TWMS J. App. Eng. Math. V.2, N.2, 2012, pp. 195-209

A NUMERICAL SOLUTION OF AN INVERSE PARABOLIC PROBLEM

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Abstract. In this paper, we will first study the existence and uniqueness of the solution of an inverse problem for a linear equation with non-linear boundary conditions (radiation terms), via an auxiliary problem. Furthermore, a stable numerical algorithm based on the use of the solution to the auxiliary problem as a basis function is proposed. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization method to obtain the stable numerical approximation to the solution. Some numerical experiments confirm the utility of this algorithm as the results are in good agreement with the exact data.

Keywords: Inverse problem, Radiation terms, Existence and Uniqueness, Stability, Tikhonov regularization method.

AMS Subject Classification: 65M32, 35K05

The problem of determining unknown coefficients in parabolic partial differential equations has been treated by many authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], and among the most versatile methods the following can be mentioned: Tikhonov regularization [14], iterative regularization [2], mollification [15], base function method (BFM) [9], semi finite difference method (SFDM) [7], and the function specification method (FSM) [4]. Dowding and Beck [6] addressed a sequential gradient method for two dimensional inverse heat conduction problems (IHCPs) with and without function specification, additionally using the conventional regularization method.

When the radiation of heat from a solid is considered, the heat flux is often taken to be proportional to the fourth power of difference of the boundary temperature of the solid with the temperature of the surroundings, [5, Pages: 72-75]. When the thermo-physical properties are independent of position and temperature, the heat transfer problem in this situation may be derived as

$$U_t(x,t) = U_{xx}(x,t), \qquad 0 < x < 1, \qquad 0 < t < T, \tag{1a}$$

$$U(x,0) = r(x), \qquad 0 \le x \le 1, \tag{1b}$$

$$U(0,t) = \phi(U_x(0,t)) + \zeta(t), \qquad 0 \le t \le T, \tag{1c}$$

$$U(1,t) = \psi(U_x(1,t)) + \eta(t), \qquad 0 \le t \le T, \tag{1d}$$

and, for a fixed point a, $0 < a < 1$, the overspecified conditions

$$U(a,t) = f(t), \quad U_x(a,t) = g(t), \qquad 0 \le t \le T \tag{2}$$

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§ Manuscript received May 07, 2012.

where T is a given constant, $\zeta(t)$ and $\eta(t)$ are given functions, $r(x)$ is the initial temperature of solid, $\phi(U_x(0,t)) + \zeta(t)$ and $\psi(U_x(1,t)) + \eta(t)$ represent a general radiation law.

In the problem (1), if the functions ϕ and ψ are given, then we deal with a direct problem. On the other hand, when ϕ and ψ are known a priori, then under certain conditions there may exist a unique solution for the problem (1a)-(1d) and this solution may not satisfy the overspecified conditions (2).

In the problem (1), if the functions ϕ and ψ are unknown, one deals with an inverse heat transfer problem. For unknown functions ϕ and ψ, we must provide additional information, namely (2), to provide a unique solution $(U,(\phi,\psi))$ to the inverse problem (1).

In this paper, a numerical approach based on the use of the solution to the auxiliary problem as a basis function is proposed.

1. Existence and uniqueness of solution

To investigate the existence and uniqueness of solution of IHCP (1), let us consider the following auxiliary Cauchy problem:

$$U_t(x,t) = U_{xx}(x,t), \qquad 0 < x < 1, \qquad 0 < t < T, \tag{3a}$$
$$U(a,t) = f(t), \qquad 0 \le t \le T, \tag{3b}$$
$$U_x(a,t) = g(t), \qquad 0 \le t \le T, \tag{3c}$$

where f, g are given in (2). Following the same method as in [5, page 25], we assume that a solution of (3) is represented as a power series:

$$U(x,t) = \sum_{n=0}^{\infty} a_n(t)(x-a)^n,$$

where the coefficients $a_n(t)$ are to be determined. By substituting U into $U_t = U_{xx}$, we get

$$0 = U_t - U_{xx} = \sum_{n=0}^{\infty} \big[a_n'(t) - (n+1)(n+2)a_{n+2}(t)\big](x-a)^n,$$

and thus

$$a_{n+2}(t) = \frac{a_n'(t)}{(n+1)(n+2)}, \qquad (n = 0,1,2,\dots). \tag{4}$$

From the recurrent relation (4) and the fact that $f(t) = U(a,t) = a_0(t)$, $g(t) = U_x(a,t) = a_1(t)$, we see that

$$a_{2n}(t) = \frac{f^{(n)}(t)}{(2n)!}, \quad a_{2n+1}(t) = \frac{g^{(n)}(t)}{(2n+1)!}, \qquad (n = 0,1,2,\dots).$$

