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ON THE EXISTENCE OF SOLUTION FOR AN INVERSE PROBLEM

C. TAJANI[1], J. ABOUCHABAKA[1], N. SAMOUH[2] §

Abstract. We consider a boundary detection problem. We present physical motivations. We formulate the problem as a shape optimization problem by introducing the Neumann condition of the accessible part in a cost functional to be minimized, which complicates the study of continuity state that requires more regularity of the free boundary. We show the existence of the optimal solution of the problem by the J. Haslinger and P. Neittaanmäki principle.

Keywords: Inverse problem, Boundary detection problem, Shape optimization, Laplace equation.

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1. Introduction

Inverse problems can be found in many realistic engineering applications, such as the determination of the boundary conditions [4], [11], material properties [12], applied force [9], boundary position [8], etc.

In this paper we are interested in the inverse problem of determining the location of the unknown and damaged boundary from the data collected on the accessible part of the boundary.

In the boundary detection problem, which is also known as the geometry identification problem, the materials used as electrical conductors, electromagnetic elements are subject to wear by corrosion or by direct contact with other elements causing a material loss or cracks, as for instance pipes transporting water, gas, chemically aggressive fluids or bodywork of aircraft, cars, etc, whose surfaces have been damaged by a corrosion attack. A very important issue in the nondestructive testing of materials [2], [5], [10] is the ability to detect possible defects (cracks, fractures for example) inside the material. In practice, it often happens that such surfaces are not accessible to direct inspection, hence in order to detect the possible presence of corrosion one has to rely on measurements only performed on the accessible part of the specimen surface. Our problem is to estimate this loss, or place of crack which is to determine the unknown part of the boundary that has suffered corrosion by making measurements of voltage and current on the known parts of the boundary.

This type of problem is known to be severely ill-posed, whose solution does not depend

[1] Ibn Tofail University, Faculty of Sciences, Department of Mathematics and Infirmatics, Kenitra, Morocco
e-mail: chakir_Tajani@hotmail.fr and abouchabaka3@yahoo.fr

[2] Moulay Ismail University, Faculty of Sciences, Department of Mathematics, Meknes, Morocco
e-mail: ssamouh@yahoo.fr

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continuously on the boundary data, i.e. a small error in the measured data may result an enormous error in the numerical solution.
In this paper, the boundary detection problem is governed by the Laplace equation, the Cauchy data is given on part of the boundary Γ_1 and the Robin boundary condition on the two other parts of the boundary Γ_0 and Σ, whose spatial position of Γ_0 is unknown a priori, and we are interested in determining the location of the unknown and damaged boundary Γ_0 from the data collected on the accessible part of the boundary Γ_1 by formulating the problem in a problem of shape optimization.
In many work of boundary detection problem, on the part of the boundary to be determined, called free boundary, we have two conditions, and to proceed to a formulation in shape optimization problem, we introduce one of the two conditions in a cost functional to minimize [1]. We use the same principle by introducing this time one of the two measurements obtained on the accessible part, especially the Neumann condition which complicates the study of continuity State that requires more regularity of the free boundary. Then; we show that our problem has at least one solution, which is to show that the set of the solutions of the shape optimization problem is compact and the cost functional is semi continuous inferiorly.
The second section is devoted to physical model and presentation of the mathematical formulation of the boundary detection problem. In section 3; we formulate the problem in a shape optimization problem. Section 4 presents the existence of the optimal solution of the problem based on the principle of J. Haslinger & P. Neittaanmäki.

2. Mathematical formulation

2.1. **The physical model.** We consider a perfect dielectric materiel damaged, represented by a bounded domain in two dimension ($\Omega \subset \mathbb{R}^2$).
$\partial\Omega$ is the boundary of Ω, where $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Sigma$.
Γ_1 and Σ are the known parts of the boundary $\partial\Omega$,
Γ_0 is the unknown part of the boundary $\partial\Omega$,
Γ_0, Γ_1 and Σ are disjoints.
To determine material loss occurring on the part $\Gamma_0 \subset \partial\Omega$, measurements of tension are taken on the accessible part of the boundary $\partial\Omega$. i.e. We want to calculate the electric field in the concerned domain. The problem is modeled by Maxwell's equations that are written in the form:

$$\left\{\begin{array}{ll} divD = \rho & (1.1) \\ rotE = 0 & (1.2) \end{array}\right. \quad \text{in } \Omega \tag{1}$$

where ρ is the density of electric charge E and D is the induction electric (or electrical displacement).
We add to these two equations, the constitutive law for a perfect medium: $D = \varepsilon E$,
where ε is the constant that characterizes the medium in question called dielectric permittivity of the medium. We can reduce the problem (1) into scalar problem by remarking that (1.2) implies the existence of a function u called potentiel such that:

$$E = -grad\ u.$$

Substituting this equation in (1.1) and taking into account the constitutive equation, we get:

$$-div(\varepsilon gradu) = \rho$$

since ε is a constant, we obtain the Poisson equation:

$$-\Delta u = \frac{\rho}{\varepsilon}$$

which becomes a Laplace equation in the absence of electrical source,(i.e. $\rho = 0$) whether;

$$-\Delta u = 0 \quad \text{in} \quad \Omega \tag{2}$$

In this equation, we add boundary conditions:
Assuming that the part of the boundary Γ_1 bears a given density of electric charge $\rho(x)$ and the outside domain of Ω is a perfect conductor, leads to boundary conditions (transmission condition):

