

EXTENSIONS FOR CERTAIN SUBORDINATION RELATIONS

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ABSTRACT. For some complex number γ which has a positive real part, a certain subordination relation concerned with the Bernardi integral operator \mathbf{I}_γ was proven by D. J. Hallenbeck and St. Ruscheweyh (Proc. Amer. Math. Soc. **52**(1975), 191-195). By considering the analyticity of the functions defined by the Bernardi integral operator \mathbf{I}_γ for some non-zero complex number γ with $\text{Re } \gamma \leq 0$, an extension for certain subordination relation are discussed.

Keywords: Analytic function, Univalent function, Integral operator, Subordination.

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1. INTRODUCTION

For a positive integer n and a complex number a , let $\mathcal{H}[a, n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let \mathcal{A} be the class of analytic functions $f(z)$ which are normalized by $f(0) = f'(0) - 1 = 0$.

Let $p(z)$ and $q(z)$ be analytic in \mathbb{U} . Then the function $p(z)$ is said to be subordinate to $q(z)$ in \mathbb{U} , written by

$$p(z) \prec q(z), \tag{1}$$

if there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), and such that $p(z) = q(w(z))$ ($z \in \mathbb{U}$). From the definition of the subordinations, it is easy to show that the subordination (1) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{2}$$

In particular, if $q(z)$ is univalent in \mathbb{U} , then the subordination (1) is equivalent to the condition (2).

For the functions $p(z) \in \mathcal{H}[a, n]$ and $h(z) \in \mathcal{H}[a, 1]$, Hallenbeck and Ruscheweyh [3] (also Miller and Mocanu [6]) considered the following first-order differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \tag{3}$$

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where γ is complex number with $\gamma \neq 0$, and proved the following subordination result.

Lemma 1.1. *Let n be a positive integer, and let γ be a complex number with $\operatorname{Re} \gamma > 0$. Also, let $h(z)$ be analytic and convex univalent in \mathbb{U} with $h(0) = a$. If $p(z) \in \mathcal{H}[a, n]$ satisfies the differential subordination (3), then $p(z) \prec q(z)$, where*

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1} dt. \quad (4)$$

For the function $f(z) \in \mathcal{H}[0, n]$, the Bernardi integral operator [2] is defined by

$$\mathbf{I}_\gamma[f](z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1} dt, \quad (5)$$

where $\gamma = 0, 1, 2, \dots$. In particular, the integral operator \mathbf{I}_0 is well-known as the Alexander integral operator [1]. The integral operator \mathbf{I}_γ is well-defined on $\mathcal{H}[0, n]$ and maps $f(z)$ into $\mathcal{H}[0, n]$. Specially, we note that $\mathbf{I}_\gamma[f](z) \in \mathcal{A}$ for $f(z) \in \mathcal{A}$. Next lemma [6] shows that the Bernardi integral operator can be extended for certain complex values of γ .

Lemma 1.2. *Let m be an integer with $m \geq 0$, and let γ be a complex number with $\operatorname{Re} \gamma > -m$. If $f(z) = \sum_{k=m}^{\infty} a_k z^k$ is analytic in \mathbb{U} , and $F(z)$ is defined by*

$$F(z) = \frac{1}{z^\gamma} \int_0^z f(t)t^{\gamma-1} dt \quad (6)$$

then $F(z) = \sum_{k=m}^{\infty} \frac{a_k}{\gamma+k} z^k$ is analytic in \mathbb{U} .

Remark 1.1. Let us consider the analyticity of the function $F(z)$ defined by (6). If $f(z) \in \mathcal{H}[a, 1]$ with $a \neq 0$, then by Lemma 1.2 with $m = 0$, the function $F(z)$ is analytic in \mathbb{U} for some complex number γ such that $\operatorname{Re} \gamma > 0$. On the other hand, considering the case $m = 1$ in Lemma 1.2, we find that the function $F(z)$ with $f(z) \in \mathcal{H}[0, 1]$ is analytic in \mathbb{U} for some complex number γ such that $\operatorname{Re} \gamma > -1$.

