

## SPECTRAL ANALYSIS OF ELASTIC WAVEGUIDES

Mahir HASANSOY \*

### ABSTRACT

This paper deals with the spectral analysis of elastic waveguides shaped like an infinite cylinder with a bounded or unbounded cross section. Note that waveguides with bounded cross sections were studied in the framework of the operator and operator pencil theory. In this paper we extend this theory on elastic waveguides with a unbounded cross section. The problem reduces to the spectral theory of a one-parameter family of unbounded operators. The spectral structure, the asymptotics of the eigenvalues, comparisons between the solutions of different problems and the existence of special guided modes for these operators are the main questions that we study in this paper. We use an operator approach to solve these problems, and on the basis of this approach, we suggest alternative methods to solve spectral problems arising in the theory of both closed elastic waveguides and elastic waveguides with unbounded cross sections.

**Keywords:** *Eigenvalue, spectrum, operator, wave solution, elastic waveguide, perturbation.*

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\*Beykent University, Faculty of Arts and Sciences, Department of Mathematics, Ayazağa Mahallesi, Hadım Koryolu  
Cd. No:19, Sariyer İstanbul, Turkey (mahirhasansoy@beykent.edu.tr) (ORCID ID: 0000-0002-9080-4242)  
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## ELASTİK DALGA KLAVUZLARININ SPEKTRAL ANALİZİ

Mahir HASANSOY \*

### ÖZ

Bu makalede, sınırlı veya sınırsız kesite sahip sonsuz bir silindir biçiminde şekillendirilmiş elastik dalga kılavuzlarının spektral analizi ele alınmıştır. Sınırlı kesitli dalga kılavuzları operatör ve operatör demet teorisi çerçevesinde çalışılmıştır. Bu çalışmada, bu teori sınırsız kesite sahip elastik dalga kılavuzları üzerine genişletilmiştir. Problem, sınırsız operatörlerden oluşan tek parametrelili bir operatör sınıfının spektral teorisine indirgenmiştir. Spektral yapı, özdeğerlerin asimptotik davranışları, farklı problemlerin çözümleri arasındaki karşılaştırmalar ve bu operatörler için özel dalga çözümünün varlığı burada çalışılan ana konulardır. Bu problemleri çözmek için bir operatör yaklaşımı kullanılmış ve bu yaklaşıma dayanılarak, hem kapalı elastik dalga kılavuzlarının hem de sınırsız kesitli elastik dalga kılavuzlarının teorisinde ortaya çıkan spektral problemleri çözmek için alternatif yöntemler önerilmiştir.

**Anahtar Kelimeler:** Özdeğer, spektrum, operatör, dalga çözümü, elastik dalga kılavuzu, pertürbasyon.

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\*Beykent University, Faculty of Arts and Sciences, Department of Mathematics, Ayazağa Mahallesi, Hadım Koruyolu  
Cd. No:19, Sariyer İstanbul, Turkey (mahirhasansoy@beykent.edu.tr) (ORCID ID: 0000-0002-9080-4242)  
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**1 Introduction and Preliminary Facts**

The main subject of this paper is the wave solutions of the elastodynamic equation in the following form, and the related spectral problems (here, we follow the notation of [6]):

$$\begin{cases} \operatorname{div} \sigma(U) = \tilde{\rho} \partial^2 U / \partial t^2 & \text{in } \tilde{\Omega}, \\ \sigma(U) \tilde{\nu} = 0 & \text{on } \partial \tilde{\Omega}, \end{cases} \quad (1.1)$$

where  $t \in \mathbb{R}$ ,  $\tilde{\nu}$  is the normal vector,

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

$U := U(x_1, x_2, x_3, t)$  and  $(x_1, x_2, x_3) \in \tilde{\Omega} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x = (x_1, x_2) \in \Omega\}$ . Throughout this paper,  $\Omega = \Omega_0 \cup \mathcal{O}$ , where  $\mathcal{O}$  is a bounded connected open set with a smooth boundary such that  $\mathcal{O} \subset \{x \in \mathbb{R}^2 : x_2 > 0\}$  and  $\Omega_0 = \{x \in \mathbb{R}^2 : x_2 < 0\}$  (see Figure 1).  $\sigma(U)$  is the stress tensor, defined by

$$\sigma_{ij}(U) = \tilde{\lambda} \operatorname{div}(U) \delta_{ij} + \tilde{\mu}(U_{i,j} + U_{j,i}),$$

where  $\delta_{ij}$  denotes the Kronecker symbol and  $U_{i,j} = \partial U_i / \partial x_j$  (see [6, 8]). In what follows we will consider unbounded waveguides described in the following figure.

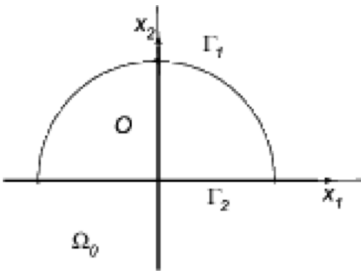


Fig 1. A bounded deformation of the half space

We are looking for wave solutions to (1.1) in the form

$$U = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ \frac{u_3(x_1, x_2)}{i} \end{pmatrix} e^{i(kx_3 - wt)},$$

$k \in \mathbb{C}, w \in \mathbb{C}$  and  $u \in H^1(\Omega)^3$ , which denotes the Sobolev space of three-dimensional vector fields (see [1]). We are particularly interested in solutions with real and positive  $k$  and  $w$ . By inserting this particular solution into (1.1) and following the notation of [6], we obtain the following system of equations in vector form:

$$\begin{pmatrix} \lambda(\operatorname{div}_k u) + 2\mu u_{1,1} + 2\mu \frac{(u_{1,2} + u_{2,1})}{2} + k2\mu \frac{(u_{3,1} - ku_1)}{2} \\ 2\mu \frac{(u_{2,1} + u_{1,2})}{2} + (\lambda(\operatorname{div}_k u) + 2\mu u_{2,2}) + k2\mu \frac{(u_{3,2} - ku_2)}{2} \\ 2\mu \frac{(u_{3,1} - ku_1)}{2} + 2\mu \frac{(u_{3,2} - ku_2)}{2} - k(\lambda(\operatorname{div}_k u) + 2\mu ku_3) \end{pmatrix} = -w^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (1.2)$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$\operatorname{div}_k u = u_{1,1} + u_{2,2} + ku_3$ ,  $\lambda = \tilde{\lambda} / \tilde{\rho}$  and  $\mu = \tilde{\mu} / \tilde{\rho}$ . This system allows us to rewrite (1.1) in

the form

$$\begin{cases} -\operatorname{div}_k^* \sigma^k(u) = w^2 u & \text{in } \Omega, \\ \sigma^k(u) \tilde{\nu} = 0 & \text{on } \partial \Omega, \end{cases} \quad (1.3)$$