$$\left\{ \begin{array}{ll} E \wedge \nu_{/\Gamma_1} = 0 & (3.1) \\ D.\nu = -\rho(x) & (3.2) \end{array} \right. \quad \text{on } \Gamma_1 \tag{3}$$

where ν is the unit outward normal vector Γ_1.
(3.1) shows that the tangential component of u in Γ_1 is zero, i.e. that u must remain constant on Γ_1. Hence an inhomogeneous boundary condition on Γ_1 ($u = f$).
(3.2) expresses that the normal component of u is continuous at the traversal of Γ_1. Hence an inhomogeneous Neumann condition on Γ_1 ($\frac{\partial u}{\partial n} = g$).
On Σ and Γ_0, we consider a mixed condition which expresses that the given potential by the system is proportional to the difference between the potential of the system and that of the external environment.
Hence;

$$\left\{ \begin{array}{lll} \alpha_0 u + \beta_0 \frac{\partial u}{\partial n} = h & \text{on} & \Sigma \\ \alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = q & \text{on} & \Gamma_0 \end{array} \right. \tag{4}$$

where α_i and β_i , for $i = 0, 1$ are the exchange coefficients.

2.2. **Formulation of the inverse problem.** In this study, the boundary detection problem considered is governed by the two-dimensional Laplace's equation. The governing equation and the corresponding boundary conditions are demonstrated as follows:

For $f \in L^2(\Gamma_1)$, $g \in L^2(\Gamma_1)$, $h \in L^2(\Sigma)$, $q \in L^2(\Gamma_0)$;

$$\left\{ \begin{array}{lll} -\Delta u = 0 & \text{in} & \Omega \\ u = f, \frac{\partial u}{\partial n} = g & \text{on} & \Gamma_1 \\ \alpha_0 u + \beta_0 \frac{\partial u}{\partial n} = h & \text{on} & \Sigma \\ \alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = q & \text{on} & \Gamma_0 \end{array} \right. \tag{5}$$

where $f, g, h, q, \alpha_0, \beta_0, \alpha_1$ and β_1 are a given functions.
$\partial u / \partial n$ is the normal derivative of u, Ω is the computational domain and $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Sigma$ (Γ_1, Σ and Γ_0 are disjoints).
The spatial position of Γ_0 is unknown apriori. Then, the purpose of the boundary detection problem is to find the solution of the Laplace's problem u, and the spatial position of the boundary portion Γ_0 .

3. Shape optimization formulation

We propose a formulation in a shape optimization problem which consists in including the Neuman condition on Γ_1 in a cost functional and in varying the domain Ω in class of domain, which will be define later by θ_{ad}.
A formulation of the problem (5) in shape optimization can be written as:
Find $\Omega^* \in \theta_{ad}$ solution of:

$$\left\{\begin{array}{ll} \jmath(\Omega^*) = \min_{\Omega\in\theta_{ad}} \jmath(\Omega) & \text{where } \jmath(\Omega) = \int_{\Gamma_1} (\frac{\partial u_\Omega}{\partial n} - g)^2 d\sigma \\ \text{and } u_\Omega \text{ solution of :} & \\ (P.E) \left\{\begin{array}{lll} -\Delta u = 0 & \text{in} & \Omega \\ u = f & \text{on} & \Gamma_1 \\ \alpha_0 u + \beta_0 \frac{\partial u}{\partial n} = h & \text{on} & \Sigma \\ \alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = q & \text{on} & \Gamma_0 \end{array}\right. & \end{array}\right. \tag{6}$$

The problem (6) is well-posed if for any element of θ_{ad} , the state equation (P.E) has a unique solution and if $\jmath(\Omega)$ is well defind.
$\jmath(\Omega)$ is well defind assuming that $\frac{\partial u}{\partial n} \in L^2(\partial\Omega)$.

3.1. **Study of the state problem.**

We will show that the state problem (P.E) has a unique solution.

3.1.1. *The variational form:* Let D be a bounded open domain in $\mathbb{R}^2$ such that $\Omega \subset D$ and u the solution of the problem (P.E).
We take $h^1 = f$ on Γ_1 and suppose that $f \in H^{\frac{1}{2}}(\Gamma_1)$.
By utilizing the trace application in $H^{\frac{1}{2}}(\partial D)$, it exist $U_0 \in H^1(D)$ such that $U^0 = h^1$ on Γ_1.
We define the space $H_D(\Omega) = \{v \in H^1(\Omega)/v_{/\Gamma_1} = 0\}$.
Assume that $u \in H^1(\Omega)$, by applying Green's formula, we get:
$\forall v \in H_D(\Omega)$,

$$\int_\Omega \nabla u.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma u.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_0} u.v d\sigma = \int_\Sigma \frac{h}{\beta_0}.v d\sigma + \int_{\Gamma_0} \frac{q}{\beta_1}.v d\sigma.$$

