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ASYMPTOTIC EXPANSIONS FOR THE ERGODIC MOMENTS OF A SEMI-MARKOVIAN RANDOM WALK WITH A GENERALIZED DELAYING BARRIER

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ABSTRACT. In this study, a semi-Markovian random walk process $(X(t))$ with a generalized delaying barrier is considered and the ergodic theorem for this process is proved under some weak conditions. Then, the exact expressions and asymptotic expansions for the first four ergodic moments of the process $X(t)$ are obtained.

Keywords: Semi - Markovian random walk, delaying barrier, ergodic distribution, ergodic moments, asymptotic expansion, ladder height.

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1. Introduction

Many interesting problems of stochastic finance, mathematical biology, reliability, queuing theory, stock control theories and mathematical insurance can be expressed by means of random walk process or its modifications. These kinds of problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, stochastic finance, mathematical insurance, etc. In particular, a number of very interesting problems of stock control, queuing and reliability theories are expressed by means of random walk with various types of barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite (or infinite) volumes or queuing systems with finite (or infinite) waiting time or sojourn time by means of random walk with delaying barriers (or barrier). Furthermore, the functioning of stochastic systems with spare equipment can be given by random walk with barriers, one of them is delaying and the other one is any barrier. In this topic, there are many interesting studies in literature (e.g., [1]-[4],[7]-[11]).

Unfortunately, the results of these studies are not readily applicable to real – world problems because the probability characteristics of considered processes have very complex mathematical structure. For avoiding this difficulty, in recent years, the asymptotic

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methods for investigation of the processes arising in the fields of queuing, reliability, stock control, etc. theories are intensively developed. In these topics, there are also some important researches in literature (e.g., [1], [2], [3], [4], [7], [9], [10]).

However, these studies have only dealt with the boundary functionals of random walk. Doubtlessly, the boundary functionals of stochastic processes are extremely important. But own characteristics of the random walk are also important for solving various applied problems. For this reason, we are interested in ergodic distribution of the semi – Markovian random walk $(X(t))$ with a generalized delaying barrier. Namely, in this study we investigate the asymptotic behavior of the ergodic moments of the process $X(t)$.

Before giving the main results of this study, mathematical construction of the investigated stochastic process is considered.

2. The Mathematical Construction of the Process $X(t)$

Let $\{(\xi_n, \eta_n)\}$, $n = 1, 2, \ldots$ be a sequence of independent and identically distributed random pairs defined on a probability space $(\Omega, \Im, P)$, where ξ_ns take only positive, η_ns take both negative and positive values. Suppose that the random variables of ξ_ns and η_ns are mutually independent and distribution functions of ξ_n and η_n are known:

$$\Phi(t) = P\{\xi_n \leq t\}, \quad F(x) = P\{\eta_n \leq x\}, \quad t \geq 0, x \in \mathbb{R}.$$

Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0, \quad n = 1, 2, \ldots$$

Moreover, define the sequences of random variables N_n, L_n, ζ_n and S_{N_n} as follows:

$N_0 = 0,\ L_0 = 0,\ \zeta_0 = z,\ S_{N_0} = 0,$

$N_1 \equiv N_1(\lambda z) \equiv N(\lambda z) = \inf\{n \geq 1 : \lambda z - S_n < 0\},$

$L_1 = \inf\{n \geq 1 : -\eta_{N_1+n} > 0\},\ \zeta_1 = -\eta_{N_1+L_1},\ S_{N_1} = \sum_{i=1}^{N_1} \eta_i,$

$N_n \equiv N_n(\lambda\zeta_{n-1}) = \inf\left\{k \geq 1 : \lambda\zeta_{n-1} - \left(S_{N_1+L_1+\ldots+N_{n-1}+L_{n-1}+k} - S_{N_1+L_1+\ldots+N_{n-1}+L_{n-1}}\right) < 0\right\},$

$L_n = \inf\{k \geq 1 : -\eta_{N_1+L_1+\ldots+N_n+k} > 0\},\ \zeta_n = -\eta_{N_1+L_1+\ldots+N_n+L_n},\ S_{N_n} = \sum_{i=1}^{N_n} \eta_i,\ n \geq 1.$

