TWMS J. App. Eng. Math. V.4, No.1, 2014, pp. 50-55.

ON A CRITERION FOR MULTIVALENT HARMONIC FUNCTIONS

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ABSTRACT. For normalized harmonic functions $f(z) = h(z) + \overline{g(z)}$ in the open unit disk, a criterion on the analytic part $h(z)$ for $f(z)$ to be p-valent and sense-preserving is discussed. Furthermore, several illustrative examples and images of $f(z)$ satisfying the obtained condition are enumerated.

Keywords: Harmonic function, Multivalent function, Univalent function.

AMS Subject Classification: 30C45, 58E20.

1. INTRODUCTION AND DEFINITIONS

For a fixed p $(p = 1, 2, 3, \cdots)$, a meromorphic function $f(z)$ in a domain $\mathbb{D}$ is said to be p-valent (or multivalent of order p) in $\mathbb{D}$ if for each w_0 the equation $f(z) = w_0$ has at most p roots in $\mathbb{D}$ where the roots are counted in accordance with their multiplicity and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p roots in $\mathbb{D}$. In particular, $f(z)$ is said to be univalent in $\mathbb{D}$ when $p = 1$. A complex-valued harmonic function $f(z)$ in $\mathbb{D}$ is given by

$$f(z) = h(z) + \overline{g(z)} \tag{1.1}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$. We call $h(z)$ and $g(z)$ the analytic part and co-analytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in $\mathbb{D}$ is $|h'(z)| > |g'(z)|$ for all $z \in \mathbb{D}$ (see [2] or [8]). Let $\mathcal{H}(p)$ denote the class of functions $f(z)$ of the from

$$f(z) = h(z) + \overline{g(z)} = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \overline{\sum_{n=p}^{\infty} b_n z^n} \tag{1.2}$$

which are harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We next denote by $\mathcal{S}_{\mathcal{H}}(p)$ the class of functions $f(z) \in \mathcal{H}(p)$ which are p-valent and sense-preserving in $\mathbb{U}$. Then, we say that $f(z) \in \mathcal{S}_{\mathcal{H}}(p)$ is a p-valently harmonic function in $\mathbb{U}$.

In the present paper, we discuss a sufficient condition about $h(z)$ for $f(z) \in \mathcal{H}(p)$ given by (1.2), satisfying

$$g'(z) = z^{m-1} h'(z) \tag{1.3}$$

for some m $(m = 2, 3, 4, \cdots)$, to be in the class $\mathcal{S}_{\mathcal{H}}(p)$.

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§ Submitted for GFTA'13, held in Işık University on October 12, 2013.

2. Main Result

Our result is contained in

Theorem 2.1. *Let* $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ *be analytic in the closed unit disk* $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ *with* $H(z) = h'(z)/z^{p-1} \neq 0 \ \ (z \in \overline{\mathbb{U}})$ *and let*

$$F(t) = (2p + m - 1)t + 2\arg\left(H(e^{it})\right) \qquad (-\pi \leq t < \pi) \tag{2.1}$$

for some m $(m = 2, 3, 4, \cdots)$*. If for each* $k \in K = \left\{0, \pm 1, \pm 2, \cdots, \pm\lfloor \frac{2p+m+1}{2} \rfloor\right\}$ *where* $\lfloor \ \ \rfloor$ *is the floor function, the equation*

$$F(t) = 2k\pi \tag{2.2}$$

has at most a single root in $[-\pi, \pi)$ *and for all* $k \in K$ *there exist exactly* $2p + m - 1$ *such roots in* $[-\pi, \pi)$*, then the harmonic function* $f(z) = h(z) + \overline{g(z)}$ *with* $g'(z) = z^{m-1}h'(z)$ *belongs to the class* $\mathcal{S}_{\mathcal{H}}(p)$ *and maps* $\mathbb{U}$ *onto a domain surrounded by* $2p + m - 1$ *concave curves with* $2p + m - 1$ *cusps.*

Remark 2.1. *If we take* $p = 1$ *in Theorem 2.1, then we readily arrive at the univalence criterion for harmonic functions due to Hayami and Owa* [5, Theorem 2.1] *(see also* [10]*).*

3. Some Illustrative Examples and Image Domains

We discuss harmonic functions $f(z) = h(z) + \overline{g(z)}$ which satisfy the conditions of Theorem 2.1 and their image domains.