where

$$\sigma^k(u) = \begin{pmatrix} \lambda(\operatorname{div}_k u) + 2\mu u_{1,1} & 2\mu \frac{(u_{1,2} + u_{2,1})}{2} & 2\mu \frac{(u_{3,1} - ku_1)}{2} \\ 2\mu \frac{(u_{2,1} + u_{1,2})}{2} & \lambda(\operatorname{div}_k u) + 2\mu u_{2,2} & 2\mu \frac{(u_{3,2} - ku_2)}{2} \\ 2\mu \frac{(u_{3,1} - ku_1)}{2} & 2\mu \frac{(u_{3,2} - ku_2)}{2} & \lambda(\operatorname{div}_k u) + 2\mu ku_3 \end{pmatrix},$$

$$\tilde{\nu} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ 0 \end{pmatrix}$$

is the normal vector,  $\operatorname{div}_k^* \sigma^k(u)$  is equal to the left-hand side of (1.2), and the asterisk indicates a minus sign in front of  $\lambda(\operatorname{div}_k u) + 2\mu ku_3$  in the third line of (1.2). The entries of the matrix  $\sigma^k(u)$  can be written in the following form:

$$\sigma_{ij}^k(u) = \lambda(\operatorname{div}_k u) \delta_{ij} + 2\mu \varepsilon_{ij}^k(u),$$

where  $\varepsilon_{ij}^k(u)$  are the entries of the matrix

$$\varepsilon^k(u) = \begin{pmatrix} u_{1,1} & \frac{u_{1,2} + u_{2,1}}{2} & \frac{u_{3,1} - ku_1}{2} \\ \frac{u_{2,1} + u_{1,2}}{2} & u_{2,2} & \frac{u_{3,2} - ku_2}{2} \\ \frac{u_{3,1} - ku_1}{2} & \frac{u_{3,2} - ku_2}{2} & ku_3 \end{pmatrix}.$$

Note that problem (1.3) is a two-parameter eigenvalue problem. Throughout this paper, we fix  $k$  and study the eigenvalue problem (1.3) with respect to  $w^2$ . The main question studied in this paper is this: for what values of the parameter  $w^2$  do there exist nontrivial solutions to problem (1.3) for a fixed  $k \in \mathbb{R}$ ?

In addition to problem (1.3), the following Dirichlet and Neumann-type boundary value problems will be considered as auxiliary problems:

$$\begin{cases} -\operatorname{div}_k^* \sigma^k(u) = w^2 u & \text{in } \mathcal{O}, \\ u|_{\partial \mathcal{O}} = 0, \end{cases} \quad (1.4)$$

and

$$\begin{cases} -\operatorname{div}_k^* \sigma^k(u) = w^2 u & \text{in } \mathcal{O}, \\ \sigma^k(u) \tilde{\nu} = 0 & \text{on } \partial \mathcal{O} \text{ (free boundary)}. \end{cases} \quad (1.5)$$

These problems are useful for studying problem (1.3) via comparison. The solutions to problems (1.3), (1.4) and (1.5) have to be understood in the weak sense. Below, we give exact definitions of these solutions.

In what follows, the main space for all the problems that we study will be  $L^2(\Omega)^3$ , and  $(\cdot, \cdot)$  denotes the scalar product in this space. We shall also use energetic spaces, which are specifically defined for given problems. For example, the energetic spaces for problems (1.3), (1.4) and (1.5) are  $H^1(\Omega)^3$ ,  $H_0^1(\mathcal{O})^3$  and  $H_0^1(\mathcal{O})^3$  respectively.

Let us first consider problem (1.3). By applying Green's formula to (1.3) and using the boundary condition  $\sigma^k(u)\bar{v} = 0$ ,  $x \in \partial\Omega$ , we obtain

$$\begin{aligned} & -(\operatorname{div}_k^* \sigma^k(u), v) = \\ & = \int_{\Omega} [\lambda(\operatorname{div}_k u) + 2\mu u_{1,1} \bar{v}_{1,1} dx + 2\mu \int_{\Omega} \frac{u_{1,2} + u_{2,1}}{2} \bar{v}_{1,2} dx - 2\mu \int_{\Omega} \frac{u_{3,1} - k u_1}{2} k \bar{v}_1 dx \\ & + 2\mu \int_{\Omega} \frac{u_{2,1} + u_{1,2}}{2} \bar{v}_{2,1} dx + \int_{\Omega} [\lambda(\operatorname{div}_k u) + 2\mu u_{2,2} \bar{v}_{2,2} dx - 2\mu \int_{\Omega} \frac{u_{3,2} - k u_2}{2} k \bar{v}_2 dx \\ & + 2\mu \int_{\Omega} \frac{u_{3,1} - k u_1}{2} \bar{v}_{3,1} dx + 2\mu \int_{\Omega} \frac{u_{3,2} - k u_2}{2} \bar{v}_{3,2} dx + \int_{\Omega} [\lambda(\operatorname{div}_k u) + 2\mu k u_3 k \bar{v}_3 dx \\ & - \int_{\partial\Omega} \sigma^k(u) \bar{v} \cdot \bar{v} dx = \lambda \int_{\Omega} \operatorname{div}_k u \operatorname{div}_k \bar{v} dx + 2\mu \int_{\Omega} \varepsilon_{ij}^k(u) \varepsilon_{ij}^k(\bar{v}) dx. \end{aligned}$$

This formula allows us to define the Friedrichs extension of the symmetric (for real  $k$ ) operator  $-\operatorname{div}_k^* \sigma^k(u)$  from the initial domain  $\mathcal{D} = \{u | u \in C^2(\Omega)^3, \sigma^k(u)\bar{v} = 0, x \in \partial\Omega\}$  to  $H^1(\Omega)^3$  in the following way: the Friedrichs extension of the operator given in problem (1.3) is the operator (more precisely, the operator pencil)  $L(\Omega; k)$  defined on  $H_0^1(\Omega)^3 \subset L_2(\Omega)^3$  and associated with the bilinear form

$$(L(\Omega; k)u, v) := \lambda \int_{\Omega} \operatorname{div}_k u \operatorname{div}_k \bar{v} dx + 2\mu \int_{\Omega} \varepsilon_{ij}^k(u) \varepsilon_{ij}^k(\bar{v}) dx, \tag{1.6}$$

where  $u, v \in H^1(\Omega)^3$ . In what follows, we shall use the notation  $L(k) := L(\Omega; k)$ . We note that  $L^*(k) = L(\bar{k})$ , i.e.,  $L(k)$  is a self - adjoint operator pencil (see [19]). In particular, it is self-adjoint when considered as an operator for each fixed real  $k$ .

We can define the Friedrichs extension of the operator given in problem (1.4) in the same way. This extension is the operator (or operator pencil)

$L_0(\mathcal{O}; k)$  defined on  $H_0^1(\mathcal{O})^3 \subset L_2(\Omega)^3$  and associated with the bilinear form

$$(L_0(\mathcal{O}; k)u, v) := \lambda \int_{\mathcal{O}} \operatorname{div}_k u \operatorname{div}_k \bar{v} dx + 2\mu \int_{\mathcal{O}} \varepsilon_{ij}^k(u) \varepsilon_{ij}^k(\bar{v}) dx, \tag{1.7}$$

where  $u, v \in H_0^1(\mathcal{O})^3$ .