Put

$$\theta_1 = \sum_{i=1}^{L_1} \xi_{N_1+i}, \ldots, \theta_n = \sum_{i=1}^{L_n} \xi_{N_1+L_1+\cdots+N_{n-1}+L_{n-1}+i}, \quad n = 1, 2, \ldots$$

and

$$\tau_n = T_{N_1+L_1+\ldots+N_{n-1}+L_{n-1}+N_n}, \gamma_n = \tau_n + \theta_n, \quad n = 1, 2, \ldots, \gamma_0 = \tau_0 = 0.$$

Moreover, define $\nu(t) = \max\{n \geq 0 : T_n \leq t\}, \quad t > 0.$

We can now construct desired stochastic process $X(t)$ in the following form :

$$X(t) = \max\left\{0, \lambda\zeta_n - \left(S_{\nu(t)} - S_{N_1+L_1+\cdots+N_n+L_n}\right)\right\}$$

for each $t \in [\gamma_n, \gamma_{n+1})$, $n = 0, 1, 2, \ldots$, where λ is a positive constant.

One of trajectories for the process $X(t)$ is given as in Figure 1:

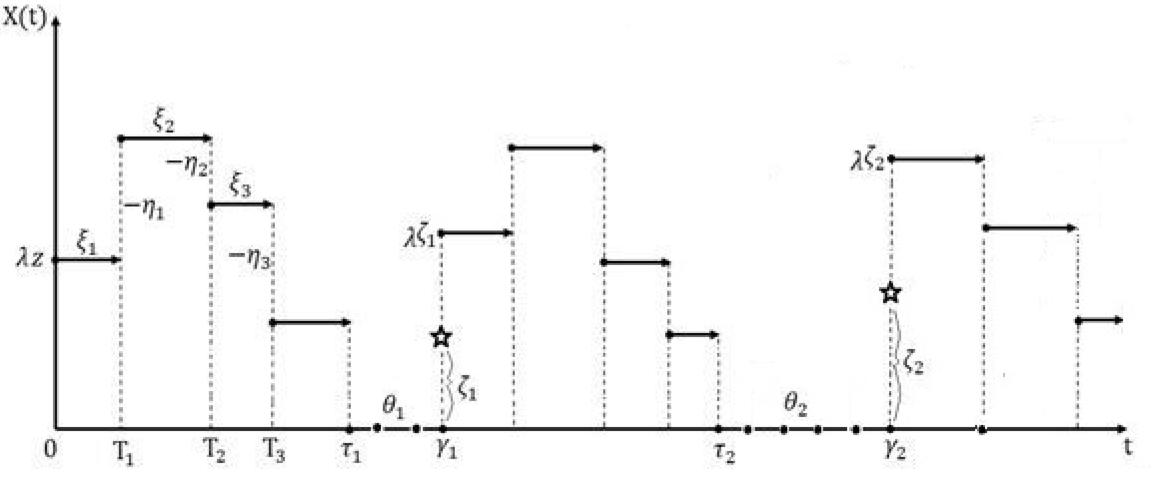

FIGURE 1. One of trajectories for the process $X(t)$

Note that, $\zeta_0 = z > 0$ and $\zeta_1, \zeta_2, \ldots$ are independent and identically distributed positive valued random variables having the following distribution function $\pi(z)$:

$$\pi(z) \equiv P\{\zeta_n \leq z\} = \frac{F(0) - F(-z)}{F(0)}, \quad F(z) = P\{\eta_n \leq z\}, \quad z \geq 0.$$

The process $X(t)$ is called as "Semi-Markovian Random Walk Process with a Generalized Delaying Barrier".

The main purpose of this study is to investigate the asymptotic behavior of the moments of ergodic distribution of the process $X(t)$, as $\lambda \to \infty$.

3. ERGODICITY OF THE PROCESS $X(t)$

Firstly, we state the following theorem on the ergodicity of the process $X(t)$.