Thus, we obtain the following formal expression for $U(x,t)$:

$$U(x,t) = \sum_{n=0}^{\infty} \Big[f^{(n)}(t)\frac{(x-a)^{2n}}{(2n)!} + g^{(n)}(t)\frac{(x-a)^{2n+1}}{(2n+1)!}\Big]. \tag{5}$$

The power series in (5), however, may not be uniformly convergent. But, according to [5, Theorem 2.3.1], if f and g are infinitely differentiable functions that are of the *Holmgren class*, in the sense of [5, Definition 2.2.1], then the power series in (5) is uniformly convergent and U in (5) is a solution of (3). Now, take

$$p(t) = U_x(0,t), \quad q(t) = U_x(1,t). \tag{6}$$

Equation (1c) implies that

$$\phi(p(t)) = U(0,t) - \zeta(t) = \sum_{n=0}^{\infty}\Big[f^{(n)}(t)\frac{a^{2n}}{(2n)!} - g^{(n)}(t)\frac{a^{2n+1}}{(2n+1)!}\Big] - \zeta(t). \tag{7}$$

Similarly, we obtain the following result for ψ:

$$\begin{aligned}\psi(q(t)) &= U(1,t) - \eta(t)\\ &= \sum_{n=0}^{\infty}\Big[f^{(n)}(t)\frac{(1-a)^{2n}}{(2n)!} + g^{(n)}(t)\frac{(1-a)^{2n+1}}{(2n+1)!}\Big] - \eta(t). \end{aligned}\tag{8}$$

To uniquely determine ϕ and ψ from equations (7) and (8), the functions $p(t)$ and $q(t)$ are required to be invertible. Therefore, we need to impose some more conditions on f and g so that $p' > 0$ [or $p' < 0$] and $q' > 0$ [or $q' < 0$].

Theorem 1.1. *Suppose that f, g are infinitely differentiable functions that are of the Holmgren class, in the sense of* [5, Definition 2.2.1]. *Suppose that $g'(t) > 0$ for all t or $g'(t) < 0$ for all t, and f and g satisfy the following conditions:*

$$|f^{(n)}(t)| < \frac{|g'(t)|}{e-1}, \quad |g^{(n)}(t)| < \frac{|g'(t)|}{e-1}, \qquad (t > 0,\ n \geq 2). \tag{9}$$

Then p and q in (6) *are invertible and thus ϕ and ψ are uniquely determined by* (7) *and* (8)*, respectively.*

Proof. We prove that $p'(t) > 0$ for all t or $p'(t) < 0$ for all t depending weather $g'(t) > 0$ or $g'(t) < 0$. A simple calculation shows that

$$p'(t) = g'(t) + \sum_{n=1}^{\infty}\Big[g^{(n+1)}(t)\frac{a^{2n}}{(2n)!} - f^{(n+1)}(t)\frac{a^{2n-1}}{(2n-1)!}\Big]. \tag{10}$$

Now, we find an upper bound for the absolute value of the series in (10):

$$\begin{aligned}&\Big|\sum_{n=1}^{\infty}\Big[g^{(n+1)}(t)\frac{a^{2n}}{(2n)!} - f^{(n+1)}(t)\frac{a^{2n-1}}{(2n-1)!}\Big]\Big|\\ &\qquad\leq \frac{|g'(t)|}{e-1}\sum_{n=1}^{\infty}\frac{a^n}{n!} = \frac{|g'(t)|}{e-1}(e^a - 1) < |g'(t)|.\end{aligned}$$

Therefore $p'(t) \neq 0$, for all t and the continuity of $p'(t)$ implies that $p'(t) > 0$ for all t, or $p'(t) < 0$ for all t. A similar argument for $q(t)$ shows that $q'(t) > 0$ for all t, or $q'(t) < 0$ for all t. □

Remark 1.1. *Conditions* (9) *which guarantee the invertibility of p and q, are not necessary. This means that there are functions f and g that may not satisfy* (9) *and yet p, q are invertible. For example, if $f^{(n)} \equiv 0$ for $n \geq 2$ and all of subsequent derivatives of g are positive (resp. negative) then $p'(t) > 0$, $q'(t) > 0$ (resp. $p'(t) < 0$, $q'(t) < 0$), or if $g^{(n)} \equiv 0$ for $n \geq 1$ and $f^{(n)} > 0$ (resp. $f^{(n)} > 0$) for $n \geq 2$, then $p'(t) < 0$, $q'(t) > 0$ (resp. $p'(t) > 0$, $q'(t) < 0$).*