Example 3.1. *Let* $h(z) = z^p$*. Then we easily see that the equation* (2.2) *becomes*

$$(2p + m - 1)t = 2k\pi \qquad \left(k = 0, \pm 1, \pm 2, \cdots, \pm\lfloor \frac{2p + m + 1}{2} \rfloor\right) \tag{3.1}$$

which satisfies the conditions of Theorem 2.1. Hence, the function

$$f(z) = h(z) + \overline{g(z)} = z^p + \overline{\frac{p}{p + m - 1} z^{p+m-1}} \qquad (g'(z) = z^{m-1}h'(z)) \tag{3.2}$$

belongs to the class $\mathcal{S}_{\mathcal{H}}(p)$ *and it maps* $\mathbb{U}$ *onto a domain surrounded by* $2p+m-1$ *concave curves with* $2p + m - 1$ *cusps. Taking* $p = 2$ *and* $m = 4$ *for example, we know that the function*

$$f(z) = z^2 + \frac{2}{5}\overline{z}^5 \tag{3.3}$$

is a 2-valently harmonic function in $\mathbb{U}$ *and it maps* $\mathbb{U}$ *onto the domain surrounded by* 7 *concave curves with* 7 *cusps as shown in Figure 1.*

Remark 3.1. *Since it follows that*

$$F(t) = (2p + m - 1)t + 2\mathrm{Im}\left(\log H(e^{it})\right) \tag{3.4}$$

where $F(t)$ *is given by* (2.1)*, we obtain that*

$$F'(t) = m + 1 + 2\mathrm{Re}\left(\frac{e^{it}h''(e^{it})}{h'(e^{it})}\right) \tag{3.5}$$

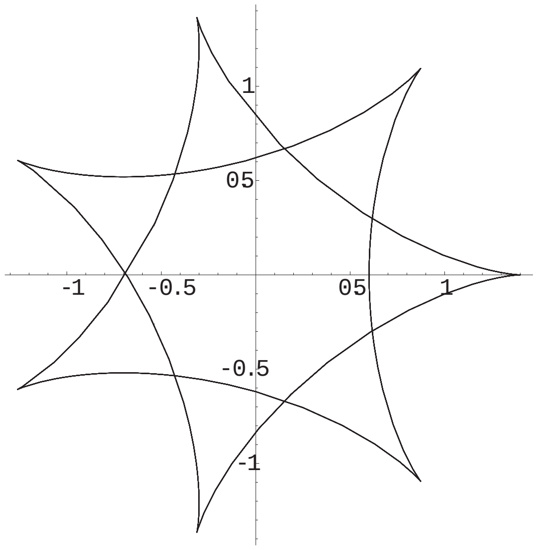

FIGURE 1. The image of $f(z) = z^2 + \frac{2}{5}\overline{z}^5$.

which implies that $F(t)$ is increasing if

$$\text{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{m-1}{2} \qquad (z \in \mathbb{U}). \tag{3.6}$$

By the above remark, we derive the following exapmle.