Theorem 3.1. *Let the initial sequence of random pairs $\{(\xi_n, \eta_n), n \geq 1\}$, satisfy the following supplementary conditions:*

(1) $E(\xi_1) < \infty$,
(2) $E(\eta_1) > 0$,
(3) $P\{\eta_1 > 0\} > 0$ *and* $P\{\eta_1 < 0\} > 0$,
(4) η_1 *is non-arithmetic random variable.*

Then, the process $X(t)$ is ergodic and for any bounded measurable function $f(x)$ defined on the interval $[0, \infty)$, the following relation holds, with probability 1:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(s))\, ds = \frac{1}{E(\gamma_1)} \int_0^\infty \int_0^\infty \int_0^\infty f(x) P_{\lambda z}\{\gamma_1 > t; X(t) \in dx\}\, dt d\pi(z) \quad (1)$$

Proof. The process $X(t)$ belongs to a wide class of processes which is called in literature as "The class of semi-Markov processes with a discrete interference of chance". General ergodic theorem of type Smith's "key renewal theorem" exists in the literature for this class (see, Gihman and Skorohold [9, p. 243]). Under the conditions of the Theorem 3.1, the assumptions of this general ergodic theorem are satisfied. Therefore, the ergodicity of the process $X(t)$ is derived from this general ergodic theorem. □

Remark 3.1. *From* (1) *can be extracted many valuable consequences. To state some of these consequences introduce the following notations:*

$$Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \le x\} \text{ and } \varphi_X(\alpha) = \lim_{t\to\infty} E(exp\{i\alpha X(t)\}), \quad \alpha \in \mathbb{R}.$$

Corollary 3.1. *The ergodic distribution function* $(Q_X(x))$ *of the process* $X(t)$ *has the following exact form:*

$$Q_X(x) = \frac{1}{E(\gamma_1)} \int_0^\infty \int_0^\infty G(t, x, \lambda z)\, dt d\pi(z),$$

where $G(t, x, \lambda z) = P_{\lambda z}\{\gamma_1 > t; X(t) \le x\}$.

Corollary 3.2. *The explicit form of characteristic function* $(\varphi_X(\alpha))$ *of the ergodic distribution of the process* $X(t)$ *can be given as :*

$$\varphi_X(\alpha) = \frac{1}{E(\gamma_1)} \int_0^\infty \int_0^\infty \int_0^\infty e^{i\alpha x} d_x G(t, x, \lambda z)\, dt d\pi(z).$$

Using the basic identity for the random walk process (see, Feller [8, p. 514]), we can obtain an alternative representation for the characteristic function $(\varphi_X(\alpha))$ of the ergodic distribution.

Theorem 3.2. *Under the assumptions of Theorem 3.1, the characteristic function* $\varphi_X(\alpha)$ *of the ergodic distribution of the process* $X(t)$ *can be expressed by means of characteristics of the boundary functional* N_1 *and* S_{N_1} *as follows* $(\alpha \in \mathbb{R}\setminus\{0\})$*:*

$$\begin{aligned}\varphi_X(\alpha) =& \frac{1}{E(N_1(\lambda\zeta_1)) + K} \int_0^\infty e^{i\alpha\lambda z} \frac{\varphi_{S_{N_1(\lambda z)}}(-\alpha) - 1}{\varphi_\eta(-\alpha) - 1} d\pi(z) \\ &+ \frac{K}{E(N_1(\lambda\zeta_1)) + K} \int_0^\infty e^{i\alpha\lambda z} \varphi_{S_{N_1(\lambda z)}}(-\alpha)\, d\pi(z),\end{aligned} \tag{2}$$

where $\varphi_{S_{N_1(\lambda z)}}(-\alpha) \equiv E\left(\exp\left\{-i\alpha S_{N_1(\lambda z)}\right\}\right)$, $\varphi_\eta(-\alpha) \equiv E(\exp\{-i\alpha\eta_1\})$*;*

$$E(N_1(\lambda\zeta_1)) \equiv \int_0^\infty E(N_1(\lambda z))\, d\pi(z); \quad K = \frac{E(\theta_1)}{E(\xi_1)} = \frac{1}{F(0)};$$

$$\pi(z) \equiv P\{\zeta_n \le z\} = \frac{F(0) - F(-z)}{F(0)}; \quad F(z) = \{\eta_n \le z\}.$$

