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IDENTITIES AND RELATIONS ON THE HERMITE-BASED TANGENT POLYNOMIALS

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ABSTRACT. In this note, we introduce and investigate the Hermite-based Tangent numbers and polynomials, Hermite-based modified degenerate-Tangent polynomials, poly-Tangent polynomials. We give some identities and relations for these polynomials.

Keywords: Bernoulli polynomials and numbers, Stirling numbers of the second kind, Tangent polynomials and numbers, polylogarithm function, Degenerate Bernoulli and Genocchi polynomials.

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1. INTRODUCTION

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler number and polynomials, Genocchi numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials, poly-Genocchi numbers and polynomials, poly-Tangent numbers and polynomials, Hermite polynomials, Hermite-based Bernoulli polynomials, Hermite-based Tangent polynomials, modified degenerate Bernoulli polynomials, modified degenerate Euler polynomials and modified degenerate Genocchi polynomials (see [1]-[20]). In this note we define the Hermite-based tangent polynomials, modified Hermite-based tangent polynomials and poly-tangent polynomials. We obtain some relations and identities for these polynomials. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and second kind $S_2(n,k)$ are defined by the relations [15]

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_2(n,k)\,(x)_k \tag{1}$$

respectively. Here, $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The number $S_2(n,m)$ also admits a representation in terms of a generating function

$$\frac{\left(e^t-1\right)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}. \tag{2}$$

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The Bernoulli polynomials $B_n^{(r)}(x)$ of order α, the Euler polynomials $E_n^{(r)}(x;\lambda)$ of order α and the Genocchi polynomials $G_n^{(r)}(x;\lambda)$ of order α are defined as respectively:

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!},\ |t|<2\pi, \tag{3}$$

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!},\ |t|<\pi \tag{4}$$

and

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!},\ |t|<\pi. \tag{5}$$

When $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$, $E_n^{(r)}(0)=E_n^{(r)}$ and $G_n^{(r)}(0)=G_n^{(r)}$ are called Bernoulli numbers of order r, Euler numbers of order r and Genocchi numbers of order r, respectively.

The familiar tangent polynomials $T_n^{(r)}(x)$ of order r are defined by the generating functions ([12]-[15], [17])

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_n^{(r)}(x)\frac{t^n}{n!},\ |2t|<\pi. \tag{6}$$

When $x=0$, $T_n^{(r)}(0)=T_n^{(r)}$ are called the tangent numbers.

2-variable Hermite-Kampéde Fériet polynomials are defined in ([5], [11]) as

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}. \tag{7}$$

Khan *et al.* in [5] defined and studied on Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials as

$$\sum_{n=0}^{\infty} {}_H\mathcal{B}_n(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt+yt^2},\ |t|<2\pi \tag{8}$$

and

$$\sum_{n=0}^{\infty} {}_H\mathcal{E}_n(x,y)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt+yt^2},\ |t|<\pi, \tag{9}$$

respectively.

Carlitz in [1] defined degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x\mid\lambda)\frac{t^n}{n!}. \tag{10}$$

When $x=0$, $\mathfrak{B}_n(\lambda)=\mathfrak{B}_n(0\mid\lambda)$ are called the degenerate Bernoulli numbers.

From (41), we can easily derive the following equation

$$\mathfrak{B}_n(x\mid\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{n-l}(\lambda)\,(x\mid\lambda)_l\,,\ n\geq 0,$$

where $(x\mid\lambda)_n = x(x-\lambda)\cdots(x-\lambda(n-1))$ and $(x\mid\lambda)_n=1$.

Dolgy *et. al.* [2] defined the modified degenerate Bernoulli polynomials, which are different from Carlitz's degenerate Bernoulli polynomials as

$$\frac{t}{(1+\lambda)^{t/\lambda}-1}(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{11}$$

When $x=0$, $\mathfrak{B}_{n,\lambda}=\mathfrak{B}_{n,\lambda}(0)$ are called the modified degenerate Bernoulli numbers. From (42) we note that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{t}{(1+\lambda)^{t/\lambda}-1}(1+\lambda)^{xt/\lambda}\\ &= \frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \end{aligned}\tag{12}$$

Thus, by (43)

$$\lim_{\lambda\to 0}\mathfrak{B}_{n,\lambda}(x)=B_n(x).$$

H.-In Known *et. al.* [8] defined the modified degenerate Euler polynomials as

$$\frac{2}{(1+\lambda)^{t/\lambda}+1}(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}(x)\frac{t^n}{n!} \tag{13}$$

and T. Kim *et. al.* in [6] defined the modified degenerate Genocchi polynomials as

$$\frac{2t}{(1+\lambda)^{t/\lambda}+1}(1+\lambda)^{tx/\lambda}=\sum_{n=0}^{\infty}\mathfrak{G}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{14}$$