After finding ϕ, ψ from (7) and (8), for piecewise-continuous functions $r(x)$, $\zeta(t)$ and $\eta(t)$, the unique solution $U(x,t)$ of problem (1) can be expressed in the following form [5, Page 62]

$$U(x,t) = W(x,t) - 2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\big[\phi(p(\tau))+\zeta(\tau)\big]d\tau \\ + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)\big[\psi(q(\tau))+\eta(\tau)\big]d\tau, \tag{11}$$

where

$$W(x,t) = \int_0^1 \big[\theta(x-\xi,t)-\theta(x+\xi,t)\big]r(\xi)d\xi,$$

$$\theta(x,t) = \sum_{-\infty}^{\infty} K(x+2m,t), \qquad K(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{x^2}{4t}\Big).$$

Regarding Theorem 1.1, [5, Theorem 6.3.1] and [5, Section 11.4], we can summarize the above discussion in the following statement:

Theorem 1.2. *Suppose that $r(x)$ is a piecewise continuous function in x that $\zeta(t)$, $\eta(t)$ are piecewise continuous functions in t, and that f, g are infinitely differentiable functions satisfying conditions* (9) *which are of the Holmgren class. Then the inverse problem* (1) *has a unique solution pair $(U,(\phi,\psi))$.*

2. Overview of the numerical method

Consider a one-dimensional inverse parabolic problem with initial and boundary condition described by the equations (1), where equations (1c) and (1d) are nonlinear. The application of the present numerical method will find a solution of problem (1), by using the following algorithm:

To solve the inverse problem (1), let us consider the auxiliary problem (3) with the solution (5). The solution (5) exists and is unique but it is not always stable [12]. Therefore, by using basis functions, a stable solution for the problem (3) will be presented.

Remark 2.1. *The problem* (3) *is actually uniquely solvable. So the initial condition* (1b) *can also be derived from* (3), [4].

Discretization of the initial condition (1b) and boundary conditions (2) at the points (x_j,t_j), may be obtain by the following equations [9]

$$U(x_j,0) = r(x_j), \qquad j=1,2,\dots,n, \tag{12a}$$

$$U(a,\bar{t}_{j-n}) = f_{j-n}, \qquad j=n+1,n+2,\dots,n+m, \tag{12b}$$

$$U_x(a,\hat{t}_{j-n-m}) = g_{j-n-m}, \qquad j=n+m+1,\dots,n+m+l, \tag{12c}$$

where $\bar{t}_{j-n}$ and $\hat{t}_{j-n-m}$ denote the discrete values of times.

An approximate solution of problem (3) can be expressed as the following form [3], [9]

$$\tilde{U}(x,t) = \sum_{j=1}^{n+m+l} \lambda_j\ \varphi(x-x_j,t-t_j), \tag{13}$$

where $\varphi(x,t) = U(x,t+\tau)$ for $\tau > T$, (τ is a constant), is a solution of the problem (3) and λ_j are unknown constants which remain to be determined.

Putting (13) into equations (12), we find

$$\begin{aligned}\tilde{U}(x_i,t_i) &= \sum_{j=1}^{n+m+l} \lambda_j \; \varphi(x_i-x_j,t_i-t_j) \\ &= \begin{cases} r(x_i), & i=1,2,\ldots,n \\ f_{i-n}, & i=n+1,\ldots,n+m \end{cases}, \\ \frac{\partial\tilde{U}}{\partial x}(x_i,t_i) &= \sum_{j=1}^{n+m+l} \lambda_j \; \frac{\partial\varphi}{\partial x}(x_i-x_j,t_i-t_j) \\ &= g_{i-n-m}, \qquad i=n+m+1,\ldots,n+m+l.\end{aligned} \tag{14}$$

In matrix form, we obtain the following algebraic system of equations with unknown coefficients λ_j, for $j=1,2,\ldots,n+m+l$

$$A\lambda = B, \tag{15}$$

where

$$A = \begin{pmatrix} \varphi(x_i-x_j,t_i-t_j) \\ \varphi(x_k-x_j,t_k-t_j) \\ \frac{\partial\varphi}{\partial x}(x_\gamma-x_j,t_\gamma-t_j) \end{pmatrix}, \quad A\in\mathbb{R}^{(n+m+l)\times(n+m+l)}, \tag{16}$$

and

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n+m+l} \end{pmatrix}, \quad B = \begin{pmatrix} r(x_i) \\ f_{k-n} \\ \vdots \\ g_{\gamma-n-m} \end{pmatrix}, \quad \lambda, B \in \mathbb{R}^{(n+m+l)\times 1} \tag{17}$$

where $i=1,\ldots,n$, $k=n+1,\ldots,n+m$, $\gamma=n+m+1,\ldots,n+m+l$.