4. Exact formulas for the first four moments of the ergodic distribution of the process $X(t)$

In section 3, the characteristic function of ergodic distribution of the process $X(t)$ has been expressed by means of characteristic functions of boundary functional $S_{N_1(\lambda z)}$ and random variable η_1. Using this result, it is possible to obtain some exact formulas for moments of the ergodic distribution of the process $X(t)$. Because of that in this section, we express the first four moments of ergodic distribution of the process $X(t)$ by means of the moments of η_1, $N_1(\lambda z)$ and $S_{N_1(\lambda z)}$. For this aim let us give the following notations:

$$E(X^n) = \lim_{t\to\infty} E\{(X(t))^n\}, m_n = E(\eta_1^n), m_{n1} = \frac{m_n}{nm_1},$$

$$M_n(z) \equiv E\left(S^n_{N_1(z)}\right), M_{n1}(z) = M_n(z)/nM_1(z), \quad n = 1, 2, \ldots$$

$\widetilde{\zeta}_1 = \lambda\zeta_1$, $E\left(\widetilde{\zeta}_1^n M_r\left(\widetilde{\zeta}_1\right)\right) \equiv \int_0^\infty (\lambda z)^n M_r(\lambda z)\, d\pi(z)$, $r = \overline{1,5}$, $n = 0, 1, 2, \ldots$. The main aim of this section is to express the first four ergodic moments $\left(E(X^n), n = \overline{1,4}\right)$ of the process $X(t)$ by means of moments of boundary functional $S_{N_1(\lambda z)}$ and random variable η_1.

Now, we can state the following theorem as follows.

Theorem 4.1. *Suppose that the conditions of Theorem 3.1 are satisfied and also* $E\left(|\eta_1|^3\right) < \infty$. *Then, the first and second order moments* $\left(E\left(X^n\right), n = 1, 2\right)$ *of the ergodic distribution of the process* $X(t)$ *exist and they can be expressed by means of the characteristics of boundary functional* $S_{N_1(z)}$ *and random variable* η_1 *as follows:*

$$E(X) = \frac{1}{E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1}\left[E\left(\widetilde{\zeta}_1 M_1\left(\widetilde{\zeta}_1\right)\right) - \frac{1}{2}E\left(M_2\left(\widetilde{\zeta}_1\right)\right) + A_1 E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1 E\left(\widetilde{\zeta}_1\right)\right], \tag{3}$$

$$E\left(X^2\right) = \frac{1}{E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1}\left\{E\left(\widetilde{\zeta}_1^2 M_1\left(\widetilde{\zeta}_1\right)\right) - E\left(\widetilde{\zeta}_1 M_2\left(\widetilde{\zeta}_1\right)\right) + \frac{1}{3}E\left(M_3\left(\widetilde{\zeta}_1\right)\right) + A_1\left[2E\left(\widetilde{\zeta}_1 M_1\left(\widetilde{\zeta}_1\right)\right) - E\left(M_2\left(\widetilde{\zeta}_1\right)\right)\right] + A_2 E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1 E\left(\widetilde{\zeta}_1^2\right)\right\}, \tag{4}$$

where $\widetilde{\zeta}_1 = \lambda\zeta_1$, $A_1 = m_{21} - Km_1$, $A_2 = 2m_{21}^2 - m_{31}$.

Theorem 4.2. *Suppose that the conditions of Theorem 3.1 are satisfied and also* $E\left(|\eta_1|^5\right) < \infty$. *Then, the third and fourth order moments* $\left(E\left(X^n\right), n = 3, 4\right)$ *of the ergodic distribution of the process* $X(t)$ *exist and we can expressed these moments by means of the characteristics of the boundary functional* $S_{N_1(z)}$ *and random variable* η_1 *as follows:*

