TWMS J. App. Eng. Math. V.3, N.2, 2013, pp. 133-141.

THE EXACT TRAVELING WAVE SOLUTIONS TO ONE INTEGRABLE KDV6 EQUATION

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Abstract. The traveling wave system of one integrable KdV6 equation is studied by using Cosgrove's method. Some exact explicit traveling wave solutions are obtained. The local dynamical behavior of some known equilibria are discussed.

Keywords: KdV6 equation, exact traveling wave solution, solitary wave solution.

AMS Subject Classification: 35G05, 35Q51, 35Q53.

1. Introduction

More recent years, the Painlevé analysis and other methods were used to test the complete integrability of the class of sixth-order nonlinear equations (KdV6) [1], given by

$$u_{xxxxxx} + au_x u_{xxxx} + bu_{xx}u_{xxx} + cu_x^2 u_{xx} + du_{tt} + eu_{xxxt} + fu_x u_{xt} + gu_t u_{xx} = 0 \qquad (1)$$

where a, b, c, d, e, f and g are arbitrary parameters. In [1, 2], it was indicated that there are four distinct cases of relations between the parameters for (1) to pass the Painlevé test. Three of them were previously well-known, and summarized in [1]. But the fourth was a new integrable equation, Kupershmidt [3] studied the Hamiltonian structures and conservation laws for the KdV6 equation, and Yao and Zeng [4] showed that the KdV6 equation (2) below is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources. Some forms of KdV6 equation (1) have been studied by using various methods, such as Bäcklund transformation [5], the Hirota's bilinear methods [6], the Lax pair methods [1], etc., to obtain soliton and multi-soliton solutions.

By using the Cole-Hopf transformation method combined with the Hirota's bilinear method, Wazwaz derived multiple regular soliton solutions and multiple singular soliton

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§ Manuscript received August 13, 2013.

solutions in [7] for the following three distinct integrable cases of equation (1):

$$(i)\ a=20, b=40, c=120, d=0, e=\frac{1}{12}(f+g)=1, f=8, g=4, f=2g,$$

$$u_{xxxxxxx}+20u_x u_{xxxxx}+40u_{xx}u_{xxxx}+120u_x^2 u_{xxx}+u_{xxxt}+8u_x u_{xt}+4u_t u_{xx}$$
$$=(\partial_x^3+8u_x\partial_x+4u_{xx})(u_t+u_{xxx}+6u_x^2)=0; \quad (2)$$

$$(ii)\ a=18, b=36, c=72, f=g=0, e=1, d=-2e^2+\frac{1}{2}ef-\frac{1}{36}f^2=-2,$$

$$u_{xxxxxxx}+18u_x u_{xxxxx}+36u_{xx}u_{xxxx}+72u_x^2 u_{xxx}-2u_{tt}+u_{xxxt}=0; \quad (3)$$

$$(iii)\ a=30, b=30, c=180, f=g=6, d=-\frac{1}{180}g^2=-\frac{1}{5}, e=\frac{1}{12}(f+g)=1,$$

$$u_{xxxxxxx}+30u_x u_{xxxxx}+30u_{xx}u_{xxxx}+180u_x^2 u_{xxx}-\frac{1}{5}u_{tt}+u_{xxxt}+6u_x u_{xt}+6u_t u_{xx}=0. \quad (4)$$

Gómez and Salas studied some exact solutions to (2) by using the Cole-Hopf transformation and one improved tanh-coth method in [8]. Further, Li and Zhang [9] investigated the exact traveling wave solutions to the integrable equations (2) and (3) from the point of view of geometric theory of the dynamical systems. They proved that these traveling wave solutions correspond to some orbits in the 4-dimensional phase space of two level sets defined by two first integrals.