The system of linear algebraic equations (15) cannot be solved by direct methods, such as the least squares method, since such an approach would produce a highly unstable solution due to the large value of the condition number of the matrix A which increases dramatically as the number of collocation points increases [16]. Several regularization procedures have been developed to solve such ill-conditioned system, see for example Hansen [17]. One of the most used regularization technique is the Tikhonov regularization method [18].

The Tikhonov regularized solution λ_α for system of equation (15) is defined as the solution of the following minimization problem:

$$\min_{\lambda}\{\|A\lambda - B\|^2 + \alpha^2\|\lambda\|^2\}, \tag{18}$$

where $\|\cdot\|$ denotes the usual Euclidean norm and α is called the regularization parameter. The choose of a suitable value of the regularization parameter α is crucial for the accuracy of the final numerical solution and is still under intensive research [18]. In our computation we use the L-curve scheme to determine a suitable value of α ([19]-[20]). The L-curve scheme was first applied by Lawson and Hanson [20]. To investigate the properties of regularized systems under different value of the regularization parameter α, [19]. The L-curve method is explained in the following form, [21],

Definition 2.1. *Let us consider the following curve*

$$L = \{\log\|\lambda_\alpha\|^2, \log\|A\lambda_\alpha - b\|^2,\ \alpha>0\}, \tag{19}$$

the curve is known as L-curve and a suitable regularization parameter α corresponds to a regularized solution near the corner of the L-curve [19].

Note that the Tikhonov regularized solution ([19], [20], [18]) to the system of linear algebraic equation (15) is given by

$$\lambda_\alpha = (A^T A + \alpha^2 I)^{-1} A^T B. \tag{20}$$

Furthermore, the stable solution for the problem (3) will be obtain by

$$\tilde{U}(x,t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \; \varphi(x - x_j, t - t_j), \tag{21a}$$

$$\tilde{U}(0,t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \; \varphi(0 - x_j, t - t_j), \tag{21b}$$

$$\tilde{U}(1,t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \; \varphi(1 - x_j, t - t_j), \tag{21c}$$

$$\frac{\partial \tilde{U}}{\partial x}(0,t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \; \frac{\partial \varphi}{\partial x}(0 - x_j, t - t_j), \tag{21d}$$

$$\frac{\partial \tilde{U}}{\partial x}(1,t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \; \frac{\partial \varphi}{\partial x}(1 - x_j, t - t_j). \tag{21e}$$

For evaluating ϕ and ψ we use

$$\tilde{U}(0,t_k) = \phi(\tilde{U}_x(0,t_k)) + \zeta(t_k), \tag{22}$$

$$\tilde{U}(1,t_k) = \psi(\tilde{U}_x(1,t_k)) + \eta(t_k). \tag{23}$$

Therefore

$$\phi(\tilde{U}_x(0,t_k)) = \tilde{U}(0,t_k) - \zeta(t_k), \tag{24}$$

$$\psi(\tilde{U}_x(1,t_k)) = \tilde{U}(1,t_k) - \eta(t_k). \tag{25}$$

Finally, the `MATLAB` package is used for interpolating these values and reconstructing the functions $\phi(U_x(0,t))$ and $\psi(U_x(1,t))$.

Remark 2.2. *Our proposed numerical procedure may be used to solved the problem* (1) *if the boundary conditions considered as*

$$U_x(0,t) = \phi(U(0,t)) + \zeta(t), \qquad 0 \le t \le T, \tag{26}$$

$$U_x(1,t) = \psi(U(1,t)) + \eta(t), \qquad 0 \le t \le T. \tag{27}$$

3. Numerical Results and Discussion

In this section, we demonstrate numerically some of results for the unknown boundary condition in the inverse problem (1). The purpose of this section is to illustrate the applicability of the present method described in Section 2 for solving the inverse problem (1). As expected, the IHCPs are ill-posed and therefore it is necessary to investigate the stability of the present method by giving some test problems. All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz).