Finally, the operator  $L(\mathcal{O}; k)$  associated with problem (1.5) is defined by the bilinear form

$$(L(\mathcal{O}; k)u, v) := \lambda \int_{\mathcal{O}} \operatorname{div}_k u \operatorname{div}_k \bar{v} dx + 2\mu \int_{\mathcal{O}} \varepsilon_{ij}^k(u) \varepsilon_{ij}^k(\bar{v}) dx, \tag{1.8}$$

where  $u, v \in H^1(\mathcal{O})^3$ . The two operators  $L_0(\mathcal{O}; k)$  and  $L(\mathcal{O}; k)$  are also self-adjoint for real  $k$ .

Throughout this paper, the solutions to problems (1.3), (1.4) and (1.5) are defined as the weak solutions (i.e., solutions for extended operators) given by the following definitions.

**Definition 1.1** A function  $u \neq 0$  is a solution to problem (1.3) if  $u \in H^1(\Omega)^3$  and  $L(k)u = w^2 u$ . (1.9)

**Definition 1.2** A function  $u \neq 0$  is a solution to problem (1.4) if  $u \in H_0^1(\mathcal{O})^3$  and  $L_0(\mathcal{O}; k)u = w^2 u$ . (1.10)

**Definition 1.3** A function  $u \neq 0$  is a solution to problem (1.5) if  $u \in H^1(\mathcal{O})^3$  and  $L(\mathcal{O}; k)u = w^2 u$ . (1.11)

Thus problem (1.1) leads us to spectral problems for the operators  $L(k)$ ,  $L_0(\mathcal{O}; k)$  and  $L(\mathcal{O}; k)$ . Next, we define some spectral sets that we need for further investigation.

Let  $T$  be a closed operator defined on a dense subspace of a Hilbert space  $H$ . We define the following spectral sets:

- $\rho(T) = \{\lambda \in \mathbb{C} | (T - \lambda I)^{-1} \in B(H)\}$  (the resolvent set), where  $B(H)$  denotes the space of all bounded operators in  $H$ ;
- $\sigma(T) = \mathbb{C} \setminus \rho(T)$  the spectrum);
- $\sigma_p(T) = \{\lambda \in \mathbb{C} | \operatorname{Ker}(T - \lambda I) \neq \{0\}\}$  (the point spectrum);
- $\sigma_c(T) = \{\lambda \in \mathbb{C} | \overline{R(T - \lambda I)} = R(T - \lambda I)\}$  (the continuous spectrum).

Note that we have used the definition of the continuous spectrum  $\sigma_c(T)$  given in Birman and Solomyak's book ([3]), which is slightly different from that given in almost all of the literature on functional analysis. However, it is more convenient for studying perturbation problems. In particular, with this definition, some eigenvalues may belong to  $\sigma_c(T)$ . Moreover, for normal operators,  $\sigma_c(T)$  consists of the nonisolated points in  $\sigma_c(T)$  (see [3], Chapters 3 and 9).

At this point we should note that the problem of the existence of nonisolated eigenvalues in  $\sigma_c(L(k))$  for a fixed  $k$  is an open problem in elasticity theory (see [6, 8]).

The essential spectrum (of a self-adjoint operator) is another subclass of  $\sigma(T)$ , which is defined as

$$\sigma_{ess}(T) = \sigma_c(T) \cup \sigma_p^\infty(T),$$

where  $\sigma_p^\infty(T)$  denotes the set of eigenvalues of infinite multiplicity.

The following fact is often used as an alternative definition of the essential spectrum: a point  $\lambda$  belongs to  $\sigma_{ess}(T)$  if and only if there exists a singular sequence for  $T$  at  $\lambda$  ([3], p. 207, Theorem 2). We recall the following definition.

**Definition 1.4** A sequence  $u_n \in H$  is said to be singular for a self-adjoint operator  $T$  at a point  $\lambda$  if the following conditions are satisfied:

- (a)  $\|u_n\| = 1, n = 1, 2, 3, \dots;$
- (b)  $u_n \in \mathcal{D}(T);$
- (c)  $u_n \rightarrow 0$  weakly in  $H;$
- (d)  $(T - \lambda)u_n \rightarrow 0.$

This paper consists of the present introduction and three further sections. In Section 2, we study closed waveguides (waveguides with bounded cross sections) and their eigenvalues in detail. Section 3 is devoted to waveguides with an unbounded cross section  $\Omega_0 \cup \mathcal{O}$ . A triangular deformation of the half-plane  $\Omega_0$  is considered separately in this section. An inverse eigenvalue problem and some methods for its solution are considered in Section 4.

**Note.** In this paper, we use operator theory to present some general results in the framework of elastic waveguides with an arbitrary unbounded cross section  $\Omega_0 \cup \mathcal{O}$ . However, by choosing special forms of  $\mathcal{O}$ , such as rectangular, cylindrical and triangular deformations, and special vector fields, one can obtain more specific and concrete results (see [2, 6, 8, 15, 14] and references therein.)

## 2 Eigenvalue Problems in Bounded Domains

The main concern of this section is problems (1.10) and (1.11) in a bounded domain  $\mathcal{O}$ . Actually, the central problem is problem (1.9) in an unbounded domain  $\Omega$ , where  $\Omega = \Omega_0 \cup \mathcal{O}$ . However, to study this problem we need the structure of the eigenvalues of problems (1.10) and (1.11) in bounded domains. By using (1.8), we can write

$$L(\mathcal{O}; k)u = Au + kBu + k^2Cu, \quad u \in H^1(\mathcal{O})^3,$$

where the operators  $A, B$  and  $C$  are defined by their bilinear forms as follows:

$$(Au, v) = \lambda \int_{\mathcal{O}} (u_{1,1} + u_{2,2})(\bar{v}_{1,1} + \bar{v}_{2,2}) \, dx + 2\mu \int_{\mathcal{O}} (u_{1,1}\bar{v}_{1,1} + u_{2,2}\bar{v}_{2,2}) \, dx + \mu \int_{\mathcal{O}} [(u_{1,2} + u_{2,1})(\bar{v}_{1,2} + \bar{v}_{2,1}) + u_{3,1}\bar{v}_{3,1} + u_{3,2}\bar{v}_{3,2}] \, dx, \quad (2.1)$$

$$(Bu, v) = \lambda \int_{\mathcal{O}} [(u_{1,1} + u_{2,2})\bar{v}_3 + u_3(\bar{v}_{1,1} + \bar{v}_{2,2})] \, dx - \mu \int_{\mathcal{O}} [u_{3,1}\bar{v}_1 + u_3\bar{v}_{3,1}] \, dx - \mu \int_{\mathcal{O}} [u_{3,2}\bar{v}_2 + u_2\bar{v}_{3,2}] \, dx \quad (2.2)$$

and

$$(Cu, v) = (\lambda + 2\mu) \int_{\mathcal{O}} u_3\bar{v}_3 \, dx + \mu \int_{\mathcal{O}} (u_1\bar{v}_1 + u_2\bar{v}_2) \, dx, \quad (2.3)$$

for all  $u, v \in H^1(\mathcal{O})$ .