In this paper, we will discuss the dynamical behavior of traveling wave solutions of the following case of (1):

$$(iv)\ a=15, b=\frac{75}{2}, c=45, d=32, e=\frac{1}{6}(f+g)=4\sqrt{15}, f=g=12\sqrt{15},$$

$$u_{xxxxxxx}+15u_x u_{xxxxx}+\frac{75}{2}u_{xx}u_{xxxx}+45u_x^2 u_{xxx}+32u_{tt}+4\sqrt{15}u_{xxxt}$$
$$+12\sqrt{15}u_x u_{xt}+12\sqrt{15}u_t u_{xx}=0, \quad (5)$$

which describes the propagation of waves in two opposite directions and represents bidirectional versions of the Kaup-Kupershmidt equation [10]. The exact traveling wave solutions to equation (5) will be investigated by using the method of dynamical systems and Cosgrove's work [11]. To study the traveling wave solutions of (5), setting $\xi=x-vt$ and $u(x,t)=u(x-vt)=u(\xi)$, where v is the wave speed. Under this transformation, equation (5) can be changed into the following:

$$u''''''+15u'u''''+\frac{75}{2}u''u'''+45(u')^2u''+32v^2u''-4\sqrt{15}vu''''-24\sqrt{15}vu'u''=0.$$

Integrating it with respect to ξ once and setting $\phi=u'$, we have

$$\phi''''+15\phi\phi''+\frac{45}{4}(\phi')^2-4\sqrt{15}v\phi''+15\phi^3-12\sqrt{15}v\phi^2+32v^2\phi+\beta=0 \quad (6)$$

where β is an integral constant. By making the transformation

$$y=-(\phi-\frac{4\sqrt{15}}{15}v)$$

(6) becomes the following fourth-order ordinary differential equation

$$y''''=15yy''+\frac{45}{4}(y')^2-15y^3+16v^2y+\beta \quad (7)$$

which is exactly the F-III form of Cosgrove's higher-order Painlevé equations for $\alpha=16v^2$.

Thus, if we know the parametric representations of $y(\xi)$ for the above equation, then we will obtain the exact traveling wave solutions

$$u(x,t) = u(\xi) = \int (\frac{4\sqrt{15}}{15}v - y(\xi))d\xi$$

to (5).

Following the ideas in Li and Zhang [9] and Cosgrove [11], we will investigate the exact explicit traveling wave solutions to (7) in the next two sections for the cases (i). $\beta = 0$ and (ii). $\beta = \frac{128}{9}v^3$, respectively. We will show that for the equation (7), their traveling wave solutions correspond to some orbits in a 4-dimensional phase space of a 4-dimensional dynamical system. These traveling wave solutions are new and possess complex form.

2. The Exact Traveling Wave Solutions For Case (i). $\beta = 0$ And Their Geometric Property

Let

$$x_1 = y, x_2 = y', x_3 = y'', x_4 = y''',$$

then equation (7) for $\alpha = 16v^2,\ \beta = 0$ is equivalent to the four-dimensional system

$$\begin{aligned} &x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4, \\ &x_4' = 15x_1x_3 + \frac{45}{4}(x_2)^2 - 15(x_1)^3 + \alpha x_1. \end{aligned} \tag{8}$$

In the following, we will investigate the local dynamical behavior of (8) in the $(x_1, x_2, x_3, x_4)-$ phase space, we first discuss the number and position of the equilibria of (8) in the phase space. For a known equilibrium, we compute the eigenvalues of the coefficient matrix of the linearized system of (8) at this equilibrium point in order to understand its local dynamical behavior. For system (8), it has three simple equilibrium points $E_1(0,0,0,0), E_2(-\frac{4}{\sqrt{15}}v,0,0,0), E_3(\frac{4}{\sqrt{15}}v,0,0,0)$.

Let $M_j(x_{1j},0,0,0)$ be the coefficient matrix of the linearized system of (8) at an equilibrium point $E_j(x_{1j},0,0,0)(j=1,2,3)$. Then we have

$$M_j(x_{1j},0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha - 45x_{1j}^2 & 0 & 15x_{1j} & 0 \end{pmatrix}.$$

The eigenvalues of $M_1(0,0,0,0)$ are one purely imaginary pair and one real pair, $\pm 2\sqrt{v}i,\ \pm 2\sqrt{v}$, the equilibrium point E_1 is a center-saddle.

The eigenvalues of $M_2(-\frac{4}{\sqrt{15}}v,0,0,0)$ are two purely imaginary pairs ,

$$\pm\sqrt{2\sqrt{15}+2\sqrt{7}}\sqrt{v}i,\ \pm\sqrt{2\sqrt{15}-2\sqrt{7}}\sqrt{v}i,$$

the equilibrium point E_2 is a center-center.