From (44) and (45), we get

$$\lim_{\lambda\to 0}\mathfrak{E}_{n,\lambda}(x)=E_n(x),\ \lim_{\lambda\to 0}\mathfrak{G}_{n,\lambda}(x)=G_n(x).$$

For $k\in\mathbb{Z}$, $k>1$, then k-th polylogarithm is defined by Kaneko [4] as

$$L_{i_k}(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^k}. \tag{15}$$

Thus this function is convergent for $|z|<1$, when $k=1$

$$L_{i_1}(z)=-\log(1-z). \tag{16}$$

Kim *et. al.* in [7] defined the poly-Bernoulli polynomials and the poly-Genocchi polynomials as

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{(k)}(x)\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{1-e^{-t}}e^{xt} \tag{17}$$

and

$$\sum_{n=0}^{\infty}\mathfrak{G}_n^{(k)}(x)\frac{t^n}{n!}=\frac{2L_{i_k}(1-e^{-t})}{e^t+1}e^{xt}, \tag{18}$$

respectively.

For $k=1$, by use (47) in (48) and (49), we get

$$\mathfrak{B}_n^{(1)}(x)=(-1)^{n+1}B_n(x),\ \mathfrak{G}_n^{(1)}(x)=G_n(x).$$

Hamahata [3] defined poly-Euler polynomials by

$$\sum_{n=0}^{\infty}\mathfrak{E}_n^{(k)}(x)\frac{t^n}{n!}=\frac{2L_{i_k}(1-e^{-t})}{t\left(e^t+1\right)}e^{xt}.$$

For $k=1$, we get $\mathfrak{E}_n^{(1)}(x)=E_n(x)$.

From (37), we obtain the following equalities easily

$$T_n^{(r)}(x)=\sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}x^{n-k},$$

$$\begin{aligned} T_n^{(r)}(x+y) &= \sum_{l=0}^{k}\binom{k}{l}T_k^{(r)}(x)y^{k-l}, \\ T_n^{(r_1+r_2)}(x+y) &= \sum_{k=0}^{n}\binom{n}{k}T_k^{(r_1)}(x)T_{n-k}^{(r_2)}(y) \end{aligned}$$

and

$$T_n^{(r)}(2(x+1))=2T_n^{(r-1)}(2x).$$

2. Hermite Based Tangent Polynomials

Khan *et. al.* in [5] and Ozarslan [11] introduced and investigated the Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials. They proved some identities and relations for these polynomials.

By this motivation, we define Hermite-based Tangent polynomials of order r as

$$\sum_{n=0}^{\infty} {}_HT_n^{(r)}(x,y)\frac{t^n}{n!}=\left(\frac{2}{e^{2t}+1}\right)^r e^{xt+yt^2}. \tag{19}$$

Theorem 2.1. *Let r_1, $r_2\in\mathbb{Z}_+$. We have*

$$\begin{aligned} {}_HT_n^{(r)}(x,y) &= \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(0,0)\,H_{n-k}(x,y), \\ {}_HT_n^{(r)}(x+u,y+v) &= \sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r)}(x,y)\,H_{n-k}(u,v) \end{aligned}$$

and

$${}_HT_n^{(r_1+r_2)}(x+u,y+v)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r_1)}(x,y)\ \ {}_HT_{n-k}^{(r_2)}(u,v).$$

Theorem 2.2. *Let $r\in\mathbb{Z}_+$. Then we obtain*

$${}_HT_n^{(r)}(2(x+u),2(y+v))=\sum_{m=0}^{n}\binom{n}{m}\ {}_HT_{n-m}^{(r)}(x,y)\sum_{p=0}^{m}\binom{m}{p}H_p(x,y)H_{m-p}(x,y).$$

Theorem 2.3. *There is the following implicit relation for the Hermite-based Tangent polynomials as*

$${}_HT_{n+m}^{(r)}(u,v)=\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}(v-y)^{p+q}\ \ {}_HT_{n+m-p-q}^{(r)}(x,y). \tag{20}$$

Proof. From (50), we replace t by $t+u$ and rewrite the generating function as

$$\frac{2e^{y(t+u)^2}}{e^{2t}+1}=e^{-x(t+u)}\sum_{n=0}^{\infty}T_{n+m}^{(r)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}.$$