Then, the problem (PE) is equivalent to:

$$\left\{\begin{array}{l} \text{Find } u \text{ such that } u - U^0 \in H_D(\Omega) \text{ and} \\ \int_\Omega \nabla u.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma u.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_0} u.v d\sigma = \int_\Sigma \frac{h}{\beta_0}.v d\sigma + \int_{\Gamma_0} \frac{q}{\beta_1}.v d\sigma \end{array}\right. \tag{7}$$

if we set $\omega = u - U^0$, we obtain the problem:

$$\left\{\begin{array}{l} \text{Find } \omega \in H_D(\Omega) \text{ such that} \\ \\ \int_\Omega \nabla \omega.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma \omega.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_0} \omega.v d\sigma \\ \\ = -\int_\Omega \nabla U^0.\nabla v dxdy + \frac{1}{\beta_0}\int_\Sigma (h - \alpha_0 U^0).v d\sigma + \frac{1}{\beta_1}\int_{\Gamma_0}(q - \alpha_1 U^0).v d\sigma \end{array}\right. \tag{8}$$

3.1.2. *Existence and uniqueness of the solution:* We consider the bilinear form:

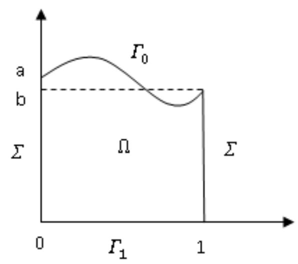
$$a(\omega, v) = \int_\Omega \nabla \omega.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma \omega.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_0} \omega.v d\sigma \tag{9}$$

and the linear form:

$$l(v) = -\int_\Omega \nabla U^0.\nabla v dxdy + \frac{1}{\beta_0}\int_\Sigma (h-\alpha_0 U^0).vd\sigma + \frac{1}{\beta_1}\int_{\Gamma_0}(q-\alpha_1 U^0).vd\sigma \qquad (10)$$

For showing the existence and the uniqueness of the problem (8), we use the Lax-Miligram lemma. Then, it suffices to show that the bilinear form a is continuous and coercive; and the linear form l is continuous in $H_D(\Omega)$ equipped with the norm $|\varphi|_{1,\Omega}$ where $|\varphi|_{1,\Omega} = (\int |\nabla\varphi|^2 dx)^{\frac{1}{2}}$.

4. Existence of the optimal solution:



The schematic diagram for the boundary detection problem

We suppose that Γ_0 is defined by a graph of a continuous function $y = \varphi(x)$.
Let $\Gamma_0 = \{(x,y)/y = \varphi(x), x \in [0,1]\}$

We define Ω by: $\Omega = \Omega(\varphi) = \{(x,y)/0 < x < 1, 0 < y < \varphi(x)\}$
and $\jmath$ is in the form of: $\jmath(\Omega(\varphi)) = \int_0^1 (\frac{\partial u_\Omega}{\partial n}(x,0) - g)^2 dx = \int_0^1 (\frac{\partial u_\Omega}{\partial n}(x) - g)^2 dx$
we define the space U_{ad} and the family of domain θ_{ad} by:

$$U_{ad} = \left\{ \begin{array}{c} \varphi \in \mathcal{C}^1[0,1]/c_1 \le \varphi(x) \le c_2 \quad for \quad x \in [0,1], \varphi(0) = a; \varphi(1) = b \\ |\varphi^{'}(x)| \le K \quad for \quad x \in [0,1] \\ |\varphi^{'}(x) - \varphi^{'}(x^{'})| < c_0 |x - x^{'}| \quad for \quad x, x^{'} \in [0,1] \end{array} \right\}$$

$\theta_{ad} = \{\Omega(\varphi)/\varphi \in U_{ad}\}$
where c_0, c_1 and K are a given strict positives constants.

4.1. **Compacity of** F_1. We define the set:

$$\mathrm{F}_1 = \{(\Omega, \omega(\Omega))/\Omega \in \theta_{ad} \text{ and } \omega(\Omega) \text{ is solution of (8) in } \Omega\} \qquad (11)$$

Then the shape optimization problem is as follows:

$$\text{Minimize } \jmath(\Omega, \omega(\Omega)) \text{ for } (\Omega, \omega(\Omega)) \in \mathrm{F}_1 \qquad (12)$$

The existence of the optimal solution of (12) is assured if F_1 is compact and if the functional $\jmath$ is semi-continuous inferiorly in F_1.

We define a topology in θ_{ad} by:

Definition 1: Let $\Omega_n = \Omega(\varphi_n)$ a sequence in θ_{ad} and $\Omega = \Omega(\varphi)$ element of θ_{ad}

$$\Omega_n \to \Omega \Leftrightarrow \varphi_n \to \varphi \text{ uniformly on } [0,1] \tag{13}$$

The domains of family θ_{ad} are Lipschitz boundary; we can uniformly extend any function ω of $H_D(\Omega)$ in a function $\tilde{\omega}$ on $H^1(D)$ [3].

Proposition 1: It exist a constant c such that $\forall\Omega \in \theta_{ad}, \forall\omega \in H_D(\Omega)$. It exist $\tilde{\omega}$ extension of ω in $H^1(D)$ that verify:

$$\|\tilde{\omega}\|_{1,D} \leq c\|\omega\|_{1,\Omega} \text{ in } \tilde{\omega}_{/\Omega} = \omega \text{ p.p on } \Omega.$$

For any sequence $(\Omega_n)_n$ of θ_{ad}, we associate the sequence of solution $\omega_n = \omega(\Omega_n)$ of (8) on Ω_n for all n. We define the convergence of ω_n to $\omega = \omega(\Omega)$ such a weak convergence of the uniform extension of ω_n to the uniform extension of ω in $H^1(D)$, and we can write:

$$\omega_n \rightharpoonup \omega \Leftrightarrow \tilde{\omega}_n \to \tilde{\omega} \text{ in } H^1(D)\text{-weak} \tag{14}$$

Then, we can define a topology on F_1 by:

Definition 2: let (Ω_n, ω_n) a sequence of (Ω_n, ω_n) and (Ω, ω) element of F_1. We define the convergence of (Ω_n, ω_n) to (Ω, ω) by:

$$(\Omega_n, \omega_n) \to (\Omega, \omega) \Leftrightarrow \begin{cases} \Omega_n \to \Omega & \text{in the sens of (13)} \\ \omega_n \rightharpoonup \omega & \text{in the sens of (14)} \end{cases} \tag{15}$$

We use the following theorem that give the existence and the solution of the problem (12)[6].