The eigenvalues of $M_3(\frac{4}{\sqrt{15}}v,0,0,0)$ are two real pairs ,

$$\pm\sqrt{2\sqrt{15}+2\sqrt{7}}\sqrt{v},\ \pm\sqrt{2\sqrt{15}-2\sqrt{7}}\sqrt{v},$$

the equilibrium point E_3 is a saddle-saddle.

System (8) has two first integrals (see [11])

$$\begin{aligned} \Phi_1(x_1,x_2,x_3,x_4) = {} & (x_4 - 12x_1x_2)^2 - 3x_1x_3^2 + (\frac{3}{2}x_2^2 + 30x_1^3)x_3 - 9x_1^2x_2^2 \\ & - 72x_1^5 - \alpha(2x_1x_3 - x_2^2 - 8x_1^3), \end{aligned}$$

$$\Phi_2(x_1,x_2,x_3,x_4) = x_1x_4^2 - (x_3+18x_1^2)x_2x_4 + \frac{1}{3}x_3^3 - 6x_1^2x_3^2 + (\frac{27}{2}x_1x_2^2 + 30x_1^4)x_3 - \frac{9}{16}x_2^4 + \frac{135}{2}x_1^3x_2^2 - 45x_1^6 - \alpha(\frac{2}{3}x_2x_4 - \frac{1}{3}x_3^2 + 2x_1^2x_3 - \frac{15}{2}x_1x_2^2 - 2x_1^4) + \frac{1}{3}\alpha^2x_1^2 - \frac{4}{81}\alpha^3.$$

For given two constants K_1 and K_2, the two level sets defined by

$$\Phi_1(x_1,x_2,x_3,x_4) = K_1$$

and

$$\Phi_2(x_1,x_2,x_3,x_4) = K_2$$

determine two families of the three-dimensional invariant manifolds of system (8). Especially, we see that

$$K_1 = \Phi_1(0,0,0,0) = 0, \quad K_2 = \Phi_2(0,0,0,0) = -(\frac{128}{9}v^3)^2.$$

Thus the two level sets defined by $\Phi_1(x_1,x_2,x_3,x_4) = 0$ and $\Phi_2(x_1,x_2,x_3,x_4) = -(\frac{128}{9}v^3)^2$ pass through the equilibrium point E_1. Hence by [11] we know that the equation (7) admits solutions as the following form:

$$y(\xi) = \frac{(U'-V')^2}{(U-V)^2} - 4(U+V) \tag{9}$$

where $U(\xi)$ and $V(\xi)$ are the Weierstrass elliptic functions defined by the differential equations

$$(U')^2 = 4U^3 - \frac{1}{12}\alpha U + \frac{1}{48}K_{12},$$

$$(V')^2 = 4V^3 - \frac{1}{12}\alpha V - \frac{1}{48}K_{12}$$

where $K_{12} = \sqrt{-K_2}$. For our case we have

$$\begin{aligned}(U')^2 &= 4U^3 - \frac{4}{3}v^2U + \frac{8}{27}v^3 = 4(U+\frac{2}{3}v)(U-\frac{1}{3}v)^2,\\ (V')^2 &= 4V^3 - \frac{4}{3}v^2V - \frac{8}{27}v^3 = 4(V-\frac{2}{3}v)(V+\frac{1}{3}v)^2.\end{aligned} \tag{10}$$

In the $(U,\dot{U})-$phase plane and $(V,\dot{V})-$phase plane, the two equations defined by (10) determine two cubic algebraic curves which are shown in Figs. $1(a)$ and $1(b)$, respectively. Clearly, the first equation of (10) gives rise to a homoclinic orbit to the equilibrium point $(\frac{1}{3}v, 0)$ and an open orbit, while the second equation gives rise to an open curve passing through the point $(\frac{2}{3}v, 0)$ and the point $(-\frac{1}{3}v, 0)$ (see figure 1).