In the same way, we may rewrite the operator  $L_0(\mathcal{O}; k)$  in the form

$$L_0(\mathcal{O}; k)u = Au + kBu + k^2Cu, \quad u \in H_0^1(\mathcal{O})^3.$$

Our further study is based on some properties of the operators  $A, B$  and  $C$ , which are given in the following theorem (see [21], Section 35, and [17]).

**Theorem 2.1** The operators  $A, B$  and  $C$  have the following properties in the space  $L^2(\mathcal{O})^3$ :

- (I)  $A$  is a self-adjoint, nonnegative operator such that  $(A + I)^{-1} \in S_{\infty}$  and  $H^1(\mathcal{O}) = \mathcal{D}((A + I)^{1/2})$ , where  $S_{\infty}$  is the set of all compact operators in  $L^2(\mathcal{O})^3$ .
- (II)  $C$  is a bounded and positive definite operator.
- (III) The operator  $B$  is symmetric, and  $(A + I)^{-1/2}B(A + I)^{-1/2} \in S_{\infty}$ . In particular, this condition means that  $\mathcal{D}((A + I)^{1/2}) \subset \mathcal{D}(B)$ .
- (IV)  $(A + I)^{-1}B \in S_{\infty}$ .

Additionally, the operator  $A$  is positive definite in  $H_0^1(\mathcal{O})$ , i.e.,  $(Au, u) \geq \delta(u, u)$  for some  $\delta > 0$  and for all  $u \in H_0^1(\mathcal{O})$ .

Theorem 2.1 means that all of the operators  $A, B$  and  $C$  are well defined on the energetic space  $\mathcal{H} = \mathcal{D}((A + I)^{1/2})$ . In particular, the operator  $A$  is extended on  $\mathcal{D}((A + I)^{1/2})$  in the following way:

$$(Au, v) = ((A + I)u, v) - (u, v) = ((A + I)^{1/2}u, (A + I)^{1/2}v) - (u, v),$$

where  $u, v \in \mathcal{D}((A + I)^{1/2})$ .

**Theorem 2.2** For all  $k \in \mathbb{C}$ , the spectrum of the operator  $L(\mathcal{O}; k)$  is discrete, i.e.,  $\sigma(L(\mathcal{O}; k))$  consists of isolated eigenvalues of finite multiplicity with a possible concentration point at infinity.

**Proof.** It follows from Theorem 2.1 that the operator  $L(\mathcal{O}; k) - w^2I$  is invertible if and only if  $(A + I)^{-1/2}(L(\mathcal{O}; k) - w^2I)(A + I)^{-1/2}$  is invertible. On the other hand,

$$\begin{aligned} & (A + I)^{-1/2}(L(\mathcal{O}; k) - w^2I)(A + I)^{-1/2} \\ &= I - ((A + I)^{-1} - k(A + I)^{-1/2}B(A + I)^{-1/2} \\ & \quad + k^2(A + I)^{-1/2}C(A + I)^{-1/2} - w^2(A + I)^{-1}) \\ &= I - T(k) - w^2(A + I)^{-1}, \end{aligned}$$

where the operators  $T(k)$  and  $(A + I)^{-1}$  are compact. It follows from this equality that the spectrum of the operator  $L(\mathcal{O}; k)$  is discrete (see [19], Theorem 17.3).

Now we establish a relation between the asymptotic behaviors of the eigenvalues of the operators  $A$  and  $L(\mathcal{O}; k)$  for a fixed real  $k$ . By Theorem 2.2, the spectrum of the operator  $L(\mathcal{O}; k)$  is discrete, i.e., it consists of a sequence

$$w_1^2(k) \leq w_2^2(k) \leq \dots \leq w_n^2(k) \leq \dots$$

First we give the following fact, due to M. G. Krein (see [9], Theorem 11.4).

**Lemma 2.1** Let a self-adjoint operator  $T$  be given in the form  $T = H(I + S)$ , where  $H, S \in S_{\infty}$ .

If either (i)  $(I + S)^{-1} \in B(H)$  or  
(ii)  $\lim_{n \rightarrow \infty} (\lambda_{n+1}(H)/\lambda_n(H)) = 1$ , then

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(T)}{\lambda_n(H)} = 1.$$

The proof of the following theorem, which is basic to our further study, is based on Lemma 2.1.

**Theorem 2.3** Let  $k \in \mathbb{R}$  be fixed and let  $\{w_n^2(k)\}_1^\infty$  be the eigenvalues of the operator  $L(O; k) = A + kB + k^2C$ . Then

- (i)  $\lim_{n \rightarrow \infty} (w_n^2(k)/\lambda_n(A)) = 1$ ;
- (ii)  $w_n^2(k) \sim cn^{2/3}$ , for some  $c > 0$ ;
- (iii) ) the same results hold for the operator  $L_0(O; k)$  too.

**Proof.** We can rewrite  $L(O;k)$  in the following form:  
 $L(O; k) = (A + I)[I - (A + I)^{-1} + k(A + I)^{-1}B + k^2(A + I)^{-1}C]$ .  
Suppose that  $0 \notin \text{Ker}(L(O; k))$ . Then, by using the fact that  $L(O;k)$  is self-adjoint, we obtain  
 $L^{-1}(O; k) = (A + I)^{-1}[I - (A + I)^{-1} + k(A + I)^{-1}B + k^2(A + I)^{-1}C]^{-1}$ .  
By using the notation  $T = L^{-1}(O; k)$ ,  $H := (A + I)^{-1}$  and  $I + S := [I - (A + I)^{-1} + k(A + I)^{-1}B + k^2(A + I)^{-1}C]^{-1}$ , we get  $T = H(I + S)$ , where, by Theorem 2.1, the operators  $T$ ,  $H$  and  $S$  satisfy the conditions of Lemma 2.1. Hence, by Lemma 2.1 we obtain the result that

$$\lim_{n \rightarrow \infty} \frac{w_n^2(k)}{\lambda_n(A)} = 1.$$

Finally, let  $0 \in \text{Ker}(L(O; k))$ . By Theorem 2.2, the spectrum of the operator  $L(O;k)$  is discrete and, consequently, there exists  $\lambda_0$  such that  $L(O; k) - \lambda_0 I$  is invertible. Therefore, by repeating the argument that we applied in the case of  $0 \notin \text{Ker}(L(O; k))$ , we get the same asymptotic relation.

(ii) The formula  $w_n^2(k) \sim cn^{2/3}$  as  $n \rightarrow \infty$  for some  $c > 0$  follows from (i) and the fact that  $\lambda_n(A) \sim cn^{2/3}$  (see [4] for details).

(iii) We can establish this statement by repeating the arguments that we applied in the proof of (i) and (ii). Next we give two inequalities that are useful for the localization of spectra.

