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ON LINE AND DOUBLE MULTIPLICATIVE INTEGRALS

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ABSTRACT. In the present paper the concepts of line and double integrals are modified to the multiplicative case. Two versions of the fundamental theorem of calculus for line and double integrals are proved in the multiplicative case.

Keywords: Multiplicative calculus, multiplicative line integral, independence on paths, Green's theorem.

AMS Subject Classification: 26B12, 97I40, 97I50.

1. INTRODUCTION

In 1972 Grossman and Katz [8] pointed out to different calculi, called non-Newtonian calculi, which modify the calculus created by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century. Since then a number of works has been done in this area. In Stanley [14], Bashirov et al. [2, 3, 4] and Riza et al. [12] a most popular non-Newtonean calculus, namely, multiplicative calculus is handled. Some elements of stochastic multiplicative calculus are concerned in the works of Karandikar [10] and Daletskii and Teterina [7]. Bashirov and Riza [5] studied complex multiplicative calculus. Another popular non-Newtonian calculus, namely, bigeometric calculus is investigated in Volterra and Hostinsky [15], Grossman [9], Aniszewska [1], Kasprzak et al. [11], Rybaczuk et al. [13], Córdova-Lepe [6].

In this paper we study functions of two variables from multiplicative point of view. Our aim is a presentation of two fundamental theorems of multiplicative calculus for line and double integrals. These theorems are useful dealing with different applications of multiplicative calculus, for example, in studies on multiplicative complex calculus.

One major notation is that the multiplicative versions of the concepts of Newtonian calculus will be called *concepts. For example, *derivative is same as multiplicative derivative, *integral as multiplicative integral etc. As always, for *derivative of the function f we use the symbol f^* distinguishing it from the ordinary derivative f'. But *integral of the function f is denoted by

$$\int_a^b f(x)^{dx},$$

which reflects the way of its definition and differs from the symbol of ordinary integral. The same kind of symbols will be used for multiplicative partial derivatives, line and double integrals.

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2. Line Multiplicative Integrals

Let f be a positive function of two variables, defined on an open connected set in $\mathbb{R}^2$, and let C be a piecewise smooth curve in the domain of f. Take a partition $\mathcal{P} = \{P_0, \ldots, P_m\}$ on C and let (ξ_k, η_k) be a point on C between P_{k-1} and P_k. Denote by Δs_k the arclength of C from the point P_{k-1} to P_k. According to the definition of *integral from Bashirov et al. [3], define the integral product

$$P(f, \mathcal{P}) = \prod_{k=1}^{m} f(\xi_k, \eta_k)^{\Delta s_k}.$$

The limit of this product when $\max\{\Delta s_1, \ldots, \Delta s_m\} \to 0$ independently on selection of the points (ξ_k, η_k) will be called a *line *integral of f in ds along C*, for which we will use the symbol

$$\int_C f(x,y)^{ds}.$$

From

$$\prod_{k=1}^{m} f(\xi_k, \eta_k)^{\Delta s_k} = e^{\sum_{k=1}^{m} \ln f(\xi_k, \eta_k) \Delta s_k},$$

it is clearly seen that the line *integral of f along C exist if f is a positive function and the line integral of $\ln f$ along C exists, and they are related as

$$\int_C f(x,y)^{ds} = e^{\int_C \ln f(x,y)\, ds}.$$

The following properties of line *integrals in ds can be proved easily:

(a) $\displaystyle\int_C (f(x,y)^p)^{ds} = \Big(\int_C f(x,y)^{ds}\Big)^p,\ p \in \mathbb{R},$

(b) $\displaystyle\int_C (f(x,y)g(x,y))^{ds} = \int_C f(x,y)^{ds} \cdot \int_C g(x,y)^{ds},$

(c) $\displaystyle\int_C (f(x,y)/g(x,y))^{ds} = \int_C f(x,y)^{ds} / \int_C g(x,y)^{ds},$

(d) $\displaystyle\int_C f(x,y)^{ds} = \int_{C_1} f(x,y)^{ds} \cdot \int_{C_2} f(x,y)^{ds},\ C = C_1 + C_2,$

where $C = C_1 + C_2$ means that the curve C is divided into two pieces at some interior point of $[a, b]$.