Thus, by using (10) to do integration, we obtain the following results.
(1) Corresponding to the homoclinic orbit in Fig.1(a), we have the parametric representation

$$U_1(\xi) = -\frac{2}{3}v + v\tanh^2(\sqrt{v}\xi).$$

(2) Corresponding to the open orbit in Fig.1(a), we have the parametric representation

$$U_2(\xi) = -\frac{2}{3}v + v\coth^2(\sqrt{v}\xi).$$

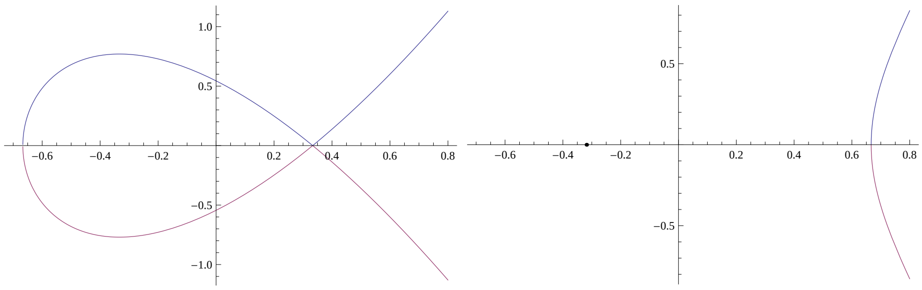

FIGURE 1. The phase curves defined by (10).

(3) Corresponding to the equilibrium point $(\frac{1}{3}v, 0)$ in Fig.1(a), we have the parametric representation

$$U_3(\xi) = \frac{1}{3}v.$$

(4) Corresponding to the equilibrium point $(-\frac{1}{3}v, 0)$ in Fig.1(b), we have the parametric representation

$$V_1(\xi) = -\frac{1}{3}v.$$

(5) Corresponding to the open orbit in Fig.1(b), we have the parametric representation

$$V_2(\xi) = \frac{2}{3}v + v\tan^2(\sqrt{v}\xi).$$

Therefore, as some intersection curves of two level manifolds

$$\Phi_1(x_1, x_2, x_3, x_4) = 0, \qquad \Phi_2(x_1, x_2, x_3, x_4) = -(\frac{128}{9}v^3)^2,$$

we obtain the exact explicit parametric representations of the nontrivial solutions to (7) as follows:

$$y_1 = x_1(\xi) = \frac{(U_1' - V_1')^2}{(U_1 - V_1)^2} - 4(U_1 + V_1) = -\frac{4v(\tanh^2(\sqrt{v}\xi) - 1)(3\tanh^2(\sqrt{v}\xi) + 1)}{(3\tanh^2(\sqrt{v}\xi) - 1)^2}, \tag{11}$$

$$y_2 = x_1(\xi) = \frac{(U_2' - V_1')^2}{(U_2 - V_1)^2} - 4(U_2 + V_1) = -\frac{4v(\coth^2(\sqrt{v}\xi) - 1)(3\coth^2(\sqrt{v}\xi) + 1)}{(3\coth^2(\sqrt{v}\xi) - 1)^2}, \tag{12}$$

$$\begin{aligned} y_3 = x_1(\xi) &= \frac{(U_1' - V_2')^2}{(U_1 - V_2)^2} - 4(U_1 + V_2) \\ &= \frac{36v(\tanh(\sqrt{v}\xi)(1 - \tanh^2(\sqrt{v}\xi)) - \tan(\sqrt{v}\xi)\sec^2(\sqrt{v}\xi))^2}{(3(\tanh^2(\sqrt{v}\xi) - \tan^2(\sqrt{v}\xi)) - 4)^2} \\ &\quad - 4v(\tanh^2(\sqrt{v}\xi) + \tan^2(\sqrt{v}\xi)), \end{aligned} \tag{13}$$

$$\begin{aligned} y_4 = x_1(\xi) &= \frac{(U_2' - V_2')^2}{(U_2 - V_2)^2} - 4(U_2 + V_2) \\ &= \frac{36v(\coth(p\xi)(\coth^2(p\xi) - 1) - \tan(\sqrt{v}\xi)\sec^2(\sqrt{v}\xi))^2}{(3(\coth^2(\sqrt{v}\xi) - \tan^2(\sqrt{v}\xi)) - 4)^2} \\ &\quad - 4v(\coth^2(\sqrt{v}\xi) + \tan^2(\sqrt{v}\xi)), \end{aligned} \tag{14}$$

$$y_5 = x_1(\xi) = \frac{(U_3' - V_2')^2}{(U_3 - V_2)^2} - 4(U_3 + V_2) = \frac{4v(3 - 4\cos^2(p\xi))}{(3 - 2\cos^2(\sqrt{v}\xi))^2}. \tag{15}$$