Example 3.1. *In this example, let us consider the following inverse problem, for estimating unknown boundary conditions ϕ and ψ*

$$U_t(x,t) = U_{xx}(x,t), \qquad 0 < x < 1, \qquad 0 < t < T, \tag{28a}$$

$$U(x,0) = x^2, \qquad 0 \le x \le 1, \tag{28b}$$

$$U(0,t) = \phi(U_x(0,t)) + 2\ t - \sin(1), \qquad 0 \le t \le T, \tag{28c}$$

$$U(1,t) = \psi(U_x(1,t)) + 2\ t + 1 - \cos(6), \qquad 0 \le t \le T, \tag{28d}$$

and, for $a = 0.5$, the overspecified conditions

$$U(0.5,t) = (0.5)^2 + 2\ t, \quad U_x(0.5,t) = 1, \qquad (0 \le t \le T).$$

The exact solution of this problem is $U(x,t) = x^2 + 2t$,

$$\phi(U_x(0,t)) = \sin((U_x(0,t))^2 + 1), \quad \psi(U_x(1,t)) = \cos((U_x(1,t))^2 + U_x(1,t)).$$

Table 1 and figure 1 show the comparison between the exact solution and approximate solution result from our method using Tikhonov regularization with noiseless data, and tables 2 and 3 and figures 2 and 3 show this comparison with noisy data (noisy data = input data$+$ $(0.01)rand(1)$) when $\tau = 1.2$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	0.841471	0.841471	0.960170	0.960170
0.1	0.841471	0.841471	0.960170	0.960170
0.2	0.841471	0.841471	0.960170	0.960170
0.3	0.841471	0.841471	0.960170	0.960170
0.4	0.841471	0.841471	0.960170	0.960170
0.5	0.841471	0.841471	0.960170	0.960170
0.6	0.841471	0.841471	0.960170	0.960170
0.7	0.841471	0.841471	0.960170	0.960170
0.8	0.841471	0.841471	0.960170	0.960170
0.9	0.841471	0.841471	0.960170	0.960170
1	0.841471	0.841471	0.960170	0.960170

TABLE 1. *The comparison between exact and approximate solutions for $\phi(U_x(0,t))$ and $\psi(U_x(1,t))$ with noiseless data when $\Delta t = 0.1$, $n = m = l = 22$ and $cond(A) = 2.770609e + 035$.*

Example 3.2. *Now, we consider the following inverse problem for estimating unknown boundary conditions ϕ and ψ*

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1, 0 < t < T, \tag{29a}$$

$$U(x,0) = \sin(x), \quad 0 \le x \le 1, \tag{29b}$$

$$U(0,t) = \phi(U_x(0,t)) - \sin(e^{-2t} + 1), \quad 0 \le t \le T, \tag{29c}$$

$$U(1,t) = \psi(U_x(1,t)) + e^{-t}\sin 1 - \cos(e^{-2t}\cos^2 1 + e^{-t}\cos 1),\ 0 \le t \le T, \tag{29d}$$

and, for $a = 0.5$, the overspecified conditions

$$U(0.5,t) = e^{-t}\sin(0.5), \quad U_x(0.5,t) = e^{-t}\cos(0.5), \qquad (0 \le t \le T).$$

The exact solution of this problem is $U(x,t) = e^{-t}\sin(x)$,

$$\phi(U_x(0,t)) = \sin((U_x(0,t))^2 + 1), \quad \psi(U_x(1,t)) = \cos((U_x(1,t))^2 + U_x(1,t)).$$

t	ϕ_{Exact}	$\phi_{approximate}$	ψ_{Exact}	$\psi_{approximate}$
0	0.841471	0.839440	0.960170	0.964099
0.1	0.841471	0.839280	0.960170	0.963940
0.2	0.841471	0.839121	0.960170	0.963780
0.3	0.841471	0.838962	0.960170	0.963621
0.4	0.841471	0.838802	0.960170	0.963462
0.5	0.841471	0.838643	0.960170	0.963302
0.6	0.841471	0.838483	0.960170	0.963143
0.7	0.841471	0.838324	0.960170	0.962983
0.8	0.841471	0.838165	0.960170	0.962824
0.9	0.841471	0.838005	0.960170	0.962665
1	0.841471	0.837846	0.960170	0.962505

TABLE 2. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 6.153262e + 034$.

t	ϕ_{Exact}	$\phi_{approximate}$	ψ_{Exact}	$\psi_{approximate}$
0	0.841471	0.840075	0.960170	0.963268
0.1	0.841471	0.839945	0.960170	0.963138
0.2	0.841471	0.839815	0.960170	0.963008
0.3	0.841471	0.839685	0.960170	0.962878
0.4	0.841471	0.839555	0.960170	0.962748
0.5	0.841471	0.839425	0.960170	0.962618
0.6	0.841471	0.839295	0.960170	0.962488
0.7	0.841471	0.839165	0.960170	0.962358
0.8	0.841471	0.839035	0.960170	0.962228
0.9	0.841471	0.838905	0.960170	0.962098
1	0.841471	0.838775	0.960170	0.961968

TABLE 3. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = 50$, $m = 30$, $l = 20$ *and* $cond(A) = 6.000048e + 052$.