Theorem 1: If F_1 is compact and the functional J is semi continuous inferiorly, then (12) admits at least one solution.

4.2. **Compacity of** F_1**.** For this, we should study the compacity of θ_{ad} for the convergence (15) and the continuity of the state equation.

4.2.1. *Compacity of* θ_{ad}. It suffices to show that U_{ad} is compact in $\mathbf{C^1}([0,1])$.
Indeed; let $(\varphi_n)_n$ a sequence of U_{ad}.
According to Ascoli-Arzela theorem [6], it exists a subsequence that we note $(\varphi_n)_n$ and a continuous function φ in $[0,1]$ such that $\varphi_n \to \varphi$ in $[0,1]$, in addition, φ is K-Lipschitzienne. More; $(\varphi_n) \in U_{ad}$, then $(\varphi_n^{'})$ is equicontinuous, therefore, relatively compact. It exists then a continuous element φ^* and a subsequence of $(\varphi_n^{'})$ also noted $(\varphi_n^{'})$ that converge to φ^*. Otherwise; (φ_n) is a sequence of derivable function in $[0,1]$, $(\varphi_n(.))$ converge to φ and $(\varphi_n^{'})$ uniformly converge in $[0,1]$. According to the theorem of derivability of sequence, we deduce $\varphi^* = \varphi^{'}$.
And we has $\varphi_n(x) \to \varphi(x)$ in $[0,1]$ since $\varphi_n(0) = a \Rightarrow \varphi(0) = a$
and $\varphi_n(1) = b \Rightarrow \varphi(1) = b$.

We deduce that $\varphi \in U_{ad}$
Thus; U_{ad} is compact in $C^1([0,1])$.

4.2.2. *Continuity of the state:* Whether $(\Omega_n)_n$ a sequence in θ_{ad}, we can extract a subsequence, denoted again (Ω_n) such that: $\Omega_n \to \Omega$ (Compacity of θ_{ad}).
We define: $H_D(\Omega_n) = \{v \in H^1(\Omega_n)/v_{/\Gamma_1} = 0\}$.

Whether $\omega_n = \omega(\Omega_n)$ solution of (8) on Ω_n, we have:
$\omega_n \in H_D(\Omega_n)$, $\forall v_n \in H_D(\Omega_n)$

$$\begin{aligned} &\int_{\Omega_n} \nabla\omega_n.\nabla\ v_n dxdy + \tfrac{\alpha_0}{\beta_0}\int_{\Sigma_n} \omega_n v_n d\sigma + \tfrac{\alpha_1}{\beta_1}\int_{\Gamma_{0,n}} \omega_n v_n d\sigma \\ &= -\int_{\Omega_n} \nabla U^0 \nabla v_n dxdy + \tfrac{1}{\beta_0}\int_{\Sigma_n}(h-\alpha_0 U^0)v_n d\sigma + \tfrac{1}{\beta_1}\int_{\Gamma_{0,n}}(q-\alpha_1 U^0)v_n d\sigma \end{aligned} \tag{16}$$

We cite the following results that will be useful later:

Proposition 2:[7] If Ω_n is a sequence of θ_{ad} and Ω element of θ_{ad} such that : $\Omega_n \to \Omega$ then:

$$\chi_{\Omega_n} \to \chi_{\Omega} \text{ in } L^\infty(D) - weak*$$

in addition, :

$$\lim_{n\to\infty} \int_D (\chi_{\Omega_n} - \chi_\Omega)^2 f dx = 0,\ \forall f \in L^1(D)$$

χ_A denote the characteristic function of a measurable set A.

Theorem 2: For $\Omega_n \in \theta_{ad}$ and for $\omega_n \in H_D(\Omega_n)$, it exists $\tilde{\omega}_n$ extention of ω_n in $H^1(D)$ and c constant such that: $\|\tilde{\omega_n}\|_{1,D} \le c$.

We define $H_0(D) = \{v \in H^1(D)/v_{\Gamma_1\cup(\partial D\backslash\partial\Omega)} = 0\}$ equipped with the norm induced by $H^1(D)$.

Lemma: $H_0(D)$ is dense in $H_D(\Omega)$ for the norm $H^1(\Omega)$.