**Proposition 2.1 (I)** The positivity of the energy: There exists  $\mu \geq 0$  such that  
 $(L(O; k)u, u) \geq \mu(u, u)$  for all  $u \in H^1(O)^3$  and  $k \in \mathbb{R}$ .  
(2.4)

(II) The energetic stability principle: There exist real numbers  $\zeta \geq 0$  and  $c_0 > 0$  such that for all  $k \in \mathbb{R}$  and all  $u \in H_0^1(O)^3$ ,  
 $(L_0(O; k)u, u) \geq (c_0^2 k^2 + \zeta)(u, u)$ . (2.5)

**Proof.** We note that (2.4) follows immediately from (1.8). Moreover, since  $\text{Ker}(A) \neq 0$  (actually,  $\dim \text{Ker}(A) = 4$ ) in the space  $H^1(O)$ , we obtain the result that  $\mu = 0$  in (2.4).

The inequality (2.5) follows from (2.1), (2.2) and (2.3) by using Korn's first inequality (see [12]) in the following form:

$$\|u\|_{H^1(O)^3}^2 \leq C \int_O \sum_{i,j=1}^3 |\varepsilon_{ij}(u)(x)|^2 dx \quad \text{for all } u \in H_0^1(O),$$

where  $\varepsilon_{ij}(u) = (u_{i,j} + u_{j,i})/2$ .

In particular, the following corollary about the real pairs  $(k, w)$  in  $\sigma_p(L_0(O; k))$  follows from (2.5).

**Corollary 2.1**

Let  $\sigma_{\mathbb{R}}(L_0(O; k)) := \{(k, w) \in \mathbb{R} \times \mathbb{R} \mid \exists u \neq 0, L_0(O; k)u = w^2 u\}$ .

Then:

- (i) The set  $\sigma_{\mathbb{R}}(L_0(O; k))$  lies inside the hyperbola  $w^2 - c_0^2 k^2 \geq \zeta$ .
- (ii)  $\sigma_p(L_0(O; k)) \subset \{\zeta + c_0^2 k^2, +\infty\}$  for each  $k \in \mathbb{R}$ , i.e.,  $\sigma_p(L_0(O; k))$  lies inside the hyperbola and on the vertical line passing through the point  $(k, 0)$ .

Corollary 2.1 yields the localization for eigenvalues of  $L_0(O; k)$  as in Figure 2.

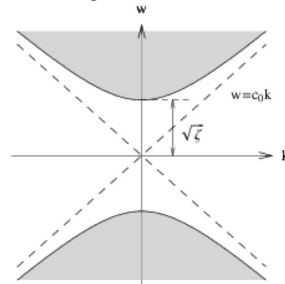


Fig 2. The domain containing the eigenvalues of operator  $L_0(O; k)$ .

By Corollary 2.1,  $0 \notin \sigma_p(L_0(O; k))$  for all  $k \in \mathbb{R}$ . Now our question is: for what values of  $k$  do we have  $0 \in \sigma_p(L(O; k))$ ? The answer is given in the following proposition.

**Proposition 2.2**  $0 \in \sigma_p(L(O; k))$  if and only if  $k = 0$ .

**Proof.** Evidently,

$$\text{Ker}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}\right\}.$$

Consequently, if  $k = 0$ , then  $L(O; 0) = A$  and  $0 \in \sigma_p(L(O; 0))$ . We shall now show that  $0 \notin \sigma_p(L(O; k)) \Rightarrow k = 0$ . We consider two cases: (a)  $u \notin \text{Ker}(A)$  and (b)  $u \in \text{Ker}(A)$ . In the case of (a), it follows from (2.4) that

$$|(Bu, u)| < 2\sqrt{(Au, u)(Cu, u)} \quad \text{for all } u \in H^1(O)^3 \setminus \text{Ker}(A). \quad (2.6)$$

On the other hand, this inequality implies that  $(L(O; k)u, u) > 0$  for all  $u \in H^1(O)^3 \setminus \text{Ker}(A)$  and  $k \in \mathbb{R}$ .

Therefore, for any  $k \in \mathbb{R}$ , a vector  $u \notin \text{Ker}(A)$  is not an eigenvector of the operator  $L(\mathcal{O};k)$  corresponding to the eigenvalue  $w=0$ . Now, let  $u \in \text{Ker}(A)$ . First, we obtain from (2.4) that

$$|(Bu, u)| \leq \sqrt{(Au, u)(Cu, u)} \text{ for all } u \in H^1(\mathcal{O})^3.$$

Since  $u \in \text{Ker}(A)$ , then  $(Bu, u) = 0$ . Hence,

$$(L(\mathcal{O};k)u, u) = (Au, u) + k(Bu, u) + k^2(Cu, u) = k^2(Cu, u) > 0, k \neq 0.$$

Thus  $0 \in \sigma_p(L(\mathcal{O};k))$  is possible only for  $k = 0$ .

### 3 The Spectrum of an Elastic Waveguide in a Domain with an Unbounded Cross Section

#### 3.1 Bounded Deformations of the Half-plane and Related Perturbations for the Operator $L(\Omega_0; k)$

The main subject of this section is the structure of the spectrum of the operator  $L(k)$  that is obtained from  $L(\Omega_0; k)$  by a bounded deformation of the half-plane  $\Omega_0$ , i.e.,  $\Omega = \Omega_0 \cup \mathcal{O}$ .

We also need the quantities  $c_p = \sqrt{\lambda + 2\mu}$ ,  $c_s = \sqrt{\mu}$  and  $c_R$ , which are called the speeds of the primary wave, the secondary wave and the Rayleigh wave, respectively. The speed of the Rayleigh wave is defined as the single root in  $(0, c_s)$  of the Rayleigh equation (see [20]),

$$4(1 - \frac{z^2}{c_p^2})^{1/2}(1 - \frac{z^2}{c_s^2})^{1/2} - (2 - \frac{z^2}{c_s^2})^{1/2} = 0.$$

The following proposition follows immediately from Definition 1.4 and the related properties of semibounded self-adjoint operators.

**Proposition 3.1**  $\sigma_{ess}(L(k)) \subset [k^2 c_R^2, +\infty)$  for all  $k \in \mathbb{R}$ .

**Proof.** By definition,  $L(k)$  is the operator defined on  $H^1(\Omega)^3 \subset L_2(\Omega)^3$  and associated with the bilinear form  $(L(k)u, v) = \lambda \int_{\Omega} \text{div}_k u \overline{\text{div}_k v} dx + 2\mu \int_{\Omega} \varepsilon_{ij}^k(u) \overline{\varepsilon_{ij}^k(v)} dx,$

$$(3.1)$$

where  $u, v \in H^1(\Omega)^3$ . Hence, we can write  $L(k) = L(\Omega_0; k) + L(\mathcal{O}; k)$ . By (3.1) and (1.8), all of the operators  $L(k)$ ,  $L(\Omega_0; k)$  and  $L(\mathcal{O}; k)$  are positive and self-adjoint. Moreover,  $L(\Omega_0; k) \geq k^2 c_R^2 I$  in  $H^1(\Omega_0)$  (see [6], Lemma 3), which implies that  $L(k) \geq L(\Omega_0; k) \geq k^2 c_R^2 I$ . This inequality yields  $\sigma_{ess}(L(k)) \subset [k^2 c_R^2, +\infty)$ .