We see that $y_1(\xi)$ and $y_2(\xi)$ are homoclinic solutions of the equation (7) and therefore $u_1(\xi) = \int(\frac{4\sqrt{15}}{15}v - y_1(\xi))d\xi$ and $u_2(\xi) = \int(\frac{4\sqrt{15}}{15}v - y_2(\xi))d\xi$ must be soliton solutions of equation (5). Notice that function $y_1(\xi)$ is discontinuous, for $y_1(\xi) \to +\infty$ as $\xi \to \pm\frac{1}{2\sqrt{v}}\ln(2+\sqrt{3})$, hence $u_1(\xi)$ must be multiple singular soliton solution or multiple cuspon solution of (5) [7]. The two solutions $y_1(\xi)$ and $y_2(\xi)$ of (7) and their first, second and third-order derivatives give respectively rise to a homoclinic orbit to the center-saddle $E_1(0,0,0,0)$.

To sum up, we have proved the following result.

Theorem 2.1. *The traveling wave equation (7) of KdV6 equation (5) has the exact explicit solutions given by (11)-(15). Geometrically, the solution curves defined by* $(x_1(\xi) = y(\xi), x_2(\xi) = y'(\xi), x_3(\xi) = y''(\xi), x_4(\xi) = y'''(\xi))$ *all lie in the intersection of two level manifolds*

$$\Phi_1(x_1,x_2,x_3,x_4) = 0, \qquad \Phi_2(x_1,x_2,x_3,x_4) = -(\frac{128}{9}v^3)^2$$

of system (8).

3. The Exact Traveling Wave Solutions For Case (*ii*). $\beta = \frac{128}{9}v^3$ And Their Geometric Property

Corresponding to the case (2). : $\beta = \frac{128}{9}v^3$, we have the 4-dimensional system :

$$\begin{aligned} &x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4, \\ &x_4' = 15x_1x_3 + \frac{45}{4}(x_2)^2 - 15(x_1)^3 + \alpha x_1 + \beta \end{aligned} \tag{16}$$

which has a unique real equilibrium point $E_1(\frac{4}{3}v,0,0,0)$ of (16) for which the eigenvalues of the coefficient matrix of the linearized system of (16) are $\pm 2\sqrt{v}$ and $\pm 4\sqrt{v}$. Hence the equilibrium E_1 is hyperbolic, i. e., a saddle-saddle. In this case, system (16) has two first integrals given by (see [11])

$$\begin{aligned} \Phi_1(x_1,x_2,x_3,x_4) &= (x_4 - 12x_1x_2)^2 - 3x_1x_3^2 + (\frac{3}{2}x_2^2 + 30x_1^3)x_3 - 9x_1^2x_2^2 \\ &\quad - 72x_1^5 - \alpha(2x_1x_3 - x_2^2 - 8x_1^3) - 2\beta(x_3 - 6x_1^2) - \frac{4}{3}\alpha\beta, \\ \Phi_2(x_1,x_2,x_3,x_4) &= x_1x_4^2 - (x_3 + 18x_1^2)x_2x_4 + \frac{1}{3}x_3^3 - 6x_1^2x_3^2 + (\frac{27}{2}x_1x_2^2 + 30x_1^4)x_3 \\ &\quad - \frac{9}{16}x_2^4 + \frac{135}{2}x_1^3x_2^2 - 45x_1^6 - \alpha(\frac{2}{3}x_2x_4 - \frac{1}{3}x_3^2 + 2x_1^2x_3 - \frac{15}{2}x_1x_2^2 \\ &\quad - 2x_1^4) - \beta(2x_1x_3 - \frac{3}{2}x_2^2 - 6x_1^3) + \frac{1}{3}\alpha^2x_1^2 + \frac{2}{3}\alpha\beta x_1 - \frac{4}{81}\alpha^3 - \beta^2. \end{aligned}$$

For given two constants K_1 and K_2, the two level sets defined by

$$\Phi_1(x_1,x_2,x_3,x_4) = K_1$$

and

$$\Phi_2(x_1,x_2,x_3,x_4) = K_2$$

determine two families of the three-dimensional invariant manifolds of system (16). Especially, we see that

$$K_1 = \Phi_1(\frac{4}{3}v,0,0,0) = 0, \quad K_2 = \Phi_2(\frac{4}{3}v,0,0,0) = 0.$$