Example 3.2. *Let $h(z) = z^p + \frac{c}{p+1}z^{p+1}$ $\left(|c| \leq p - \frac{2p}{2p+m+1}\right)$. Then the equation (2.2) becomes*

$$F(t) = (2p + m - 1)t + 2\arg\left(p + ce^{it}\right). \tag{3.7}$$

Noting

$$\text{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > p + 1 - \frac{p}{p - |c|} \geq -\frac{m-1}{2} \qquad (z \in \mathbb{U}), \tag{3.8}$$

$$F(-\pi) = -(2p + m - 1)\pi - 2\arctan\left(\frac{|c|\sin\theta}{p - |c|\cos\theta}\right) \tag{3.9}$$

and

$$F(\pi) = (2p + m - 1)\pi - 2\arctan\left(\frac{|c|\sin\theta}{p - |c|\cos\theta}\right) \tag{3.10}$$

where $0 \leq \theta = \arg(c) < 2\pi$, we see that $F(t)$ satisfies the conditions of Theorem 2.1. Hence, the function

$$f(z) = h(z) + \overline{g(z)} = z^p + \frac{c}{p+1}z^{p+1} + \overline{\frac{p}{p+m-1}z^{p+m-1} + \frac{c}{p+m}z^{p+m}} \tag{3.11}$$

belongs to the class $\mathcal{S}_{\mathcal{H}}(p)$ and it maps $\mathbb{U}$ onto a domain surrounded by $2p+m-1$ concave curves with $2p+m-1$ cusps. Putting $p = 2$, $m = 4$ and $c = \frac{2}{3}i$ $\left(|c| \leq \frac{14}{9}\right)$, we know that the function

$$f(z) = z^2 + \frac{2i}{9}z^3 + \frac{2}{5}\overline{z}^5 + \frac{i}{9}\overline{z}^6 \tag{3.12}$$

is a 2-valently harmonic function in $\mathbb{U}$ and it maps $\mathbb{U}$ onto the domain surrounded by 7 concave curves with 7 cusps as shown in Figure 2.

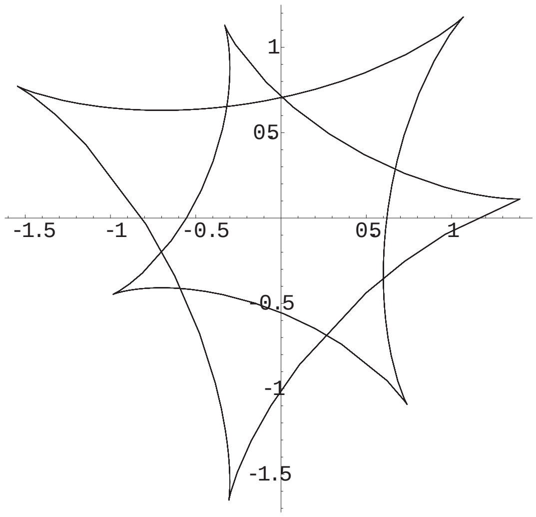

FIGURE 2. The image of $f(z) = z^2 + \frac{2i}{9}z^3 + \frac{2}{5}\overline{z}^5 + \frac{i}{9}\overline{z}^6$.

In consideration of the process of proving Theorem 2.1, we obtain the following interesting example.

Example 3.3. *If we consider special functions $h(z)$ and $g(z)$ given by*

$$h'(z) = \frac{pz^{p-1}}{1+z^{2p+m-1}} \quad \text{and} \quad g'(z) = \frac{pz^{p+m-2}}{1+z^{2p+m-1}} \qquad \left(g'(z) = z^{m-1}h'(z)\right), \tag{3.13}$$

then the function

$$f(z) = h(z) + \overline{g(z)} = \int_0^z \frac{p\zeta^{p-1}}{1+\zeta^{2p+m-1}}d\zeta + \overline{\int_0^z \frac{p\zeta^{p+m-2}}{1+\zeta^{2p+m-1}}d\zeta} \tag{3.14}$$

is a member of the class $\mathcal{S}_{\mathcal{H}}(p)$ and it maps $\mathbb{U}$ onto a domain surrounded by $2p+m-1$ straight lines with $2p+m-1$ cusps. Indeed, setting $p=2$ and $m=2$, we know that

$$f(z) = \int_0^z \frac{2\zeta}{1+\zeta^5}d\zeta + \overline{\int_0^z \frac{2\zeta^2}{1+\zeta^5}d\zeta} \tag{3.15}$$