In a similar way, we can introduce the line *integrals in dx and in dy and establish their relation to the respective line integrals in the form

$$\int_C f(x,y)^{dx} = e^{\int_C \ln f(x,y)\, dx} \quad \text{and} \quad \int_C f(x,y)^{dy} = e^{\int_C \ln f(x,y)\, dy}. \tag{1}$$

Clearly, all three kinds of line *integrals exist if f is a positive continuous function. The above mentioned properties of the line *integrals in ds are valid for line *integrals in dx and in dy as well. Additionally,

$$\int_C f(x,y)^{dx} = \Big(\int_{-C} f(x,y)^{dx}\Big)^{-1}$$

and

$$\int_C f(x,y)^{dy} = \Big(\int_{-C} f(x,y)^{dy}\Big)^{-1}$$

while

$$\int_C f(x,y)^{ds} = \int_{-C} f(x,y)^{ds},$$

where $-C$ is the curve C with the opposite orientation. Moreover, the following evaluation formulae for the line *integrals are also easily seen:

$$\text{(a)} \quad \int_C f(x,y)^{ds} = \int_a^b \left(f(x(t),y(t))^{\sqrt{x'^2+y'^2}} \right)^{dt},$$

$$\text{(b)} \quad \int_C f(x,y)^{dx} = \int_a^b \left(f(x(t),y(t))^{x'(t)} \right)^{dt},$$

$$\text{(c)} \quad \int_C f(x,y)^{dy} = \int_a^b \left(f(x(t),y(t))^{y'(t)} \right)^{dt},$$

where $\{(x(t),y(t)) : a \le t \le b\}$ is a suitable parametrization of C and $\int_a^b g(t)^{dt}$ is the *integral of g on the interval $[a,b]$ (see, Bashirov et al. [3]). It is also suitable to denote

$$\int_C f(x,y)^{dx} g(x,y)^{dy} = \int_C f(x,y)^{dx} \cdot \int_C g(x,y)^{dy}.$$

In cases when C is a closed curve we write $\oint_C$ instead of $\int_C$.

Example 2.1. *Let $c \ne 0$ and let $C = \{(x(t),y(t)) : a \le t \le b\}$ be a piecewise smooth curve. Then*

$$\int_C c^{dx} = e^{\int_C \ln c\, dx} = e^{(x(b)-x(a)) \ln c} = c^{x(b)-x(a)}.$$

In order to state and prove fundamental theorem of calculus for line *integrals, we need in partial *derivatives. Recall that *derivative of the function f of one variable, which is assumed to be positive and differentiable in the ordinary sense, is defined as

$$f^*(x) = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}},$$

and it has the following relation to the ofdinary derivative:

$$f^*(x) = e^{(\ln f(x))'} = e^{\frac{d}{dx} \ln f(x)}.$$

Based on this, it is natural to define the partial derivatives of the function f of two variables as

$$f^*_x(x,y) = e^{\frac{\partial}{\partial x} \ln f(x,y)} \quad \text{and} \quad f^*_y(x,y) = e^{\frac{\partial}{\partial y} \ln f(x,y)}.$$

Theorem 2.1 (Fundamental theorem of calculus for line *integrals)**.** *Let $D \subseteq \mathbb{R}^2$ be an open connected set and let $C = \{(x(t),y(t)) : a \le t \le b\}$ be a piecewise smooth curve in D. Assume that f is a continuously differentiable positive function on D. Then*

$$\int_C f^*_x(x,y)^{dx} f^*_y(x,y)^{dy} = f(x(b),y(b))/f(x(a),y(a)).$$

Proof. From the fundamental theorem of calculus for line integrals,

$$\begin{aligned} \int_C f^*_x(x,y)^{dx} f^*_y(x,y)^{dy} &= e^{\int_C (\ln f^*_x(x,y)\, dx + \ln f^*_y(x,y)\, dy)} \\ &= e^{\int_C ([\ln f]'_x(x,y)\, dx + [\ln f]'_y(x,y)\, dy)} \\ &= e^{\ln f(x(b),y(b)) - \ln f(x(a),y(a))} \\ &= f(x(b),y(b))/f(x(a),y(a)). \end{aligned}$$

