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# SOME PROPERTIES OF CERTAIN SUBCLASSES OF MEROMORPHIC P-VALENT INTEGRAL OPERATORS

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Abstract. For meromorphic p-valent function of the form $f_i(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=2}^{\infty} a_n^i (z-w)^n,\ \alpha < 1$, which are analytic in the punctured unit disk $z : 0 < |z-w| < 1$ with a pole of order $p$ at $w$, a class $\Gamma_\beta^p(\zeta_1, \zeta_2; \gamma)$ is introduced and some properties for $\Gamma_\alpha^p(\zeta_1, \zeta_2; \gamma)$ of $f_i(z)$ in relatioon to the coefficient bounds, convex combination and convolution were discussed.

Keywords: Analytic,Coefficient bound, Convex combination, Meromorphic, p-valent, Integral operator.

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## 1. Introduction

Let $A$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic and normalized with $f(0) = f'(0) - 1 = 0$ in the open disk $U = \{z \in C : |z| < 1\}$. In [6], Seenivasagan gave a condition of the univalence of the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} \left( \frac{f_i(s)}{s} \right)^{1/\alpha} ds \right\}^{1/\beta}$$

where $f_i(z)$ is defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \qquad (1)$$

while Makinde in [5] gave a condition for the starlikeness for the function:

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{k} \left( \frac{f_i(s)}{s} \right)^{1/\alpha} ds, \alpha \in C \qquad (2)$$

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where $f_i(z)$ is defined by (1).
Also, Kanas and Ronning [2] introduced the class of function of the form

$$f(z) = (z-w) + \sum_{n=2}^{\infty} a_n^i (z-w)^n$$

where $w$ is a fixed point in the unit disk normalized with $f(w) = f'(w) - 1 = 0$.
We define $f_i(z)$ by

$$f_i(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=2}^{\infty} a_n^i (z-w)^n, \ \alpha < 1 \tag{3}$$

where $w$ is an arbitrary fixed point in the $D$, and $F_{w,\alpha}(z)$ is defined by

$$F_{w,\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s-w)}{s-w}\right)^{1/\alpha} ds, \alpha \in C \tag{4}$$

Furthermore, Xiao-Feili et al [7] denote $L_1^*(\beta_1, \beta_2, \lambda)$ as a subclass of $A$ such that:

$$L_1^*(\beta_1, \beta_2, \gamma) = \left\{ f \in A : \left| \frac{f'(z) - 1}{\beta_1 f'(z) + \beta_2} \leq \lambda \right| \right\}, \ 0 \leq \beta_1 \leq 1; \ 0 < \beta_1 \leq 1; \ 0 < \lambda \leq 1$$

for some $\beta_1, \ \beta_2$ and for some real $\lambda$. Also, he denoted $T$ to be the subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0$$

and $L^*(\beta_1, \beta_2, \lambda)$ denotes the subclass of $L_1^*(\beta_1, \beta_2, \lambda)$ defined by:

$$L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \bigcap T$$

for some real number, $0 \leq \beta_1 \leq 1; \ 0 < \beta_2 \leq 1; \ 0 < \lambda \leq 1$
The class $L^*(\beta_1, \beta_2, \lambda)$ was studied by Kim and Lee in [3], see also [1], [2], [7].
Let $F_\alpha(z)$ be defined by (2), then

$$\frac{zF''_\alpha(z)}{F'_\alpha(z)} = \sum_{i=1}^{k} \frac{1}{\alpha} \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right)$$

Let $G(z)$ be denoted by

$$G(z) = \sum_{i=1}^{k} \frac{1}{\alpha} \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right) \tag{5}$$

The class

$$\Gamma_\alpha(\zeta_1, \zeta_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\}$$

was studied by Makinde and Oladipo [sientia magna accepted]
We define

$$\Gamma_\alpha^p(\zeta_1, \zeta_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\} \tag{6}$$

for some complex $\zeta_1, \ \zeta_2, \ \alpha$ and for some real $\gamma, \ 0 \leq |\zeta_1| \leq 1, \ 0 < |\zeta_2| \leq 1, \ |\alpha| \leq 1$ and $0 < \gamma \leq 1$ with $G(z)$ as in (5) and $f_i(z)$ as in (3).
Let $f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n$ and $g_i(z) = z + \sum_{n=2}^{\infty} b_n^i z^n$, we define the convolution of $f_i(z)$ and $g_i(z)$ by

$$f_i(z) * g_i(z) = (f_i * g_i)(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n \tag{7}$$

We shall now present the main results of this paper.