Theorem 3:(Theorem of continuity) There exists an extension $\tilde{\omega}_n$ of ω_n in $H^1(D)$ which converge weakly in $H^1(D)$ to a limit which we denote $\tilde{\omega}$ such that its restriction on Ω is a solution of (8) in Ω.
i.e. There exists $\tilde{\omega}_n$ uniform extension of ω_n in $H^1(D)$ such that:

$$\tilde{\omega}_n \rightharpoonup \tilde{\omega} \text{ weak - } H^1(D) \text{ and } \tilde{\omega}_{/\Omega} = \omega \in H_D(\Omega)$$

and ω satisfies the variational formulation of (8) $\forall v \in H_D(\Omega)$.
Therefore; $\tilde{\omega_n} + U^0 \rightharpoonup \tilde{\omega} + U^0$ in $H^1(D)$ weak.
$\tilde{U}_n = \tilde{\omega}_n + U^0$ (resp. $\tilde{U} = \tilde{\omega} + U^0$) is solution of (7) in (Ω_n) (resp. in Ω)

Proof. From the previous theorem $(\tilde{\omega}_n)_n$ is uniformly bounded.
So; we can extract a subsequence, still noted $(\tilde{\omega}_n)$, which converges weakly to a limit denoted $\tilde{\omega}$.
That, it suffices to show that: $\tilde{\omega}_{/\Omega} = \omega$ is a solution of the variational formulation.
To do this, we will show that both assertions are true:

$$(i) \quad \tilde{\omega}_{/\Omega} = \omega \in H_D(\Omega)$$

$$(ii) \quad \int_\Omega \nabla\omega.\nabla v dxdy + \tfrac{\alpha_0}{\beta_0}\int_\Sigma \omega.v d\sigma + \tfrac{\alpha_1}{\beta_1}\int_{\Gamma_0}\omega.v d\sigma$$

$$= -\int_\Omega \nabla U^0.\nabla v dxdy + \tfrac{1}{\beta_0}\int_\Sigma (h-\alpha_0 U^0) v d\sigma + \tfrac{1}{\beta_1}\int_{\Gamma_0}(q-\alpha_1 U^0) v d\sigma, \forall v \in H_0(D)$$

(i) We have $\tilde{\omega}_{/\Omega} = \omega \in H^1(\Omega)$.
In addition; we have:

$$\tilde{\omega}_n \rightharpoonup \tilde{\omega} \text{ in } H^1(D)\text{- weak.}$$

and using the continuity and the linearity of the trace application from $H^1(D)$ to $L^2(\Gamma_1)$ we have:

$$\tilde{\omega}_{n/\Gamma_1} \rightharpoonup \tilde{\omega}_{/\Gamma_1} \text{ in } L^2(\Gamma_1)\text{-weak.}$$

i.e.

$$\int_{\Gamma_1}\tilde{\omega}_n.v d\sigma \to \int_{\Gamma_1}\tilde{\omega}.v d\sigma \ , \forall v \in H^1(D)$$

and since;

$$\int_{\Gamma_1}\tilde{\omega}_n.v d\sigma \to 0 \text{ then } \int_{\Gamma_1}\tilde{\omega}.v d\sigma \to 0$$

we have;

$$\tilde{\omega}_{/\Gamma_1} = 0$$

and then;

$$\omega \in H_D(\Omega)$$

(ii) Remain to prove that ω verifies the variational formulation for $v \in H_0(D)$.
For every $v \in H_0(D)$ and any n, we have $v \in H_D(\Omega_n)$.
Therefore we have:

$$\int_{\Omega_n}\nabla\tilde{\omega}_n.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma \tilde{\omega}_n.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_{0,n}}\tilde{\omega}_n.v d\sigma$$

$$= -\int_{\Omega_n}\nabla U^0.\nabla v dxdy + \frac{1}{\beta_0}\int_\Sigma (h-\alpha_0 U^0).v d\sigma + \frac{1}{\beta_1}\int_{\Gamma_{0,n}}(q-\alpha_1 U^0).v d\sigma,$$

$\forall v \in H_0(D)$

By passing to the limit, when $n \to \infty$, we get: $\tilde{\omega}_\Omega$ solution of :

$$\int_\Omega \nabla\tilde{\omega}.\nabla v dxdy + \frac{\alpha_0}{\beta_0}\int_\Sigma \tilde{\omega}.v d\sigma + \frac{\alpha_1}{\beta_1}\int_{\Gamma_0}\tilde{\omega}.v d\sigma$$

$$= -\int_\Omega \nabla U^0.\nabla v dxdy + \frac{1}{\beta_0}\int_\Sigma (h-\alpha_0 U^0).v d\sigma + \frac{1}{\beta_1}\int_{\Gamma_0}(q-\alpha_1 U^0).v d\sigma,$$

$\forall v \in H_0(D)$.
Indeed; $\forall v \in H_0(D)$, we put:

$$\begin{array}{lcl} I_1 & = & \int_{\Omega_n} \nabla\tilde{\omega}_n.\nabla v dxdy - \int_{\Omega} \nabla\tilde{\omega}.\nabla v dxdy \\ I_2 & = & \int_{\Sigma} \tilde{\omega}_n.v d\sigma - \int_{\Sigma} \tilde{\omega}.v d\sigma \\ I_3 & = & \int_{\Gamma_{0,n}} \tilde{\omega}_n.v d\sigma - \int_{\Gamma_0} \tilde{\omega}.v d\sigma \\ I_4 & = & \int_{\Omega_n} \nabla U^0.\nabla v dxdy - \int_{\Omega} \nabla U^0.\nabla v dxdy \\ I_5 & = & \int_{\Gamma_{0,n}} (q-\alpha_1 U^0).v d\sigma - \int_{\Gamma_0} (q-\alpha_1 U^0).v d\sigma \end{array} \tag{17}$$