This proves the theorem. □

3. Green's Theorem in Multiplicative Form

As far as line *integrals are concerned, we can present another fundamental theorem of *calculus related to line *integrals, that is the Green's theorem in *form. Let f be a bounded positive function f, defined on the Jordan set $D \subseteq \mathbb{R}^2$. Let $\mathcal{Q} = \{D_k : k = 1, \ldots, m\}$ be a partition of D. Take any $(\xi_k, \eta_k) \in D_k$ and let A_k be the area of D_k. Define the integral product

$$P(f, \mathcal{Q}) = \prod_{k=1}^{m} f(x(t), y(t))^{A_k}.$$

Denote by d_k the diameter of D_k, that is, $d_k = \sup \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where the supremum is taken over all $(x_1, y_1), (x_2, y_2) \in D_k$. The limit of the integral product when $\max\{d_1, \ldots, d_m\} \to 0$ independently on selection of the points (ξ_k, η_k) will be called a *double *integral of f on D*, for which we will use the symbol

$$\iint_D f(x, y)^{dA}.$$

A relation between double integrals and double *integrals can be easily derived as

$$\iint_D f(x, y)^{dA} = e^{\iint_D f(x,y) dA}.$$

The following properties of double *integrals can also be proved easily:

(a) $\displaystyle\iint_D (f(x, y)^p)^{dA} = \Big(\iint_D f(x, y)^{dA}\Big)^p, \ p \in \mathbb{R},$

(b) $\displaystyle\iint_D (f(x, y)g(x, y))^{dA} = \iint_D f(x, y)^{dA} \cdot \iint_D g(x, y)^{dA},$

(c) $\displaystyle\iint_D (f(x, y)/g(x, y))^{dA} = \iint_D f(x, y)^{dA} / \iint_D g(x, y)^{dA},$

(d) $\displaystyle\iint_D f(x, y)^{dA} = \iint_{D_1} f(x, y)^{dA} \cdot \iint_{D_2} f(x, y)^{dA}, \ D = D_1 + D_2,$

where $D = D_1 + D_2$ means that D_1 and D_2 are two non-overlapping Jordan sets with $D_1 \cup D_2 = D$.

Theorem 3.1 (Green's theorem in *form)**.** *Let f and g be continuously differentiable positive functions on a simply connected Jordan set $D \subseteq \mathbb{R}^2$ with the piecewise smooth and positively oriented boundary C. Then*

$$\oint_C f(x, y)^{dx} g(x, y)^{dy} = \iint_D (g_x^*(x, y)/f_y^*(x, y))^{dA}.$$

Proof. From the Green's theorem,

$$\begin{aligned}\oint_C f(x, y)^{dx} g(x, y)^{dy} &= e^{\oint_C (\ln f(x,y)\, dx + \ln g(x,y)\, dy)} \\ &= e^{\iint_D ([\ln g]'_x(x,y) - [\ln f]'_y(x,y))\, dA} \\ &= e^{\iint_D (\ln g_x^*(x,y) - \ln f_y^*(x,y))\, dA} \\ &= \iint_D (g_x^*(x, y)/f_y^*(x, y))^{dA}.\end{aligned}$$

This proves the theorem. □

4. Conclusion

Multiplicative calculus is an alternative to Newtonian calculus. The growth related problems in economics, finance, actuarial science etc. are better described and solved in terms of multiplicative calculus rather than Newtonian calculus. Since its establishment as one of the non-Newtonian calculi, a number of works are devoted to different applications of multiplicative calculus. In this paper the concepts of partial derivative, line and double integrals are considered from multiplicative point of view. Two fundamental theorems of calculus for line and double integrals are interpreted in terms of multiplicative calculus.

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