Thus the two level sets defined by $K_1 = \Phi_1(\frac{4}{3}v,0,0,0) = 0$ and $K_2 = \Phi_2(\frac{4}{3}v,0,0,0) = 0$ pass through the equilibrium point E_1. Their intersection lies in the homoclinic manifold of $E_1(\frac{4}{3}v, 0, 0, 0)$. Hence by [11] we know that the equation (7) admits solutions as the following form:

$$y(\xi) = \frac{(qU'U''+1)^2}{q^2(U')^4} - 8U \tag{17}$$

where $U(\xi)$ is the elliptic function defined by

$$(U')^2 = 4U^3 - \frac{1}{12}\alpha U + \frac{1}{48}\beta$$

and $q(\xi)$ is given by

$$\int (4U^3 - \frac{1}{12}\alpha U + \frac{1}{48}\beta)^{-1} d\xi + C$$

where C is an integral constant and the later integral in general can be evaluated in terms of the Weierstrass zeta function. Now we have

$$(U')^2 = 4U^3 - \frac{4}{3}v^2 U + \frac{8}{27}v^3 = 4(U + \frac{2}{3}v)(U - \frac{1}{3}v)^2.$$

This equation has two nontrivial solutions as follows:

$$U_1(\xi) = -\frac{2}{3}v + v\tanh^2(\sqrt{v}\xi),$$

$$U_2(\xi) = -\frac{2}{3}v + v\coth^2(\sqrt{v}\xi).$$

Hence

$$U_1' = 2(\sqrt{v})^3 \tanh(\sqrt{v}\xi)\cosh^{-2}(\sqrt{v}\xi), \quad U_1'' = 2v^2(3\cosh^{-4}(\sqrt{v}\xi) - 2\cosh^{-2}(\sqrt{v}\xi)),$$

and

$$U_2' = -2(\sqrt{v})^3 \coth(\sqrt{v}\xi)\sinh^{-2}(\sqrt{v}\xi), \quad U_2'' = 2v^2(3\sinh^{-4}(\sqrt{v}\xi) + 2\sinh^{-2}(\sqrt{v}\xi)).$$

The corresponding q can be easily evaluated by hyperbolic integral as follows:

$$q_1 = \int \frac{1}{(U_1')^2} d\xi + C_1 = \frac{1}{4(\sqrt{v})^7} \int [\frac{1}{\sinh^2(\sqrt{v}\xi)} + 3 + 3\sinh^2(\sqrt{v}\xi) + \sinh^4(\sqrt{v}\xi)] d(\sqrt{v}\xi) + C_1$$
$$= \frac{1}{4(\sqrt{v})^7}[-\coth(\sqrt{v}\xi) + \frac{15}{8}(\sqrt{v}\xi) + \frac{9}{8}\sinh(\sqrt{v}\xi)\cosh(\sqrt{v}\xi) + \frac{1}{4}\sinh^3(\sqrt{v}\xi)\cosh(\sqrt{v}\xi) + C_1]$$

and

$$q_2 = \int \frac{1}{(U_2')^2} d\xi + C_2 = \frac{1}{4(\sqrt{v})^7} \int [\frac{-1}{\cosh^2(\sqrt{v}\xi)} + 3 - 3\cosh^2(\sqrt{v}\xi) + \cosh^4(\sqrt{v}\xi)] d(\sqrt{v}\xi) + C_2$$
$$= \frac{1}{4(\sqrt{v})^7}[-\tanh(\sqrt{v}\xi) + \frac{15}{8}(\sqrt{v}\xi) - \frac{9}{8}\sinh(\sqrt{v}\xi)\cosh(\sqrt{v}\xi) + \frac{1}{4}\sinh(\sqrt{v}\xi)\cosh^3(\sqrt{v}\xi) + C_2].$$

Hence by (17) we get that system (16) has the following two classes of exact solutions:

$$y_1(\xi) = \frac{(q_1U_1'U_1''+1)^2}{q_1^2(U_1')^4} - 8U_1, \tag{18}$$

$$y_2(\xi) = \frac{(q_2U_2'U_2''+1)^2}{q_2^2(U_2')^4} - 8U_2. \tag{19}$$

Letting $\xi \to \pm\infty$, noticing that $\cosh(\sqrt{v}\xi) \to +\infty$, and that $\sinh(\sqrt{v}\xi) \to \pm\infty$, we see that