$$E\left(X^3\right) = \frac{1}{E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1}\left\{E\left(\widetilde{\zeta}_1^3 M_1\left(\widetilde{\zeta}_1\right)\right) - \frac{3}{2}E\left(\widetilde{\zeta}_1^2 M_2\left(\widetilde{\zeta}_1\right)\right) + E\left(\widetilde{\zeta}_1 M_3\left(\widetilde{\zeta}_1\right)\right) - \frac{1}{4}E\left(M_4\left(\widetilde{\zeta}_1\right)\right) + A_1\left[3E\left(\widetilde{\zeta}_1^2 M_1\left(\widetilde{\zeta}_1\right)\right) - 3E\left(\widetilde{\zeta}_1 M_2\left(\widetilde{\zeta}_1\right)\right) + E\left(M_3\left(\widetilde{\zeta}_1\right)\right)\right] + 3A_2\left[E\left(\widetilde{\zeta}_1 M_1\left(\widetilde{\zeta}_1\right)\right) - \frac{1}{2}E\left(M_2\left(\widetilde{\zeta}_1\right)\right)\right] + 3A_3 E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1 E\left(\widetilde{\zeta}_1^3\right)\right\}, \tag{5}$$

$$E\left(X^4\right) = \frac{1}{E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1}\left\{E\left(\widetilde{\zeta}_1^4 M_1\left(\widetilde{\zeta}_1\right)\right) - 2E\left(\widetilde{\zeta}_1^3 M_2\left(\widetilde{\zeta}_1\right)\right) + 2E\left(\widetilde{\zeta}_1^2 M_3\left(\widetilde{\zeta}_1\right)\right) - E\left(\widetilde{\zeta}_1 M_4\left(\widetilde{\zeta}_1\right)\right) + \frac{1}{5}E\left(M_5\left(\widetilde{\zeta}_1\right)\right) + A_1\left[4E\left(\widetilde{\zeta}_1^3 M_1\left(\widetilde{\zeta}_1\right)\right) - 6E\left(\widetilde{\zeta}_1^2 M_2\left(\widetilde{\zeta}_1\right)\right) + 4E\left(\widetilde{\zeta}_1 M_3\left(\widetilde{\zeta}_1\right)\right) - E\left(M_4\left(\widetilde{\zeta}_1\right)\right)\right] + 2A_2\left[3E\left(\widetilde{\zeta}_1^2 M_1\left(\widetilde{\zeta}_1\right)\right) - 3E\left(\widetilde{\zeta}_1 M_2\left(\widetilde{\zeta}_1\right)\right) + E\left(M_3\left(\widetilde{\zeta}_1\right)\right)\right] + 6A_3\left[2E\left(\widetilde{\zeta}_1 M_1\left(\widetilde{\zeta}_1\right)\right) - E\left(M_2\left(\widetilde{\zeta}_1\right)\right)\right] + 3A_4 E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1 E\left(\widetilde{\zeta}_1^4\right)\right\}, \tag{6}$$

where $A_3 = \frac{1}{3}m_{41} - 2m_{21}m_{31} + 2m_{21}^3$, $A_4 = 4m_{21}^4 - 6m_{21}^2 m_{31} + m_{31}^2 + \frac{4}{3}m_{21}m_{41} - \frac{1}{6}m_{51}$.

Proof. (Proofs of the Theorems 4.1 and 4.2). Conditions of the Theorems 4.1 and 4.2 provide the existence and finiteness of first five moments of the boundary functional $S_{N_1(z)}$ (see, Feller [8, p. 514]). In this case, Taylor expansion of the characteristic functions of variables $S_{N_1(z)}$ and η_1 can be written as follows, when $\alpha \to 0$:

$$\varphi_{S_{N_1(\lambda z)}}(-\alpha) - 1 = -i\alpha M_1(\lambda z) + \frac{(i\alpha)^2}{2!} M_2(\lambda z) - \frac{(i\alpha)^3}{3!} M_3(\lambda z) + \frac{(i\alpha)^4}{4!} M_4(\lambda z) + o(\alpha^4), \quad (7)$$

$$\varphi_\eta(-\alpha) - 1 = -i\alpha m_1 + \frac{(i\alpha)^2}{2!} m_2 - \frac{(i\alpha)^3}{3!} m_3 + \frac{(i\alpha)^4}{4!} m_4 + o(\alpha^4). \quad (8)$$