It suffices to prove that:

$$\begin{array}{lcl} lim_{n\to\infty} I_1 = 0 & ; & lim_{n\to\infty} I_2 = 0 \quad lim_{n\to\infty} I_3 = 0 \\ lim_{n\to\infty} I_4 = 0 & ; & lim_{n\to\infty} I_5 = 0 \end{array} \tag{18}$$

- For (I_1); we have :

$$I_1 = \int_D (\chi_{\Omega_n} - \chi_\Omega)\nabla\tilde{\omega}_n.\nabla v dxdy + \int_D \chi_\Omega(\nabla\tilde{\omega}_n - \nabla\tilde{\omega}).\nabla v dxdy$$

Then :

$$|I_1| \leq |\int_D (\chi_{\Omega_n} - \chi_\Omega)\nabla\tilde{\omega}_n.\nabla v dxdy| + |\int_D \chi_\Omega(\nabla\tilde{\omega}_n - \nabla\tilde{\omega}).\nabla v dxdy|$$

On the one hand by Holder's inequality;

$$\begin{array}{lcl} |\int_D (\chi_{\Omega_n} - \chi_\Omega)\nabla\tilde{\omega}_n.\nabla v dxdy| & \leq & \int_D |\chi_{\Omega_n} - \chi_\Omega||\nabla\tilde{\omega}_n||\nabla v| dxdy \\ & \leq & [\int_D |\nabla\tilde{\omega}_n|^2]^{\frac{1}{2}} [\int_D (\chi_{\Omega_n} - \chi_\Omega)^2 |\nabla v|^2 dxdy]^{\frac{1}{2}} \\ & \leq & \|\tilde{\omega}_n\|_{1,D} [\int_D (\chi_{\Omega_n} - \chi_\Omega)^2 |\nabla v|^2 dxdy]^{\frac{1}{2}} \end{array}$$

According to the previous proposition and using the previous theorem, we have:

$$\lim_{n\to\infty} \int_D (\chi_{\Omega_n} - \chi_\Omega)\nabla\tilde{\omega}_n.\nabla v dxdy = 0$$

Moreover; since we have the convergence: $\tilde{\omega}_n \rightharpoonup \tilde{\omega}$ in $H^1(D)$- weak
And using the linearity of the application of gradient $H^1(D)$ in $L^2(D)$, we also have :

$$\nabla\tilde{\omega}_n \rightharpoonup \nabla\tilde{\omega} \text{ in } L^2(D)\text{- weak ;}$$

And since $\chi_\Omega \nabla v \in L^2(D)$, we have:

$$\int_D \chi_\Omega(\nabla\tilde{\omega}_n - \nabla\tilde{\omega}).\nabla v dxdy = 0$$

Accordingly;

$$lim_{n\to\infty} I_1 = 0.$$

- For (I_2):
We have: $I_2 = \int_\Sigma \tilde{\omega}_n.v d\sigma - \int_\Sigma \tilde{\omega}.v d\sigma = \int_\Sigma (\tilde{\omega}_n - \tilde{\omega}).v d\sigma$.
Then, according to the continuity and the linearity of the trace application of $H^1(D)$ in $L^2(\Sigma)$; we have:

$$lim_{n\to\infty} I_2 = 0$$

- For (I_4):
Applying the inequality of Holder to the following inequality:

$$|I_4| \leq |\int_D (\chi_{\Omega_n} - \chi_\Omega)\nabla U^0.\nabla v dxdy|$$

We obtain:

$$|I_4| \leq \|U^0\|_{1,D}[\int_D (\chi_{\Omega_n} - \chi_\Omega)^2 |\nabla v|^2 dxdy]^{\frac{1}{2}}$$

And since $U^0 \in H^1(D)$ it exists c such that: $\|U^0\|_{1,D} \leq c$
And according to previous proposition, we have:

$$lim_{n\to\infty} I_4 = 0$$

- For (I_3) ,we have :

$$\begin{aligned} I_3 &= \int_{\Gamma_{0,n}} \tilde{\omega}_n.vd\sigma - \int_{\Gamma_0} \tilde{\omega}.vd\sigma \\ &= \int_{\Gamma_0} (\tilde{\omega}_n - \tilde{\omega}).vd\sigma + \int_{\Gamma_{0,n}} \tilde{\omega}_n.vd\sigma - \int_{\Gamma_0} \tilde{\omega}_n.vd\sigma \end{aligned}$$

Using the linearity and the continuity of the trace application of $H^1(D)$ in $L^2(\Gamma_0)$, we have:

$$\tilde{\omega}_{n/L^2(\Gamma_0)} \rightharpoonup \tilde{\omega}_{/L^2(\Gamma_0)} \text{ in } L^2(\Gamma_0)\text{- weak}$$

hence;

$$\lim_{n\to\infty} \int_{\Gamma_0} (\tilde{\omega}_n - \tilde{\omega}).vd\sigma \to 0$$

on the other hand,
$\int_{\Gamma_{0,n}} \tilde{\omega}_n.vd\sigma - \int_{\Gamma_0} \tilde{\omega}_n.vd\sigma$
$= \int_0^1 \tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))\sqrt{1+\varphi_n'(x)^2}dx - \int_0^1 \tilde{\omega}_n(x,\varphi(x))v(x,\varphi(x))\sqrt{1+\varphi'(x)^2}dx$
$= \int_0^1 \tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))(\sqrt{1+\varphi_n'(x)^2} - \sqrt{1+\varphi'(x)^2})dx$
$+ \int_0^1 (\tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x)) - \tilde{\omega}_n(x,\varphi(x))v(x,\varphi(x)))\sqrt{1+\varphi_n'(x)^2}dx$