## 2. Main Results

**Theorem 2.1.** *Let $f_i(z)$ be as in (3) and $F_{w,\alpha}$ be as in (4). Then $f_i(z)$ is in the class $\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$if and only if*

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]|a_n^i| \le \gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)|-|(1-\alpha)(-p-\alpha)|, \quad (8)$$

$0\le\zeta_1\le 1,\ 0<\zeta_2\le 1,\ 0<\alpha\le 1$

*Proof.* From (6), we have

$$\left|\frac{G(z)+\frac{1}{\alpha}-1}{\zeta_1(G(z)+\frac{1}{\alpha})+\zeta_2}\right| = \left|\frac{\frac{\sum_{i=1}^{k}\left(-p\frac{1-\alpha}{(z-w)^p}\right)+\sum_{n=2}^{\infty}na_n^i(z-w)^n}{\sum_{i=1}^{k}\alpha\left(\frac{1-\alpha}{(z-w)^p}+\sum_{n=2}^{\infty}na_n^i(z-w)^n\right)}-1}{\frac{\sum_{i=1}^{k}\zeta_1\left(-p\frac{1-\alpha}{(z-w)^p}\right)+\sum_{n=2}^{\infty}na_n^i(z-w)^n}{\sum_{i=1}^{k}\alpha\left(\frac{1-\alpha}{(z-w)^p}+\sum_{n=2}^{\infty}na_n^i(z-w)^n\right)}+\zeta_2}\right|$$
$$\le \frac{|(1-\alpha)(-p-\alpha)|+\sum_{i=1}^{k}\sum_{n=2}^{\infty}(n-\alpha)|a_n^i|}{|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)-\sum_{i=1}^{k}\sum_{n=2}^{\infty}(n+\alpha\zeta_2)|a_n^i||}$$

Let $f_i(z)$ satisfy the inequality (8), the $f_i(z)\in\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$. Conversely, let the function $f_i(z)\in\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$, then

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]|a_n^i| \le \gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)|-|(1-\alpha)(-p-\alpha)|$$

□

**Corollary 2.1.** *Let $f_i(z)\in\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$, then*

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}|a_n^i| \le \frac{\gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)|-|(1-\alpha)(-p-\alpha)|}{\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]}.$$

**Theorem 2.2.** *Let $f_i(z)\in\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$ and the function $g_i(z)$ defined by $g_i(z)=z+\sum_{n=2}^{\infty}b_n^iz^i$ be in the same $\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$. Then the function $\Omega_i(z)$ defined by*

$$\Omega_i(z)=(1-\lambda)f_i(z)+\lambda g_i(z)=z+\sum_{n=2}^{\infty}C_n^iz^i$$

*is also in the class $\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$, where*

$$C_n^i=(1-\lambda)a_n^i+\lambda b_n^i,\ 0\le\lambda\le 1.$$

*Proof.* Let $f_i(z),g_i(z)$ be in $\Gamma^p_\alpha(\zeta_1,\zeta_2,\gamma)$. Then by (8) and following the proof of Theorem 2.1, we have

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]|C_n^i| \le \gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)|-|(1-\alpha)(-p-\alpha)|$$

This shows that the convex combination $f_i(z), g_i(z)$ is in the class $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$
This concludes the proof of the Theorem 2.1.

□

**Theorem 2.3.** *Let $f_i(z)$ belong to the class $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$ and $g_i(z)$ belong to the class $\Gamma^p_\alpha(\beta_1, \beta_2, \gamma)$, then $(f_i * g_i)(z)$ belong to the class $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma) \subset \Gamma^p_\alpha(\beta_1, \beta_2, \gamma)$.*

*Proof.* $f_i(z)$ belong to the class $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$ implies

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]|a_n^i| \le \gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)| - |(1-\alpha)(-p-\alpha)|,$$

$$0 \le \zeta_1 \le 1,\ 0 < \zeta_2 \le 1,\ 0 < \alpha \le 1$$

Similarly, $g_i(z)$ belong to the class $\Gamma^p_\alpha(\beta_1, \beta_2, \gamma)$ implies

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\beta_1)-\alpha(1+\gamma\beta_2)\big]|a_n^i| \le \gamma|(1-\alpha)(-p\beta_1+\alpha\beta_2)| - |(1-\alpha)(-p-\alpha)|,$$

$$0 \le \beta_1 \le 1,\ 0 < \beta_2 \le 1,\ 0 < \alpha \le 1$$

But

$$\begin{aligned}(f_i * g_i)(z) &= \sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\beta_1)-\alpha(1+\gamma\beta_2)\big]|a_n^i||b_n^i| \\ &\le \sum_{i=1}^{k}\sum_{n=2}^{\infty}\big[n(1-\gamma\zeta_1)-\alpha(1+\gamma\zeta_2)\big]|a_n^i| \\ &\le \gamma|(1-\alpha)(-p\zeta_1+\alpha\zeta_2)| - |(1-\alpha)(-p-\alpha)|\end{aligned}$$

which implies that

$$(f_i * g_i)(z) \in \Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma) \subset \Gamma^p_\alpha(\beta_1, \beta_2, \gamma).$$

□

**Theorem 2.4.** *Let $\Psi_i(z) \in \Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $\upsilon_i(z)$ defined by*

$$\upsilon_i(z) = z + \sum_{n=2}^{\infty} A_n^i B_n^i z^i$$

*be in the same $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$. Then the function $\Phi_i(z)$ defined by*

$$\Phi_i(z) = (1-\lambda)\Psi_i(z) + \lambda\upsilon_i(z) = z + \sum_{n=2}^{\infty} C_n^i z^i$$

*is also in the class $\Gamma^p_\alpha(\zeta_1, \zeta_2, \gamma)$, where*

$$C_n^i = (1-\lambda)a_n^i b_n^i + \lambda A_n^i B_n^i,\ 0 \le \lambda \le 1.$$

*Proof.* The proof is similar to that the Theorem 2.2. □

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