Substituting expansions (7) and (8) in (2) and after appropriate calculation, the exact expressions from (3) to (6) for $\left(E(X^n), n = \overline{1,4}\right)$ can be obtained. Thus, the proofs of Theorem 4.1 and Theorem 4.2 are completed. □

5. Asymptotic Expansions for the Moments of $S_{N_1(z)}$

In section 4, we obtain the exact expressions for the first four ergodic moments of the process $X(t)$. However because of the complex structure of these expressions they cannot answer to the concrete needs of application. It has been begun to use the asymptotic methods for removing this difficulty. Because of that, it is useful to investigate the asymptotic behavior of the ergodic moments $(E(X^n))$ of the process $X(t)$, when $\lambda \to \infty$.

For this aim, let us firstly give the asymptotic result for the moments $(M_n(z), n \geq 1)$ of the boundary functional $S_{N_1(z)}$, which are known in literature (see, [11]).

Note that $N_1 \equiv N_1(z)$ is the first crossing time of the level z with random walk process $\{S_n\}$ and $S_{N_1(z)}$ is a value of the random walk at this time. The following lemma includes some consequences from studies of Khaniyev and Mammadova [11].

Proposition 5.1. *Under the assumptions of Theorem 4.1, the following asymptotic expansions are true for the moments of $S_{N_1(z)}$, as $z \to \infty$:*

$$M_n(z) \equiv E\left(S^n_{N_1(z)}\right) = z^n + n\mu_{21} z^{n-1} + o\left(z^{n-1}\right), \quad n = \overline{1,5},$$

where $\mu_k \equiv E\left(\chi_1^{+k}\right)$, $k \geq 1$, $\mu_{21} = \frac{\mu_2}{2\mu_1}$ and χ_1^+ is the first ascending ladder height of the random walk $\{S_n\}$, $n \geq 0$.

Proposition 5.2. *Let $g(x)$ be bounded measurable function and $\lim_{x\to\infty} g(x) = 0$. Then the following asymptotic relation is hold, when $\lambda \to \infty$:*

$$\int_0^\infty z^n g(\lambda z)\, d\pi(z) \to 0, \quad n = 0, 1, 2, \ldots.$$

Using the Proposition 5.2, we can give the following result.

Corollary 5.1. *Under the assumptions of Theorem 4.1, for each $n = 0, 1, 2, \ldots$ the following asymptotic expansions are true for the integrals from the moments of $S_{N_1(z)}$, as $\lambda \to \infty$:*

(1) $E\left(\widetilde{\zeta}_1^n M_1\left(\widetilde{\zeta}_1\right)\right) = \lambda^{n+1}\beta_{n+1} + \lambda^n \mu_{21}\beta_n + o\left(\lambda^{n-1}\right)$,

(2) $E\left(\widetilde{\zeta}_1^n M_2\left(\widetilde{\zeta}_1\right)\right) = \lambda^{n+2}\beta_{n+2} + 2\lambda^{n+1} \mu_{21}\beta_{n+1} + o\left(\lambda^{n}\right)$,

(3) $E\left(\widetilde{\zeta}_1^n M_3\left(\widetilde{\zeta}_1\right)\right) = \lambda^{n+3}\beta_{n+3} + 3\lambda^{n+2} \mu_{21}\beta_{n+2} + o\left(\lambda^{n+1}\right)$,

(4) $E\left(\widetilde{\zeta}_1^n M_4\left(\widetilde{\zeta}_1\right)\right) = \lambda^{n+4}\beta_{n+4} + 4\lambda^{n+3} \mu_{21}\beta_{n+3} + o\left(\lambda^{n+2}\right)$,

(5) $E\left(\widetilde{\zeta}_1^n M_5\left(\widetilde{\zeta}_1\right)\right) = \lambda^{n+5}\beta_{n+5} + 5\lambda^{n+4} \mu_{21}\beta_{n+4} + o\left(\lambda^{n+3}\right)$,

where $\widetilde{\zeta}_1 \equiv \lambda\zeta_1$, $\mu_{21} = \frac{\mu_2}{2\mu_1}$, $\beta_k \equiv E\left(\zeta_1^k\right)$, $k = 1, 2, \ldots$

6. Asymptotic Expansions for Ergodic Moments of the Process $X(t)$

By using the asymptotic results of Corollary 5.1 it is possible to obtain from Theorem 4.1 and Theorem 4.2 the asymptotic expansions for the first four ergodic moments $\left(E\left(X^n\right), n = \overline{1,4}\right)$ of the process $X(t)$, as $\lambda \to \infty$. Thus, the following theorem can be stated.