Let:

$$I_{3,1} = \int_0^1 \tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))(\sqrt{1+\varphi_n'(x)^2} - \sqrt{1+\varphi'(x)^2})dx$$

and,

$$I_{3,2} = \int_0^1 (\tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x)) - \tilde{\omega}_n(x,\varphi(x))v(x,\varphi(x)))\sqrt{1+\varphi'(x)^2}dx$$

We have:

$$\begin{aligned} |I_{3,1}| &\leq \sup_{x\in[0,1]}(|\varphi_n'(x) - \varphi'(x)|)\int_0^1 \tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))dx \\ &\leq \sup_{x\in[0,1]}(|\varphi_n'(x) - \varphi'(x)|)(\int_0^1 (\tilde{\omega}_n(x,\varphi_n(x)))^2 dx)^{\frac{1}{2}}(\int_0^1 (v(x,\varphi_n(x)))^2 dx)^{\frac{1}{2}} \end{aligned}$$

According to theorem of the mean;
It exists $\bar{x} \in [0,C]$ such that :$\int_0^1 \tilde{\omega}_n^2(x,\bar{x})dx = \frac{1}{C}\int_0^C \int_0^1 \tilde{\omega}_n^2(x,y)dxdy$
then:

$$\tilde{\omega}_n(x,\varphi_n(x)) = \tilde{\omega}_n(x,\bar{x}) + \int_{\bar{x}}^{\varphi_n(x)} \frac{\partial \tilde{\omega}_n}{\partial y}(x,y)dy$$

From which :

$$\begin{array}{rcl}\tilde{\omega}_n^2(x,\varphi_n(x)) & \leq & 2[\tilde{\omega}_n^2(x,\bar{x})+(\int_{\bar{x}}^{\varphi_n(x)}\frac{\partial\tilde{\omega}_n}{\partial y}(x,y)dy)^2] \\ & \leq & 2[\tilde{\omega}_n^2(x,\bar{x})+(\varphi_n(x)-\bar{x})(\int_{\bar{x}}^{\varphi_n(x)}(\frac{\partial\tilde{\omega}_n}{\partial y}(x,y))^2dy)]\end{array}$$

then :

$$\begin{array}{rcl}\int_0^1(\tilde{\omega}_n(x,\varphi_n(x)))^2dx & \leq & 2[\int_0^1\tilde{\omega}_n^2(x,\bar{x})dx+C'\int_0^1\int_{\bar{x}}^{\varphi_n(x)}(\frac{\partial\tilde{\omega}_n}{\partial y}(x,y))^2dydx] \\ & \leq & \frac{2}{c}\|\tilde{\omega}_n\|_{L^2(D)}+2C'\|\tilde{\omega}_n\|_{1,D}\end{array}$$

Using the Poincare inequality, we have:

$$\int_0^1(\tilde{\omega}_n(x,\varphi_n(x)))^2dx \leq C''\|\tilde{\omega}_n\|_{1,D} \leq k$$

As far as;

$$\int_0^1(v(x,\varphi_n(x))^2dx \leq k'$$

And since: $\sup_{x\in[0,1]}(|\varphi_n'(x)-\varphi'(x)|)\to 0$ pour $n\to\infty$; then: $lim_{n\to\infty}I_{3,1}=0$. and we have:

$$\begin{array}{rcl}|I_{3,2}| & \leq & |\int_0^1(\tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))-\tilde{\omega}_n(x,\varphi(x))v(x,\varphi(x)))\sqrt{1+\varphi'(x)^2}dx| \\ & \leq & c\int_0^1|\tilde{\omega}_n(x,\varphi_n(x))v(x,\varphi_n(x))-\tilde{\omega}_n(x,\varphi(x))v(x,\varphi(x))|dx \\ & \leq & c(\int_0^1|\tilde{\omega}_n(x,\varphi_n(x))(v(x,\varphi_n(x))-v(x,\varphi(x)))| \\ & & +\int_0^1|(\tilde{\omega}_n(x,\varphi_n(x))-\tilde{\omega}_n(x,\varphi(x)))v(x,\varphi(x))|) \\ & \leq & c(\int_0^1(\tilde{\omega}_n(x,\varphi_n(x))\int_{\varphi(x)}^{\varphi_n(x)}\frac{\partial v(x,y)}{\partial y}dy)dx+\int_0^1(\int_{\varphi(x)}^{\varphi_n(x)}\frac{\partial\tilde{\omega}_n(x,y)}{\partial y}dy)v(x,\varphi(x))dx)\end{array}$$

By using Holder, we will have:

$$\begin{array}{rcl}|I_{3,2}|^2 & \leq & 2c^2\sup_{x\in[0,1]}|\varphi_n(x)-\varphi(x)|(\|\tilde{\omega}_n(.,\varphi_n(.))\|_{L^2([0,1])}\|v\|_{1,D} \\ & & +\|\tilde{\omega}\|_{1,D}\|v(.,\varphi(.))\|_{L^2([0,1])}) \\ & \leq & c'sup_{x\in[0,1]}|\varphi_n(x)-\varphi(x)|\end{array}$$

The Uniform convergence of φ_n to φ in $[0,1]$ then: $lim_{n\to\infty}I_{3,2}=0$
Hence :

$$lim_{n\to\infty}I_3=0$$

And similarly, we show that: $lim_{n\to\infty}I_5=0$

4.3. **Semi-continuity of the cost functional.** Considering $(\Omega_k)_k$ a minimizing sequence of $\jmath$ on θ_{ad}
i.e.