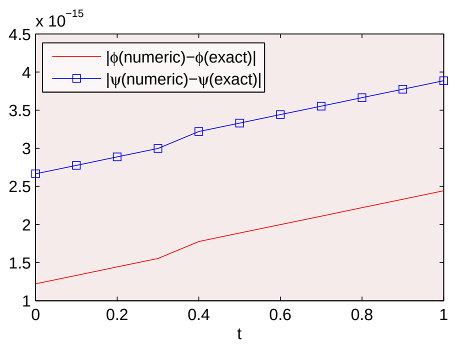

FIGURE 1. *The comparison between the exact results and the present numerical results of the problem (28) with discrete noiseless data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.770609e + 035$.

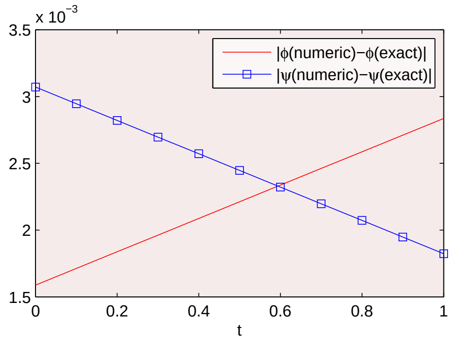

FIGURE 2. *The comparison between the exact results and the present numerical results of the problem* (28) *with discrete noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 6.153262e + 034$.

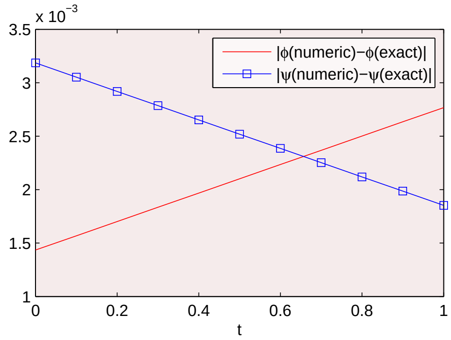

FIGURE 3. *The comparison between the exact results and the present numerical results of the problem* (28) *with discrete noisy data when* $\Delta t = 0.1$, $n = 50,\ m = 30,\ l = 20$ *and* $cond(A) = 6.000048e + 052$.

Table 4 and figure 4 show the comparison between the exact solution and approximate solution result from our method using Tikhonov regularization with noiseless data, and table 5 and 6 and figures 5 and 6 show this comparison with noisy data (noisy data $=$ *input data* $+ (0.01)rand(1)$*) when* $\tau = 1.2$.

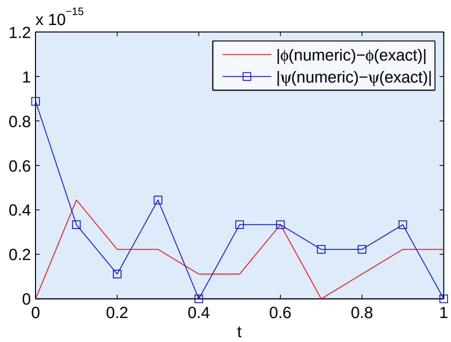

FIGURE 4. *The comparison between the exact results and the present numerical results of the problem* (29) *with discrete noiseless data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.022777e + 034$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	0.909297	0.909297	0.673229	0.673229
0.1	0.969421	0.969421	0.746577	0.746577
0.2	0.995052	0.995052	0.803261	0.803261
0.3	0.999758	0.999758	0.847001	0.847001
0.4	0.992632	0.992632	0.880753	0.880753
0.5	0.979483	0.979483	0.906826	0.906826
0.6	0.963877	0.963877	0.927005	0.927005
0.7	0.947906	0.947906	0.942656	0.942656
0.8	0.932725	0.932725	0.954829	0.954829
0.9	0.918906	0.918906	0.964322	0.964322
1	0.906676	0.906676	0.971747	0.971747

TABLE 4. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noiseless data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.022777e + 034$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	0.909297	0.906972	0.673229	0.677702
0.1	0.969421	0.967192	0.746577	0.750498
0.2	0.995052	0.992910	0.803261	0.806683
0.3	0.999758	0.997695	0.847001	0.849972
0.4	0.992632	0.990640	0.880753	0.883315
0.5	0.979483	0.977555	0.906826	0.909019
0.6	0.963877	0.962007	0.927005	0.928863
0.7	0.947906	0.946089	0.942656	0.944212
0.8	0.932725	0.930956	0.954829	0.956111
0.9	0.918906	0.917180	0.964322	0.965356
1	0.906676	0.904989	0.971747	0.972556