Theorem 6.1. *Under the assumptions of Theorem 4.1 and Theorem 4.2, the following asymptotic expansions can be written for the first four ergodic moments of the process* $X(t)$*, as* $\lambda \to \infty$*:*

$$E(X) = \lambda\beta_{21} + D_1 + o(1), \tag{9}$$
$$E\left(X^2\right) = \lambda^2\beta_{31} + \lambda D_2 + o(\lambda), \tag{10}$$
$$E\left(X^3\right) = \lambda^3\beta_{41} + \lambda^2 D_3 + o\left(\lambda^2\right), \tag{11}$$
$$E\left(X^4\right) = \lambda^4\beta_{51} + \lambda^3 D_4 + o\left(\lambda^3\right), \tag{12}$$

where $D_1 = [m_{21} - c_{21}B_1]$, $D_2 = [2m_{21}\beta_{21} - c_{31}B_1]$, $D_3 = [3m_{21}\beta_{31} - c_{41}B_1]$, $D_4 = [4m_{21}\beta_{41} - c_{51}B_1]$, $B_1 = \mu_{21} + Km_1$, $c_{n1} = \frac{\beta_{n1}}{\beta}$, $n = \overline{2,5}$, $\beta_{n1} = \frac{\beta_n}{n\beta_1}$, $\beta_1 \equiv \beta = E(\zeta_1)$, $\beta_n = E(\zeta_1^n)$, $m_n = E(\eta_1)^n$, $m_{n1} = \frac{m_n}{nm_1}$, $\mu_n = E\left(\chi_1^+\right)^n$, $\mu_{n1} = \frac{\mu_n}{n\mu_1}$, $K = \frac{1}{F(0)}$, $F(0) = P\{\eta_1 < 0\}$.

Proof. Firstly, we will obtain the asymptotic expansion for the expectation of ergodic distribution of the process $X(t)$, when $\lambda \to \infty$. Remind that for $E(X)$ has obtained the exact formula Eq.(3) in Theorem 4.1. For the simplicity, represent Eq.(3) as follows:

$$E(X) = I_1(\lambda)\left\{I_2(\lambda) + I_3(\lambda)\right\}, \tag{13}$$

where, $I_1(\lambda) = \frac{1}{\left\{E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1\right\}}$, $I_2(\lambda) = E\left(\widetilde{\zeta}_1 M_1\left(\widetilde{\zeta}_1\right)\right) - \frac{1}{2}E\left(M_2\left(\widetilde{\zeta}_1\right)\right)$,
$I_3(\lambda) = A_1 E\left(M_1\left(\widetilde{\zeta}_1\right)\right) + Km_1 E\left(\widetilde{\zeta}_1\right)$.

By using the asymptotic results of Corollary 5.1, we get the following asymptotic expansion for $I_1(\lambda)$, when $\lambda \to \infty$:

$$I_1(\lambda) = \frac{1}{\lambda\beta}\left(1 - \frac{B_1}{\lambda\beta} + o\left(\frac{1}{\lambda}\right)\right), \tag{14}$$

where $B_1 = \mu_{21} + Km_1$.

Similarly we can obtain:

$$I_2(\lambda) = \frac{1}{2}\lambda^2\beta_2 + o(1), \tag{15}$$

$$I_3(\lambda) = m_{21}\lambda\beta + A_1\mu_{21} + o\left(\frac{1}{\lambda}\right), \tag{16}$$

where $A_1 = m_{21} - Km_1$.

Substituting asymptotic expansions (14), (15) and (16) in (13), finally we get the asymptotic expansion (9) for $E(X)$, as $\lambda \to \infty$.