Next we give a stronger result established in [6].

**Proposition 3.2** ([6], Theorem 1) For every  $k \in \mathbb{R}$ ,  $\sigma_{ess}(L(k)) = [k^2 c_R^2, +\infty)$ .

A proof of this proposition was given in [6], and for this reason we shall not give an alternative proof. We note also that throughout this paper we use operator and operator pencil techniques, which are quite different from the methods that were used

in [6] and related papers. Particularly, Proposition 3.2 may be proved by using the generalized Weyl criterion (see [3], p. 207, Theorem 4 and [16]) and some known facts from perturbation theory.

The generalized Weyl criterion. Let  $T, S$  be self-adjoint operators. Suppose that for some point  $\lambda_0 \in \rho(T) \cap \rho(S)$  the difference of the resolvents is compact, i.e.,

$$(T - \lambda)^{-1} - (S - \lambda)^{-1} \in \sigma_{com};$$

then  $\sigma_{ess}(T) = \sigma_{ess}(S)$ .

As one can see from Proposition 3.2, the problem of the existence of the eigenvalues and (if there are any) their dependence on  $k$  and on the shape of the deformation is one of the main questions in the spectral theory of unbounded elastic waveguides. There may be two kinds of eigenvalues: (i) nonisolated eigenvalues in  $[k^2 c_R^2, +\infty)$ , as pointed out in the introduction; and (ii) isolated eigenvalues located strictly below  $k^2 c_R^2$ .

We set

$$\lambda_n(k) = \inf_{L \subset H^1(\Omega)} \max_{\substack{u \in L \\ u \neq 0}} \frac{(L(k)u, u)}{(u, u)}, n = 1, 2, \dots, (3.2)$$

and

$$N_\lambda(k) := N_\lambda(L(k)) = \max\{n | \lambda_n(k) < \lambda\}. (3.3)$$

Evidently, all eigenvalues (if any) located strictly below  $k^2 c_R^2$ , are among the numbers  $\lambda_n(k)$ . Our first observation is given in the following proposition.

**Proposition 3.3** For any  $k \in \mathbb{R}$ :

- (a)  $\lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_n(k) \leq \dots \leq k^2 c_R^2$ , i.e.,  $\lambda_n(k)$  either is an eigenvalue located strictly below  $k^2 c_R^2$  or is equal to  $k^2 c_R^2$ , which belongs to  $\sigma_{ess}(L(k))$ .
- (b)  $\lambda_n(k)$  is an eigenvalue if and only the infimum in (3.2) is attained.
- (c)  $N_\lambda(k) = +\infty$  for  $\lambda > k^2 c_R^2$ .

- (d)  $N_\lambda(k) = \max\{\dim\{E(\lambda)u, u\} < \lambda \|u\|^2, u \in E \setminus \{0\}\}$  for  $\lambda \leq k^2 c_R^2$ . In particular,  $N_{k^2 c_R^2}(k) = \max\{\dim\{E(L(k)u, u) < k^2 c_R^2 \|u\|^2, u \in E \setminus \{0\}\}$ .

**Proof.**

(a) This statement follows from (3.2) and Proposition 3.2.

(b) The proof is based on the spectral expansion

$$L(k) = \int_0^\infty \lambda dE_\lambda(k),$$

where  $E_\lambda(k)$  is the spectral measure for the operator  $L(k)$ , and on the fact that  $L(k)$  is a semibounded self-adjoint operator.

(c) This follows from (3.3) and Proposition 3.2.

(d) This fact is known from the spectral theory of self-adjoint operators (see [3]), which is also based on the spectral expansion given above.

Now, in addition to problems (1.4) and (1.5), we consider the following Zaremba-type boundary value problem (see [13] for similar problems).

The Zaremba problem: 
$$\begin{cases} L(k)u = w^2u & \text{in } \mathcal{O}, \\ \sigma^k(u)\bar{v} = 0 & \text{on } \Gamma_1, \\ u|_{\Gamma_2} = 0, \end{cases}$$

where  $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$  (see Figure 1). The operator associated with the Zaremba problem will be denoted by  $L_Z(\mathcal{O}; k)$ . We denote the energetic spaces of problem (1.4), the Zaremba problem and problem (1.5) by  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ , respectively. As mentioned in the introduction,  $\mathcal{H}_1 = H_0^1(\mathcal{O})$  and  $\mathcal{H}_3 = H^1(\mathcal{O})$ . Hower  $\mathcal{H}_2 = \{u \in H^1(\mathcal{O}) | u|_{\Gamma_2} = 0\}$ .

By Theorem 2.2, for all  $k \in \mathbb{R}$ , the spectra of all these problems are discrete, i.e., they consist of isolated eigenvalues of finite multiplicity with a concentration point at infinity. We denote by  $\lambda_n^D(\mathcal{O}, k)$ ,  $\lambda_n^Z(\mathcal{O}, k)$  and  $\lambda_n^F(\mathcal{O}, k)$  the eigenvalues of the operators  $L_0(\mathcal{O}; k)$ ,  $L_Z(\mathcal{O}; k)$  and  $L(\mathcal{O}; k)$ , respectively. Then we have

$$\lambda_n^D(\mathcal{O}, k) = \inf_{L \subset H_0^1(\mathcal{O})} \max_{\dim L = n} \max_{u \in L, u \neq 0} \frac{(L_0(\mathcal{O}; k)u, u)}{(u, u)}, \quad n = 1, 2, \dots, \quad (3.4)$$

$$\lambda_n^Z(\mathcal{O}, k) = \inf_{L \subset \mathcal{H}_2} \max_{\dim L = n} \max_{u \in L, u \neq 0} \frac{(L_Z(\mathcal{O}; k)u, u)}{(u, u)}, \quad n = 1, 2, \dots, \quad (3.5)$$

$$\lambda_n^F(\mathcal{O}, k) = \inf_{L \subset H^1(\mathcal{O})} \max_{\dim L = n} \max_{u \in L, u \neq 0} \frac{(L(\mathcal{O}; k)u, u)}{(u, u)}, \quad n = 1, 2, \dots \quad (3.6)$$

Thus, the inclusion  $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3$  and (3.4), (3.5) and (3.6) yield  $\lambda_n^F(\mathcal{O}, k) \leq \lambda_n^Z(\mathcal{O}, k) \leq \lambda_n^D(\mathcal{O}, k)$ . (3.7)

Finally, the following corollary follows from (3.7), Theorem 2.3 and the fact that  $\mathcal{H}_2 \hookrightarrow H^1(\Omega)$ .