TABLE 5. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.744951e + 033$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	0.909297	0.909424	0.673229	0.673678
0.1	0.969421	0.971112	0.746577	0.749949
0.2	0.995052	0.998158	0.803261	0.809279
0.3	0.999758	1.004146	0.847001	0.855412
0.4	0.992632	0.998178	0.880753	0.891330
0.5	0.979483	0.986078	0.906826	0.919363
0.6	0.963877	0.971421	0.927005	0.941315
0.7	0.947906	0.956309	0.942656	0.958571
0.8	0.932725	0.941904	0.954829	0.972195
0.9	0.918906	0.928789	0.964322	0.983002
1	0.906676	0.917194	0.971747	0.991616

TABLE 6. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = 50,\ m = 30,\ l = 20$ *and* $cond(A) = 1.925513e + 034$.

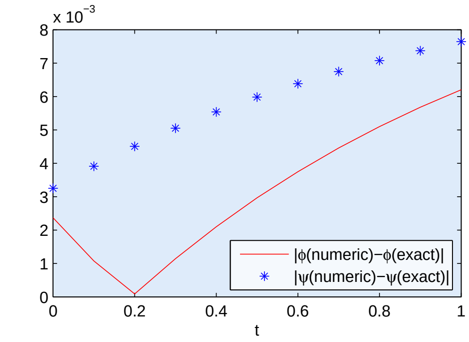

FIGURE 5. *The comparison between the exact results and the present numerical results of the problem* (29) *with discrete noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.744951e + 033$.

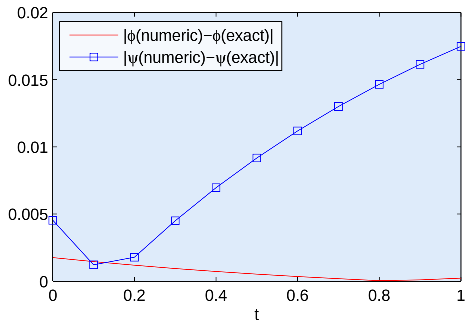

FIGURE 6. *The comparison between the exact results and the present numerical results of the problem* (29) *with discrete noiseless data when* $\Delta t = 0.1$, $n = 50$, $m = 30$, $l = 20$ *and* $cond(A) = 1.925513e + 034$.

Example 3.3. *Now, we consider the following inverse problem for estimating unknown boundary conditions* ϕ *and* ψ

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1, \quad 0 < t < T, \tag{30a}$$

$$U(x,0) = \sin(\pi x) + \cos(\pi x), \quad 0 \le x \le 1, \tag{30b}$$

$$U_x(0,t) = \phi(U(0,t)) + \pi e^{-\pi^2 t} + e^{-4\pi^2 t}, \quad 0 \le t \le T, \tag{30c}$$

$$U_x(1,t) = \psi(U(1,t)) - \pi e^{-\pi^2 t} - e^{-4\pi^2 t}, \ 0 \le t \le T, \tag{30d}$$

and, for $a = 0.5$, *the overspecified conditions*

$$U(0.5,t) = e^{-\pi^2 t}, \quad U_x(0.5,t) = -\pi e^{-\pi^2 t}, \quad (0 \le t \le T).$$

The exact solution of this problem is $U(x,t) = e^{-\pi^2 t}(\sin(\pi x) + \cos(\pi x))$,

$$\phi(U(0,t)) = -U^4(0,t), \quad \psi(U(1,t)) = U^4(1,t).$$

Table 7 and figure 7 show the comparison between the exact solution and approximate solution result from our method using Tikhonov regularization with noiseless data, and table 8 and 9 and figures 8 and 9 show this comparison with noisy data (noisy data $=$ *input data* $+ (0.01)rand(1)$*) when* $\tau = 1.2$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	−1.000000	−1.000000	1.000000	1.000000
0.1	−0.019296	−0.019296	0.019296	0.019296
0.2	−0.000372	−0.000372	0.000372	0.000372
0.3	−0.000007	−0.000007	0.000007	0.000007
0.4	−0.000000	−0.000000	0.000000	0.000000
0.5	−0.000000	−0.000000	0.000000	0.000000
0.6	−0.000000	−0.000000	0.000000	0.000000
0.7	−0.000000	+0.000000	0.000000	0.000000
0.8	−0.000000	+0.000000	0.000000	0.000000
0.9	−0.000000	+0.000000	0.000000	0.000000
1	−0.000000	+0.000000	0.000000	0.000000