Similarly, the asymptotic expansions for moments $\left(E\left(X^n\right), n = 2, 3, 4\right)$ of the ergodic distribution of the process $X(t)$ can be derived. This completes the proof of Theorem 6.1. □

7. A Special Case

In this section, explicit expressions for the above mentioned coefficients of the asymptotic expansions are established as an example.

Let the random variable X_1 and X_2 be independent random variable and they have exponential distribution with parameters a and b, respectively, i.e., the density function of X_1 and X_2 are as follows:

$$f_{X_1}(x) = \begin{cases} ae^{-ax} & ,x \geq 0 \\ 0 & ,x < 0 \end{cases}, \quad f_{X_2}(x) = \begin{cases} be^{-bx} & ,x \geq 0 \\ 0 & ,x < 0 \end{cases}, \quad 0 < a < b < \infty.$$

By using X_1 and X_2, the random variable η_1 define as in the following form:

$$\eta_1 = X_1 - X_2.$$

Since X_1 and X_2 has an exponential distribution with parameters a and b, respectively, their moments are given as follows:

$$E(X_1^n) = \frac{n!}{a^n}, \quad E(X_2^n) = \frac{n!}{b^n}.$$

On the other hand, since the memoryless property of the exponential distribution, ζ_1 has the same distribution with the random variable X_2. In this case we have:

$$\beta_n \equiv E(\zeta_1^n) = E(X_2^n) = \frac{n!}{b^n}.$$

In a similar way, by using the memoryless property of exponential distribution, it is not difficult to see that χ_1^+ has the same distribution of X_1. Therefore:

$$\mu_n \equiv E\left(\left(\chi_1^+\right)^n\right) = E(X_1^n) = \frac{n!}{a^n}.$$

Thus, the following asymptotic expansion for the first ergodic moment of the proved $X(t)$ can be written:

$$E(X) = \frac{1}{b}\lambda + D + o(1),$$

where

$$D = \frac{a^2 + ab - b^2}{a^2(b-a)}.$$

Similarly, it is possible to obtain the following asymptotic expansions for higher order ergodic moments of the process $X(t)$:

$$E\left(X^2\right) = \frac{2}{b^2}\lambda^2 + \frac{2}{b}\lambda D + o(\lambda),$$
$$E\left(X^3\right) = \frac{6}{b^3}\lambda^3 + \frac{6}{b^2}\lambda^2 D + o\left(\lambda^2\right),$$
$$E\left(X^4\right) = \frac{24}{b^4}\lambda^4 + \frac{24}{b^3}\lambda^3 D + o\left(\lambda^3\right).$$

Let consider a special case in this example. Assume that $b = 2a$, $a \in (0,1)$. In this case, the following asymptotic expansions can be written:

$$E(X) = \frac{1}{2a}\lambda - \frac{1}{a} + o(1), \quad E\left(X^2\right) = \frac{1}{2a^2}\lambda^2 - \frac{1}{a^2}\lambda + o(\lambda),$$
$$E\left(X^3\right) = \frac{3}{4a^3}\lambda^3 - \frac{3}{2a^3}\lambda^2 + o\left(\lambda^2\right), \quad E\left(X^4\right) = \frac{3}{2a^4}\lambda^4 - \frac{3}{a^4}\lambda^3 + o\left(\lambda^3\right).$$

8. Conclusion

In this study, a semi-Markovian random walk with a generalized delaying barrier is considered and the ergodicity of this process is proved. Then, the characteristic function of the ergodic distribution of the process $X(t)$ is expressed by characteristic function of a boundary functional $S_{N_1(z)}$. The asymptotic expansions are obtained for the first four moments of the ergodic distribution of $X(t)$ as $\lambda \to \infty$. The obtained asymptotic expansions allow us to observe how the initial random variables ξ_1, η_1 and ζ_1 influence to the stationary characteristics of the process $X(t)$. Finally, in this study one special case was considered and asymptotic coefficients were obtained.

Note that, the asymptotic approach method considered here can be also used for obtaining approximation formulas which are simple enough for the ergodic distribution of the random walk process with other types of barrier (e.g., reflecting, elastic, absorbing, etc.).

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