**Corollary 3.1** (a) For each  $k \in \mathbb{R}$ ,

$\lambda_n(k) \leq \lambda_n^Z(\mathcal{O}, k)$  and  $\lambda_n^Z(\mathcal{O}, k) \sim cn^{2/3}$ , where the constant  $c$  does not depend on  $k$ ;

(b) for all  $\lambda \leq k^2 c_{\mathbb{R}}^2$  and  $k \in \mathbb{R}$ ,

$$N_{\lambda}^Z(\mathcal{O}, k) \leq N_{\lambda}(k) \leq N_{\lambda}^F(\mathcal{O}, k),$$

where  $N_{\lambda}^Z(\mathcal{O}, k)$  and  $N_{\lambda}^F(\mathcal{O}, k)$  denote the spectral distribution functions of the operators  $L_Z(\mathcal{O}; k)$  and  $L(\mathcal{O}; k)$ , respectively.

We recall that one of the main problems in the spectral theory of elastic waveguides is that of the nonemptiness of the set

$$N_{k^2 c_{\mathbb{R}}^2}(k) := \max\{n | \lambda_n(k) < k^2 c_{\mathbb{R}}^2\},$$

which is the set of all eigenvalues (if any) of the operator  $L(k)$  located strictly below  $k^2 c_{\mathbb{R}}^2$ . This is equivalent to the existence of a guided mode with a speed lower than the speed of the Rayleigh wave. The obvious result is that  $N_{k^2 c_{\mathbb{R}}^2}^D(\mathcal{O}, k) = 0$  for all  $k \in \mathbb{R}$ , where  $N_{\lambda}^D(\mathcal{O}, k)$  is the spectral distribution function for operator  $L_0(\mathcal{O}, k)$ . The following theorem contains such a non-trivial result for  $N_{\lambda}^Z(\mathcal{O}, k)$ .

**Theorem 3.1** There exists  $k^* > 0$  such that

$$N_{k^2 c_{\mathbb{R}}^2}^Z(\mathcal{O}, k) = 0 \quad \text{for } k < k^*.$$

**Proof.** By the definition  $N_{\lambda}^Z(\mathcal{O}, k)$  is the spectral distribution function for the operator  $L_Z(\mathcal{O}, k)$ .

Then

$$N_{\lambda}^Z(\mathcal{O}, k) = \max \dim \{E \subset \mathcal{H}_2 | (Au, u) + k(Bu, u) + k^2(Cu, u) < \lambda\}.$$

Hence,

$$N_{k^2 c_{\mathbb{R}}^2}^Z(\mathcal{O}, k) = \max \dim \{E \subset \mathcal{H}_2 | (Au, u) + k(Bu, u) + k^2((C - c_{\mathbb{R}}^2)u, u) < 0\},$$

i.e.,  $N_{k^2 c_{\mathbb{R}}^2}^Z(\mathcal{O}, k)$  is the number of the negative eigenvalues of the operator

$$M_Z(k) := M_Z(\mathcal{O}, k) = A + kB + k^2((C - c_{\mathbb{R}}^2)I), \quad \mathcal{D}(M_Z(k)) = \mathcal{H}_2.$$

Thus the set of all eigenvalues (if any) of operator  $L_Z(\mathcal{O}, k)$  located strictly below  $k^2 c_{\mathbb{R}}^2$  is the same as the number of the negative eigenvalues of the operator  $(M_Z(k))$ . Now, we show that there exists  $k^* > 0$  such that if  $k < k^*$  then  $N_0(M_Z(k)) = 0$ . Suppose that this is not true. Then we obtain a sequence  $k_n \rightarrow 0$ ,  $k_n > 0$  such that

$$\min_{\mathcal{H}_2 \setminus \{0\}} \frac{(M_Z(k_n)u, u)}{\|u\|_{H^1(\mathcal{O})}} < 0.$$

Therefore there exists  $u_n \in \mathcal{H}_2$ ,  $\|u_n\|_{H^1(\mathcal{O})} = 1$  satisfying

$$(M_Z(k_n)u_n, u_n) = \tau_n < 0.$$

By using the compact embedding (see [1])

$$\mathcal{H}_2 \hookrightarrow L_2(\mathcal{O}) \quad \text{we get } u_n \rightarrow u_0 \neq 0 \text{ in } L_2(\mathcal{O}).$$

Consequently,

$$\lim_{n \rightarrow \infty} (M_Z(k_n)u_n, u_n) = (M_Z(0)u_0, u_0) = (Au_0, u_0) \leq 0.$$

This is a contradiction to the fact that  $A$  is a positive operator in  $\mathcal{H}_2$ .

#### 4 On the inverse dispersion relation $k(w)$ and a linearization method for its solution

The main target of this section is the following inverse problem: for what values of  $k$  will the given  $w^2$  be an eigenvalue of the eigenvalue problem  $L(k)u = w^2u$ ? So far, we fixed  $k$  and studied the eigenvalues of the operator  $L(k)$ . Actually, this inverse problem is also the main concern of elastic waveguides. As we have seen in the previous sections there are may be two kind of eigenvalues for operator  $L(k)$ : i) non-isolated eigenvalues in  $[k^2 c_{\mathbb{R}}^2, +\infty)$ , ii) isolated eigenvalues located strictly below  $k^2 c_{\mathbb{R}}^2$ . Although, we were mainly concentrated on the second case, however the existence or non-existence of non-isolated eigenvalues in  $[k^2 c_{\mathbb{R}}^2, +\infty)$  is another open problem for many cylindrically-perturbed elastic waveguides. Thus, if we have an eigen-pair  $(k, w^2)$  for the problem  $L(k)u = w^2u$  then  $w^2$  (for the fixed  $k$ ) is either a non-isolated eigenvalues in  $[k^2 c_{\mathbb{R}}^2, +\infty)$  or an isolated eigenvalues located strictly below  $k^2 c_{\mathbb{R}}^2$ . Clearly, this question is a typical eigenvalue problem for the operator pencil:  $L_w(k) := L(k) - w^2I$ .



The quantitative analysis of this problem via the comparison principles, as before leads to the eigenvalue problems for operator pencils

$$L_w(\mathcal{O}, k) = L(\mathcal{O}, k) - w^2 I, \text{ and } L_w^0(\mathcal{O}, k) = L_0(\mathcal{O}, k) - w^2 I.$$

There are various methods to investigate the spectrum of operator pencils (see [7], [10], [11], [17], [19] and [21]). But in this paper we are trying to stay within the framework of the operator theory. The best way to do this is to apply a linearization method. The most suitable linearization in this context is that of given by M. Krein and H. Langer in their well-known paper [18]. This linearization allows us to establish a desirable connection between the root functionals of the pencils studied in this paper and the numerical ranges of their linearizations, which is very important in the variational theory.

We start with  $L_w(\mathcal{O}, k) = A + kB + k^2 C - w^2 I$  and fix  $w$ .

By Proposition 2.1 we have

*The principle of positivity of the energy:*

There exists  $\mu \geq 0$  such that

$$(L(\mathcal{O}; k)u, u) \geq \mu(u, u) \text{ for all } u \in H^1(\mathcal{O})^3 \text{ and } k \in \mathbb{R}. \tag{4.1}$$

*The energetic stability principle:*

There exist real numbers  $\zeta \geq 0$  and  $c_0 > 0$  such that for all  $k \in \mathbb{R}$  and all  $u \in H_0^1(\mathcal{O})^3$ ,

$$(L_0(\mathcal{O}; k)u, u) \geq (c_0^2 k^2 + \zeta)(u, u). \tag{4.2}$$

We repeat a result from our paper [7].