$$lim_{k\to\infty}\jmath(\Omega_k)=\min_{\Omega^*\in\theta_{ad}}\jmath(\Omega^*)$$

Based to the above, there exists a subsequence still noted $(\Omega_k)_k$ and an element $\Omega\in\theta_{ad}$ such that $\Omega_k\to\Omega$.
The functional $\jmath$ definied on θ_{ad} by: $\jmath(\Omega)=\jmath(\Omega,u(\Omega))=\int_{\Gamma_1}(\frac{\partial u}{\partial n}-g)^2d\sigma$ is semi-continuous inferiorly on θ_{ad}

Indeed;

$$\begin{aligned} \jmath(\Omega_k)-\jmath(\Omega) &= \int_{\Gamma_1}(\frac{\partial u_k}{\partial n}-g)^2 d\sigma - \int_{\Gamma_1}(\frac{\partial u}{\partial n}-g)^2 d\sigma \\ &= \int_{\Gamma_1}(\frac{\partial u_k}{\partial n}-\frac{\partial u}{\partial n})(\frac{\partial u_k}{\partial n}+\frac{\partial u}{\partial n}-2g)d\sigma \\ &\leq (\int_{\Gamma_1}(\frac{\partial u_k}{\partial n}-\frac{\partial u}{\partial n})^2 d\sigma)^{\frac{1}{2}}(\int_{\Gamma_1}(\frac{\partial u_k}{\partial n}+\frac{\partial u}{\partial n}-2g)^2 d\sigma)^{\frac{1}{2}} \\ &\leq \|(\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n})\|_{L^2(\Gamma_1)}(\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}+\frac{\partial \tilde{u}}{\partial n}-2g)^2 d\sigma)^{\frac{1}{2}} \end{aligned}$$

We have: $\|(\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n})\|_{L^2(\Gamma_1)} \leq c\|\tilde{u}_k-\tilde{u}\|_{H^1(D)}$
And since $\tilde{u}_k \to \tilde{u}$ in $H^1(D)$
then $\|(\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n})\|_{L^2(\Gamma_1)} \to 0$
On the other hand;

$$\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}+\frac{\partial \tilde{u}}{\partial n}-2g)^2 d\sigma \leq 3[\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n})^2 d\sigma + \int_{\Gamma_1}(\frac{\partial \tilde{u}}{\partial n})^2 d\sigma + \int_{\Gamma_1} 4g^2 d\sigma]$$

And since

$$\begin{aligned} \int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n})^2 d\sigma &\leq 2[\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}-g)^2 d\sigma + \int_{\Gamma_1} g^2 d\sigma] \\ &\leq c_1 \end{aligned}$$

($\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}-g)^2 d\sigma = \jmath(\Omega_k)$ is bounded et $g \in L^2(\Gamma_1)$)
and

$$\begin{aligned} \int_{\Gamma_1}(\frac{\partial \tilde{u}}{\partial n})^2 d\sigma &\leq 2[\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n})^2 d\sigma + \int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n})^2 d\sigma] \\ &= 2[\|\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n}\|^2_{L^2(\Gamma_1)} + \|\frac{\partial \tilde{u}_k}{\partial n}\|^2_{L^2(\Gamma_1)}] \\ &\leq 2c_2 \end{aligned}$$

($\|\frac{\partial \tilde{u}_k}{\partial n}-\frac{\partial \tilde{u}}{\partial n}\|^2_{L^2(\Gamma_1)} \to 0$ et $\|\frac{\partial \tilde{u}_k}{\partial n}\|^2_{L^2(\Gamma_1)} \leq c_2$)
then; $\int_{\Gamma_1}(\frac{\partial \tilde{u}_k}{\partial n}+\frac{\partial \tilde{u}}{\partial n}-2g)^2 d\sigma \leq C$
Hence;

$$\jmath(\Omega_k)-\jmath(\Omega) \to 0$$

Hence; the semi-continuity inferior of the cost functional.

5. Conclusion

In this paper, we have considered a boundary detection problem governed by Laplace's equation, with a Cauchy conditions in the accessible part of the boundary and Robin condition on the inaccessible part and the other part of the boundary. We have proposed a formulation of the problem in a shape optimization problem by introducing the Neumann condition of the accessible part in a cost functional to be minimized. The existence of the problem has been shown.

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Chakir Tajani was born in Meknes, Morocco. He received the degree of Doctor of Science, discipline applied mathematics from Ibn Tofail University, Faculty of sciences, Kenitra, Morocco in 2012. His research interest area includes inverse problem, shape optimization and computer science.

Jaafar Abouchabaka is a Professor in the computer science Department at Ibn Tofail University, Kenitra, Morocco, where he directs the unit in image processing and multimedia. He received his Ph.D. in applied mathematics in 2001. He taught many computer science courses for students since 1992. He had coordinate two formations, the first in the software engineering and the second in Networks and operating system. He had some publications in applied mathematics, in parallel computing and in Computer Science. His current research interests are in optimization, parallel algorithms, Multi-agent, Image processing

Nadia Samouh is a Professor in the mathematics Department at Moulay Ismail University, Meknes, Morocco. He received his Ph.D in applied mathematics in 2002. His research interest area includes inverse problem and shape optimization.