TABLE 7. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noiseless data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 2.131381e + 039$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	−1.000000	−0.994772	1.000000	1.005561
0.1	−0.019296	−0.013964	0.019296	0.024753
0.2	−0.000372	+0.004999	0.000372	0.005790
0.3	−0.000007	+0.005379	0.000007	0.005411
0.4	−0.000000	+0.005391	0.000000	0.005398
0.5	−0.000000	+0.005394	0.000000	0.005396
0.6	−0.000000	+0.005394	0.000000	0.005395
0.7	−0.000000	+0.005395	0.000000	0.005395
0.8	−0.000000	+0.005395	0.000000	0.005395
0.9	−0.000000	+0.005395	0.000000	0.005395
1	−0.000000	+0.005395	0.000000	0.005395

TABLE 8. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 3.945205e + 034$.

t	ϕ_{Exact}	$\phi_{Approximate}$	ψ_{Exact}	$\psi_{Approximate}$
0	−1.000000	−0.994768	1.000000	1.006881
0.1	−0.019296	−0.013547	0.019296	0.024660
0.2	−0.000372	+0.005569	0.000372	0.006543
0.3	−0.000007	+0.006006	0.000007	0.006106
0.4	−0.000000	+0.006040	0.000000	0.006072
0.5	−0.000000	+0.006050	0.000000	0.006062
0.6	−0.000000	+0.006054	0.000000	0.006058
0.7	−0.000000	+0.006055	0.000000	0.006057
0.8	−0.000000	+0.006056	0.000000	0.006057
0.9	−0.000000	+0.006056	0.000000	0.006056
1	−0.000000	+0.006056	0.000000	0.006056

TABLE 9. *The comparison between exact and approximate solutions for* $\phi(U_x(0,t))$ *and* $\psi(U_x(1,t))$ *with noisy data when* $\Delta t = 0.1$, $n = 50$, $m = 30$, $l = 20$ *and* $cond(A) = 1.074841e + 034$.

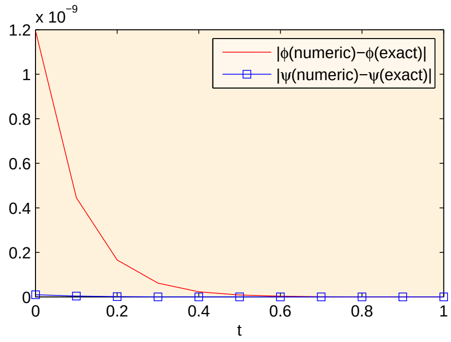

FIGURE 7. *The comparison between the exact results and the present numerical results of the problem* (30) *with discrete noiseless data.*

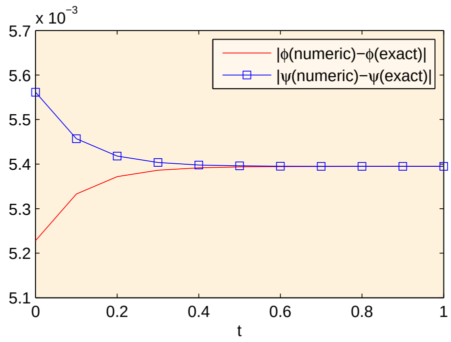

FIGURE 8. *The comparison between the exact results and the present numerical results of the problem* (30) *with discrete noisy data when* $\Delta t = 0.1$, $n = m = l = 22$ *and* $cond(A) = 3.945205e + 034$.

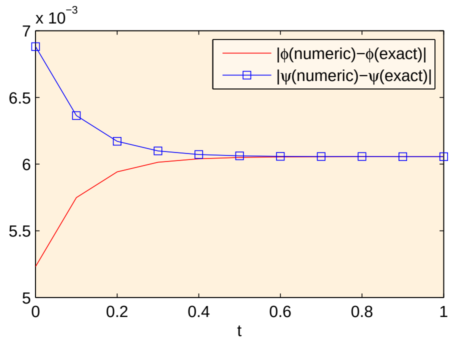

FIGURE 9. *The comparison between the exact results and the present numerical results of the problem* (30) *with discrete noiseless data when* $\Delta t = 0.1$, $n = 50$, $m = 30$, $l = 20$ *and* $cond(A) = 1.074841e + 034$.

4. CONCLUSION

A numerical method to estimate unknown boundary conditions is proposed for these kinds of IHCPs and the following results are obtained:

(1) The present study successfully applies the numerical method to IHCPs.
(2) Numerical results show that a good estimation can be obtained within a couple of minutes CPU time at Pentium(R) 4 CPU 3.2 GHz.

(3) The present method has been found stable with respect to small perturbation in the input data.

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