To prove the subordination relation in Lemma 1.1, Miller and Mocanu [6] discussed the analyticity of the solution $q(z)$ of the following first-order differential equation

$$q(z) + \frac{nzq'(z)}{\gamma} = h(z) \quad (z \in \mathbb{U}) \quad (7)$$

with $q(0) = h(0)$, where γ is complex number with $\gamma \neq 0$. Note that the solution $q(z)$ of the differential equation (7) is given by (4). Let $h(z) \in \mathcal{H}[h(0), 1]$. According to the first assertion in Remark 1.1, the function $q(z)$ given in (4) is analytic in \mathbb{U} for some complex number γ with $\operatorname{Re} \gamma > 0$, because $h(z) \in \mathcal{H}[h(0), 1]$.

On the other hand, let us define the functions $h_0(z)$ and $q_0(z)$ by

$$h_0(z) = h(z) - h(0) \quad \text{and} \quad q_0(z) = q(z) - q(0) \quad (8)$$

for $z \in \mathbb{U}$, where $q(0) = h(0)$. Then the differential equation (7) is equivalent to

$$q_0(z) + \frac{nzq_0'(z)}{\gamma} = h_0(z) \quad (z \in \mathbb{U}) \quad (9)$$

with $q_0(0) = h_0(0) = 0$. Noting that the solution $q_0(z)$ of the differential equation (9) is given by

$$q_0(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h_0(t) t^{\frac{\gamma}{n}-1} dt \quad (z \in \mathbb{U}),$$

it follows from the equalities in (8) that the solution $q(z)$ of the differential equation (7) can be represented by

$$q(z) = h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - h(0)) t^{\frac{\gamma}{n}-1} dt \quad (z \in \mathbb{U}). \quad (10)$$

Then since $h(z) - h(0) \in \mathcal{H}[0, 1]$, the second assertion in Remark 1.1 leads us that the function $q(z)$ given by (10) is analytic in \mathbb{U} for some complex number γ with $\operatorname{Re} \left(\frac{\gamma}{n} \right) > -1$.

From the above-mentioned, we expect that the subordination relation in Lemma 1.1 can be discussed for some complex number γ with $\gamma \neq 0$ and $\operatorname{Re} \gamma > -n$ by replacing the conclusion in Lemma 1.1 with the following subordination

$$p(z) \prec h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - h(0)) t^{\frac{\gamma}{n}-1} dt.$$

In the present paper, by considering some properties for the function $q(z)$ given in (10), we will discuss the following subordination relation :

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad \text{implies} \quad p(z) \prec h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - h(0)) t^{\frac{\gamma}{n}-1} dt \quad (11)$$

for some complex number γ with $\gamma \neq 0$ and $\operatorname{Re} \gamma > -n$.

2. PRELIMINARY RESULTS

Miller and Mocanu [6] developed a lemma which is well-known as the open door lemma. By considering a simpler version of the open door function, Kuroki and Owa [4] provided the better result for the open door lemma.

Definition 2.1 (Simpler version of open door function). *Let c be a complex number with $\operatorname{Re} c > 0$. Then the open door function $R_c(z)$ is defined by*

$$R_c(z) = -\bar{c} - \frac{1}{1-z} + \frac{2\operatorname{Re} c + 1}{1 + \frac{c}{z}} \quad (z \in \mathbb{U}). \quad (12)$$

The function $R_c(z)$ is analytic and univalent in \mathbb{U} with $R_c(0) = c$. In addition, $R_c(z)$ maps \mathbb{U} onto the complex plane with slits along the half-lines ℓ_c^+ and ℓ_c^- , where

$$\ell_c^+ = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \text{ and } \operatorname{Im} w \geq \frac{1}{\operatorname{Re} c} \left(|c| \sqrt{2\operatorname{Re} c + 1} - \operatorname{Im} c \right) \right\}$$

and

$$\ell_c^- = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \text{ and } \operatorname{Im} w \leq -\frac{1}{\operatorname{Re} c} \left(|c| \sqrt{2\operatorname{Re} c + 1} + \operatorname{Im} c \right) \right\}.$$

Note that the slit domain $\mathbb{C} \setminus \{\ell_c^+ \cup \ell_c^-\}$ is not symmetric with respect to the real axis (see [4]).

Lemma 2.1 (Open door lemma). *Let c be a complex number with $\operatorname{Re} c > 0$, and let $R(z) \in \mathcal{H}[c, 1]$ satisfy the subordination $R(z) \prec R_c(z)$, where $R_c(z)$ is defined by (12). If $p(z) \in \mathcal{H}[\frac{1}{c}, 1]$ satisfies the differential equation*

$$zp'(z) + R(z)p(z) = 1 \quad (z \in \mathbb{U}),$$

then $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$).