**Theorem 4.1** Let  $L_w(\mathcal{O}, k)$  be an operator pencil of w.g.t. satisfying the energetic stability condition then for all  $w \in \mathbb{C}$  the spectrum

$\sigma(L_w(\mathcal{O}, k)) := \{k \in \mathbb{C} | L_w(\mathcal{O}, k) \text{ is not invertible}\}$  is discrete, i.e.  $\sigma(L_w(\mathcal{O}, k))$  consists of isolated eigenvalues of finite multiplicity.

The following properties of  $\sigma(L_w(\mathcal{O}, k))$  follow from the inequalities (4.1) and (4.2).

**Corollary 4.1**

- If  $\text{Im } w \neq 0$  then  $\text{Im } k \neq 0$ .
- If  $\text{Im } w = 0$  and  $|w| < \mu$  then  $\text{Im } k \neq 0$ .

Consequently, real wavenumbers exist only at real frequencies satisfying the inequality  $|w| \geq \mu$ .

Finally, we study real eigenvalues in  $\sigma(L_w(\mathcal{O}, k))$ . According to Theorem 2.1, applying the operator  $(A + I)^{-\frac{1}{2}}$  to both the sides of  $L_w(\mathcal{O}, k)$  reduces it to the bounded operator pencil of the form (see our the previous paper [11]):

$$T_w(\mathcal{O}, \lambda) = \lambda^2 M(w) + 2\lambda N(w) + I,$$

where  $M(w), N(w) \in S_\omega$ ,  $M(w) > 0$ ,  $N(w) = N(w)^*$  and  $\lambda = k + k^*$  for some  $k \in \mathbb{R}$ . We fix  $w$  and rewrite  $T_w(\mathcal{O}, \lambda)$  in the form

$$T(\mathcal{O}, \lambda) = \lambda^2 M + 2\lambda N + I$$

Notice that many properties of this pencil were studied in the paper of M. Krein and H. Langer [18].

A simple connection between the eigenvalues of  $T(\mathcal{O}, \lambda)$  and  $L_w(\mathcal{O}, k)$  is that  $\lambda_0$  is an eigenvalue for  $T(\mathcal{O}, \lambda)$  if and only if  $\lambda_0 + k^*$  is an eigenvalue for  $L_w(\mathcal{O}, k)$ .

In the following discussion, our starting point is the pencil

$$T(\mathcal{O}, \lambda) = \lambda^2 M + 2\lambda N + I.$$

and its linearization:  $L(\lambda) = I - \lambda T$  in the space  $H^2 = H \oplus H$ , where

$$\mathbb{T} = \begin{pmatrix} 0 & M^{\frac{1}{2}} \\ -M^{\frac{1}{2}} & -2N \end{pmatrix}.$$

A proof of the following proposition is based on the definitions of eigenvectors and associated vectors (see [19] for the definitions) by using the above given linearization.

**Proposition 4.1**  $\sigma(T) = \sigma(L)$ . Moreover, if the vectors  $u_0, u_1, \dots, u_n$  form a chain of eigenvectors and associated vectors (e.a.v.) corresponding to the eigenvalue  $\lambda_0$  of the pencil  $T(\lambda)$ , then the vectors

$$\begin{bmatrix} \lambda_0 M^{\frac{1}{2}} u_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} \lambda_0 M^{\frac{1}{2}} (\lambda_0 u_1 + u_0) \\ u_1 \end{bmatrix}, \dots, \begin{bmatrix} M^{\frac{1}{2}} (\lambda_0 u_n + u_{n-1}) \\ u_n \end{bmatrix}$$

form a chain of e.a.v. corresponding to the same eigenvalue  $\lambda_0$  of the linear pencil  $L(\lambda) = I - \lambda T$ . Conversely, if

$$\begin{bmatrix} u_0^1 \\ u_0^2 \end{bmatrix}, \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix}, \dots, \begin{bmatrix} u_n^1 \\ u_n^2 \end{bmatrix}$$

form a chain of e.a.v. corresponding to the eigenvalue  $\lambda_0$  of  $L(\lambda) = I - \lambda T$  then the vectors  $u_0^2, u_1^2, \dots, u_n^2$  form a chain of e.a.v. corresponding to same eigenvalue  $\lambda_0$  of the pencil  $T(\lambda)$ .

Evidently, the eigenvalues of the linear pencil  $L(\lambda) = I - \lambda T$ , and the characteristic values of the operator  $\mathbb{T}$  are the same. The operator  $\mathbb{T}$  is not self-adjoint in the Hilbert space  $H \oplus H$ . But if we define

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \text{ then } (J\mathbb{T})^* = J\mathbb{T}. \text{ This relationship}$$

means that the operator  $\mathbb{T}$  is a self-adjoint operator in the Krein space  $K := H \oplus H$  with the inner product  $[\tilde{u}, \tilde{v}]_J := (J\tilde{u}, \tilde{v})$  (see [5]). According to Proposition 4.1,  $u_0$  is an eigenvector corresponding to the eigenvalue  $\lambda_0$  of the pencil  $T(\lambda)$  if and only if the vector

$$\tilde{u}_0 = \begin{bmatrix} \lambda_0 M^{\frac{1}{2}} u_0 \\ u_0 \end{bmatrix}$$

is an eigenvector corresponding to the characteristic value  $\lambda_0$  of the operator  $\mathbb{T}$ .

We note that all the results, presented in this paper are based on the variational principles like (3.2) and (3.3).

These are variational principles for operator  $L(k)$ . However, the characterization of the eigenvalues of problem  $T(\lambda)u = 0$  is much more complicated, because this is an eigenvalue problem for the operator pencil  $T(\lambda)$ . In this case one can use the characteristic values of the self-adjoint operator  $\mathbb{T}$  in the Krein space  $K$ . Let us write  $[u, v]$  for  $[u, v]_J$  in the Krein space  $K$ . Variational principles for real eigenvalues (positive or negative type) of a self-adjoint operator  $\mathbb{T}$  in a Krein space  $K$  are given by the following formula:

If  $C^\pm = \{u \in K \mid \pm [u, u] > 0\}$  then

$$\lambda_n^\pm(\mathbb{T}) = \inf_{L, \dim L = n} \sup_{L \subset C^\pm} \frac{[\mathbb{T}u, u]}{[u, u]},$$

$$\lambda_n^\pm(\mathbb{T}) = \sup_{L, \text{codim} L = k-1} \inf_{L \cap C^\pm} \frac{[\mathbb{T}u, u]}{[u, u]}.$$

### CONCLUSION

We use an operator approach and suggest alternative methods to solve spectral problems arising in the theory of both closed elastic waveguides and elastic waveguides with unbounded cross sections. By using operator methods we give full description of the spectral sets studied in this paper. We study the inverse dispersion relation  $k(w)$  and suggest a linearization method for its solution.

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