$$\begin{aligned} q_1 U_1' U_1'' &= \frac{4(\sqrt{v})^7 \tanh(\sqrt{v}\xi)(3 - 2\cosh^2(\sqrt{v}\xi))}{\cosh^6(\sqrt{v}\xi)} q_1 \\ &= \tanh(\sqrt{v}\xi)(\frac{3}{\cosh^2(\sqrt{v}\xi)} - 2)[-\frac{\coth(\sqrt{v}\xi)}{\cosh^4(\sqrt{v}\xi)} + \frac{15x}{8\cosh^4(\sqrt{v}\xi)} + \frac{9\sinh(\sqrt{v}\xi)}{8\cosh^3(\sqrt{v}\xi)} \\ &\quad + \frac{\sinh^3(\sqrt{v}\xi)}{4\cosh^3(\sqrt{v}\xi)} + \frac{C_1}{\cosh^4(\sqrt{v}\xi)}] \to -\frac{1}{2}, \end{aligned}$$

and hence the numerator in the first term of $y_1(\xi)$ tends to $(\frac{1}{2})^2 = \frac{1}{4}$, while the denominator in the first term of $y_1(\xi)$ is

$$\begin{aligned} q_1^2 (U_1')^4 &= \frac{16v^6 \tanh^4(\sqrt{v}\xi)}{\cosh^8(\sqrt{v}\xi)} q_1 \\ &= \frac{16v^6 \tanh^4(\sqrt{v}\xi)}{16v^7}[-\frac{\coth(\sqrt{v}\xi)}{\cosh^4(\sqrt{v}\xi)} + \frac{15x}{8\cosh^4(\sqrt{v}\xi)} + \frac{9\sinh(\sqrt{v}\xi)}{8\cosh^3(\sqrt{v}\xi)} \\ &\quad + \frac{\sinh^3(\sqrt{v}\xi)}{4\cosh^3(\sqrt{v}\xi)} + \frac{C_1}{\cosh^4(\sqrt{v}\xi)}]^2 \end{aligned}$$

which tends to $\frac{1}{16v}$. Therefore $y_1(\xi) \to 4v - \frac{8}{3}v = \frac{4}{3}v$. For a similar calculation, we also have that $y_2(\xi) \to \frac{4}{3}v$.

Thus, the two solution families $y_1(\xi)$ and $y_2(\xi)$ of (6) given by (18), (19) respectively depending on one integral constant C_1 and, C_2, give rise correspondingly to uncountably infinite many solitary wave solutions $u_1(\xi) = \int(\frac{4\sqrt{15}}{15}v - y_1(\xi))d\xi$ and $u_2(\xi) = \int(\frac{4\sqrt{15}}{15}v - y_2(\xi))d\xi$ of equation (5). These two solution families $y_1(\xi)$ and $y_2(\xi)$ of equation (7) are all lying on a two-dimensional homoclinic manifold of equilibrium E_1.

To sum up, we have the following conclusion.

Theorem 3.1. *For any $C_1, C_2 \in R$, the traveling wave system (7) for $\beta = \frac{126}{9}v^3$ of KdV6 equation (5) has uncountably infinite many solutions given by (18), (19). Geometrically, in the four-dimensional phase space, these two solution curvilinear families defined by $(x_{11}(\xi) = y_1(\xi), x_{12}(\xi) = y_1'(\xi), x_{13}(\xi) = y_1''(\xi), x_{14}(\xi) = y_1'''(\xi))$ and $(x_{21}(\xi) = y_2(\xi), x_{22}(\xi) = y_2'(\xi), x_{23}(\xi) = y_2''(\xi), x_{24}(\xi) = y_2'''(\xi))$ all lie in the intersection of two level manifolds*

$$\Phi_1(x_1, x_2, x_3, x_4) = 0, \qquad \Phi_2(x_1, x_2, x_3, x_4) = 0$$

of system (16).

4. Conclusion

In this paper, we have obtained many exact explicit wave solutions for the KdV6 equation (5) by employing the Cosgrove's method. Some exact explicit traveling wave solutions are obtained. The local dynamical behavior of some known equilibria are discussed. The method can be further applied for equilibria of other integrable KdV6 equations to give traveling wave solutions. We believe that this method can be also applied to some other evolution equations and we will investigate this problem further.

Acknowledgement The authors would like to extend their gratitude to the referee of this paper for careful reading and valuable comments which led us to improve the paper.

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