More general form of this lemma for $p(z) \in \mathcal{H}[\frac{1}{c}, n]$ was given in the work [5].

3. SOME PROPERTIES FOR CERTAIN INTEGRAL OPERATOR

To considering the subordination relation (11), we need to develop some property for certain integral operator by using the open door lemma.

Theorem 3.1. *Let γ be a complex number with $\operatorname{Re} \gamma > -1$, and let $f(z) \in \mathcal{A}$ satisfy*

$$1 + \frac{zf''(z)}{f'(z)} + \gamma \prec R_{\gamma+1}(z),$$

where $R_{\gamma+1}(z)$ is the open door function defined by

$$R_{\gamma+1}(z) = -(\bar{\gamma} + 1) - \frac{1}{1-z} + \frac{2\operatorname{Re} \gamma + 3}{1 + \frac{\gamma+1}{\bar{\gamma}+1}z} \quad (z \in \mathbb{U}).$$

If $F = \mathbf{I}_\gamma[f]$ is defined by (5), then $F(z) \in \mathcal{A}$, $F'(z) \neq 0$ ($z \in \mathbb{U}$) and

$$\operatorname{Re} \left(1 + \frac{zF''(z)}{F'(z)} + \gamma \right) > 0 \quad (z \in \mathbb{U}).$$

4. MAIN RESULTS

Applying Theorem 3.1, we derive the following result concerned with the subordination relation (11).

Theorem 4.1. *Let n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and $-n < \operatorname{Re} \gamma \leq 0$. Also, let $h(z) \in \mathcal{H}[a, 1]$ satisfy the subordination*

$$1 + \frac{zh''(z)}{h'(z)} + \frac{\gamma}{n} \prec R_{\frac{\gamma}{n}+1}(z), \quad (13)$$

where $R_{\frac{\gamma}{n}+1}(z)$ is the open door function defined by

$$R_{\frac{\gamma}{n}+1}(z) = -\left(\frac{\bar{\gamma}}{n} + 1\right) - \frac{1}{1-z} + \frac{2\operatorname{Re} \left(\frac{\gamma}{n}\right) + 3}{1 + \frac{\frac{\gamma}{n}+1}{\frac{\bar{\gamma}}{n}+1}z} \quad (z \in \mathbb{U}). \quad (14)$$

If $p(z) \in \mathcal{H}[a, n]$ satisfies the differential subordination (3), then

$$p(z) \prec a + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a)t^{\frac{\gamma}{n}-1} dt. \quad (15)$$

Remark 4.1. For $h(z) \in \mathcal{H}[a, 1]$, if we define the function $H(z)$ by

$$H(z) = 1 + \frac{zh''(z)}{h'(z)} \quad (z \in \mathbb{U}), \quad (16)$$

then the assumption (13) in Theorem 4.1 can be written by

$$H(z) \prec R_{\frac{\gamma}{n}+1}(z) - \frac{\gamma}{n}, \quad (17)$$

where $R_{\frac{\gamma}{n}+1}(z)$ is defined by (14). The subordination (17) means that $H(z)$ maps \mathbb{U} onto inside of the slit domain $\mathbb{C} \setminus \{\ell^+ \cup \ell^-\}$, where

$$\ell^+ = \left\{ w \in \mathbb{C} : \operatorname{Re} w = -\frac{\operatorname{Re} \gamma}{n} \text{ and } \operatorname{Im} w \geq \frac{|\gamma + n| \sqrt{\frac{2\operatorname{Re} \gamma}{n} + 3} - (\operatorname{Im} \gamma) \left(\frac{\operatorname{Re} \gamma}{n} + 2 \right)}{\operatorname{Re} \gamma + n} \right\}$$

and

$$\ell^- = \left\{ w \in \mathbb{C} : \operatorname{Re} w = -\frac{\operatorname{Re} \gamma}{n} \text{ and } \operatorname{Im} w \leq \frac{-|\gamma + n| \sqrt{\frac{2\operatorname{Re} \gamma}{n} + 3} - (\operatorname{Im} \gamma) \left(\frac{\operatorname{Re} \gamma}{n} + 2 \right)}{\operatorname{Re} \gamma + n} \right\}.$$

