *Proceedings of the 7th Mediterranean Conference on Control and Automation, Haifa, Israel, June 28–30, 1999*, 323–343.
[13] Bashirov, A. E., and Jneid, M., (2013), On partial complete controllability of semilinear systems, *Abstract and Applied Analysis*, doi:10.1155/2013/521052, 8 pages.
[14] Curtain, R. F., and Zwart, H. J., (1995), *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, Berlin.
[15] Fattorini, H. O., (1967), Some remarks on complete controllability, *SIAM Journal of Control*, 4, 686–694.
[16] Kalman, R. E., (1960), A new approach to linear filtering and prediction problems, *Transactions of ASME, Series D, Journal of Basic Engineering*, 82, 35–45.
[17] Klamka, J., (1991), *Controllability of Dynamical Systems*, Kluwer Academic, Dordrecht.
[18] Klamka, J., (2000), Shauder's fixed point theorem in nonlinear controllability problems, *Control Cybernetics*, 29, 153–165.
[19] Li, X., and Yong, J., (1995), *Optimal Control for Infinite Dimensional Systems; Systems* & *Control: Foundations* & *Applications*, Birkhäuser, Boston.
[20] Mahmudov, N. I., (2003), Controllability of semilinear stochastic systems in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, 288, 197–211.
[21] Ren, Y., Hu, L., and Sakthivel, R., (2011), Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, *Journal of Computational and Applied Mathematics*, 235, 2603–2614.
[22] Russel, D. L., (1967), Nonharmonic Fourier series in the control theory of distributed parameter systems, *Journal of Mathematical Analysis and Applications*, 18, 542–560.
[23] Sakthivel, R., Ganesh, R., and Suganya, N., (2012), Approximate controllability of fractional neutral stochastic system with infinite delay, *Reports on Mathematical Physics*, 70, 291–311.
[24] Sakthivel, R., Ren, Y., and Mahmudov, N. I., (2011), On approximate controllability of semilinear fractional differential systems, *Computers and Mathematics with Applications*, 62, 1451–1459.
[25] Yan, Z., (2012), Approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay, *International Journal of Control*, 85, 1051–1062.
[26] Zabczyk. J., (1995), *Mathematical Control Theory: An Introduction; Systems* & *Control: Foundations* & *Applications*, Birkhäuser, Berlin.

Agamirza Bashirov, for a photography and short biography, see TWMS J. of Appl. and Eng. Math., V.1, No1.

Maher Jneid was a PhD student at Department of Mathematics of Eastern Mediterranean University. In July 11, 2014 he has successfully defended his PhD thesis. His research concerns controllability theory.