is a 2-valently harmonic function and it maps $\mathbb{U}$ onto a star as shown in Figure 3. Furthermore, if we take $p=1$ in (3.14), then we see that the function

$$f_{m+1}(z) = h(z) + \overline{g(z)} = \int_0^z \frac{1}{1+\zeta^{m+1}}d\zeta + \overline{\int_0^z \frac{\zeta^{m-1}}{1+\zeta^{m+1}}d\zeta} \tag{3.16}$$

is univalent in $\mathbb{U}$ and it maps $\mathbb{U}$ onto a $(m+1)$-sided polygon.

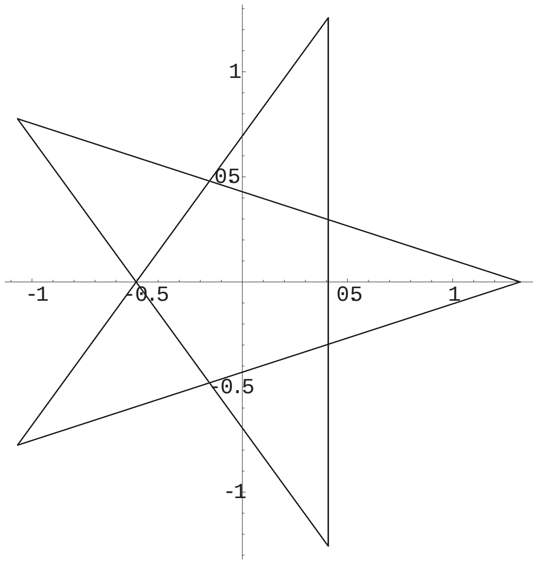

FIGURE 3. The image of $f(z)=\int_0^z \frac{2\zeta}{1+\zeta^5}d\zeta+\overline{\int_0^z \frac{2\zeta^2}{1+\zeta^5}d\zeta}$

4. Appendix

Finally, we recall here the following theorem due to Mocanu [9].

Theorem 4.1. *Let $h(z)$ and $g(z)$ be analytic functions in a domain $\mathbb{D}$. If $h(z)$ is convex in $\mathbb{D}$ and $|g'(z)|<|h'(z)|$ for $z\in\mathbb{D}$, then the harmonic function $f(z)=h(z)+\overline{g(z)}$ is univalent and sense-preserving in $\mathbb{D}$.*

In other words, if $h(z)$ and $g(z)$ satisfy

$$g'(z)=w(z)h'(z) \qquad (z\in\mathbb{D}) \tag{4.1}$$

and

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>0 \qquad (z\in\mathbb{D}) \tag{4.2}$$

for some analytic function $w(z)$ in $\mathbb{D}$ satisfying $|w(z)|<1$ $(z\in\mathbb{D})$, then $f(z)$ is univalent and sense-preserving in $\mathbb{D}$.

Bshouty and Lyzzaik [1] have shown the next theorem which is closely related to Theorem 2.1 and Remark 3.1 with $p=1$ and $m=2$ as the stronger result of the conjecture of Mocanu [10].

Theorem 4.2. *If $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$, with $h'(0)\neq 0$, which satisfy*

$$g'(z)=zh'(z) \tag{4.3}$$

and

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2} \tag{4.4}$$

for all $z\in\mathbb{U}$, then the harmonic function $f(z)=h(z)+\overline{g(z)}$ is univalent close-to-convex in $\mathbb{U}$.

These theorems motivate us to state

Conjecture 4.1. *If the function $f(z)$ given by* (1.2) *is harmonic in $\mathbb{U}$ which satisfies*

$$g'(z) = z^{m-1}h'(z) \tag{4.5}$$

and

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{m-1}{2} \qquad (z \in \mathbb{U}) \tag{4.6}$$

for some m $(m = 2, 3, 4, \cdots)$, then $f(z)$ is p-valent in $\mathbb{U}$.

The details of this article can be found in the paper [7].

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