5. AN EXTENSION OF SUBORDINATION RELATION FOR CERTAIN COMPLEX VALUES OF γ

Let ζ be a smooth arc in \mathbb{U} connecting 0 to z , and assign a value to $\lim_{t \rightarrow 0} \arg t$ ($t \in \zeta$). We define $t^\gamma = e^{\gamma \log t}$ ($t \in \zeta$) by continuation. Noting that

$$\lim_{t \rightarrow 0} |t^\gamma| = \lim_{t \rightarrow 0} |t|^{\operatorname{Re} \gamma} e^{-(\operatorname{Im} \gamma) \arg t} = 0 \quad (t \in \zeta)$$

when $\operatorname{Re} \gamma > 0$, we define $t^\gamma = 0$ at $t = 0$ ($\operatorname{Re} \gamma > 0$). Thus, a simple calculation gives that

$$\int_0^z t^{\gamma-1} dt = \int_\zeta t^{\gamma-1} dt = \left[\frac{t^\gamma}{\gamma} \right]_0^z = \frac{z^\gamma}{\gamma} \quad (\operatorname{Re} \gamma > 0).$$

Therefore, it follows from the above fact that

$$\begin{aligned} \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1} dt &= \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z a t^{\frac{\gamma}{n}-1} dt + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a)t^{\frac{\gamma}{n}-1} dt \\ &= a + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a)t^{\frac{\gamma}{n}-1} dt \quad (\operatorname{Re} \gamma > 0), \end{aligned}$$

where $h(z) \in \mathcal{H}[a, 1]$. This leads that the conclusion in Lemma 1.1 can be replaced with the subordination (15). Hence by combining the assertions in Lemma 1.1 and Theorem 4.1, we derive the subordination result for some non-zero complex number γ with $\operatorname{Re} \gamma > -n$.

Theorem 5.1. *Let n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and $\operatorname{Re} \gamma > -n$. Also, let $h(z) \in \mathcal{H}[a, 1]$, and suppose that $H(z)$ defined by (16) satisfy one of the following :*

- (i) $\operatorname{Re} H(z) > 0$ ($z \in \mathbb{U}$) when $\operatorname{Re} \gamma > 0$,
- (ii) $H(z) \prec R_{\frac{\gamma}{n}+1}(z) - \frac{\gamma}{n}$ when $-n < \operatorname{Re} \gamma \leq 0$,

where $R_{\frac{\gamma}{n}+1}(z)$ is the open door function defined by (14). Then $p(z) \in \mathcal{H}[a, n]$ satisfies the following subordination relation :

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad \text{implies} \quad p(z) \prec a + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a)t^{\frac{\gamma}{n}-1} dt.$$

If we consider the function $h(z)$ given by

$$h(z) = 1 + z \in \mathcal{H}[1, 1],$$

then, it is easy to see that $h(z)$ satisfies all assumptions in Theorem 5.1. Also, it follows that

$$q(z) = 1 + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - 1)t^{\frac{\gamma}{n}-1} dt = 1 + \frac{\gamma}{n + \gamma} z \quad (\operatorname{Re} \gamma > -n).$$

Hence by Theorem 5.1, we find the following corollary.

Corollary 5.1. *n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and $\operatorname{Re} \gamma > -n$. Then $p(z) \in \mathcal{H}[1, n]$ satisfies the following subordination relation :*

$$p(z) + \frac{zp'(z)}{\gamma} \prec 1 + z \quad \text{implies} \quad p(z) \prec 1 + \frac{\gamma}{n + \gamma} z. \quad (18)$$

Example 5.1. Let us consider the function $p(z)$ given by

$$p(z) = 1 + \frac{i}{2} z^2 \in \mathcal{H}[1, 1] \quad (19)$$

in Corollary 5.2 with $n = 1$ and $\gamma = -\frac{1}{2}(1 - i)$. A simple calculation gives that

$$p(z) + \frac{zp'(z)}{-\frac{1}{2}(1 - i)} = 1 + \frac{1}{2} z^2 \quad (z \in \mathbb{U}).$$

Then, we see that $p(z)$ defined by (19) satisfies the subordination relation (18) with $n = 1$ and $\gamma = -\frac{1}{2}(1 - i)$.

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