Replacing x by v in the above equation to the above equation, we get

$$\sum_{n,m=0}^{\infty} {}_{H}T_{n+m}^{(r)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}=e^{(t+u)(v-x)}\sum_{n,m=0}^{\infty} {}_{H}T_{n+m}^{(r)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}$$

which on using formula [19, Srivastava p. 52]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}=\sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{21}$$

The right hand side on (52) becomes

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(v-x)^{p+q}\frac{t^p}{p!}\frac{u^q}{q!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_{H}T_{n+m}^{(r)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=\sum_{n,m=0}^{\infty} {}_{H}T_{n+m}^{(r)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}.$$

By using Cauchy product and comparing the coefficients of both sides, we have (51). □

Theorem 2.4. *There is the following relation between the Hermite-based Tangent polynomials and the Hermite-based Bernoulli polynomials as*

$${}_{H}\mathfrak{B}_n^{(r)}\left(\frac{x+u}{4},\frac{y+v}{16}\right)=2^{r-n-k}\sum_{k=0}^{n}\binom{n}{k} {}_{H}T_k^{(r)}(x,y)\ {}_{H}\mathfrak{B}_{n-k}^{(r)}\left(\frac{u}{2},\frac{v}{4}\right). \tag{22}$$

Proof. From (50), we obtain

$$\sum_{n=0}^{\infty} {}_{H}\mathfrak{B}_n^{(r)}\left(\frac{x+u}{4},\frac{y+v}{16}\right)\frac{(4t)^n}{n!}=\left(\frac{2\times 4t}{e^{4t}-1}\right)^{(r)}e^{(x+u)t+(y+v)t^2}$$
$$=\left(\frac{2}{e^{2t}+1}\right)^{(r)}e^{xt+yt^2}2^r\left(\frac{2t}{e^{2t}-1}\right)^{(r)}e^{ut+vt^2}$$
$$=\sum_{n=0}^{\infty} {}_{H}T_n^{(r)}(x,y)\frac{t^n}{n!}2^r\sum_{q=0}^{\infty} {}_{H}\mathfrak{B}_q^{(r)}\left(\frac{u}{2},\frac{v}{4}\right)\frac{(2t)^n}{n!}.$$

By using Cauchy product and comparing the coefficients of both sides. We get (53). □

3. Modified Degenerate Hermite-Based Tangent Polynomials

Dolgy *et. al.* [2] introduced and investigated the modified degenerate Bernoulli polynomials. Known *et. al.* [8] defined and investigated the modified degenerate Euler polynomials. They proved some properties for these polynomials.

By these motivations, we define 2-variable fully degenerate Hermite polynomials and the fully degenerate Hermite-based Tangent polynomials of order r

$$\sum_{n=0}^{\infty} H_n(x,y:\lambda)\frac{t^n}{n!}=(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{23}$$

and

$$\sum_{n=0}^{\infty} {}_{H}T_n^{(r)}(x,y:\lambda)\frac{t^n}{n!}=\left(\frac{2}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\right)^r(1+\lambda)^{\frac{xt+yt^2}{\lambda}}, \tag{24}$$

respectively.

From (54) and (55), we get

$$\lim_{\lambda\longrightarrow 0} H_n(x,y:\lambda)=H_n(x,y),\ \lim_{\lambda\longrightarrow 0}\ {}_HT_n^{(r)}(x,y:\lambda)=\ {}_HT_n^{(r)}(x,y).$$

Similiary, we define the fully Hermite-based Bernoulli poynomials and the fully Hermite-based Euler polynomials as

$$\sum_{n=0}^{\infty}\ {}_H\mathfrak{B}_n(x,y:\lambda)\frac{t^n}{n!}=\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{25}$$

and

$$\sum_{n=0}^{\infty}\ {}_H\mathfrak{E}_n(x,y:\lambda)\frac{t^n}{n!}=\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{26}$$

respectively.

From (55), we obtain the following relations easily

$${}_HT_n^{(r_1+r_2)}(x+u,y+v:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r_1)}(x,y:\lambda)\ {}_HT_{n-k}^{(r_2)}(u,v:\lambda),$$

$${}_HT_n^{(r)}(x,y:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r)}(0,0:\lambda)H_{n-k}(x,y:\lambda),$$

$${}_HT_n^{(r)}(x+2,y:\lambda)+\ {}_HT_n^{(r)}(x,y:\lambda)=2\ {}_HT_n^{(r-1)}(x,y:\lambda)$$

for $r=1$,

$${}_HT_n(x+2,y:\lambda)+\ {}_HT_n(x,y:\lambda)=2H_n(x,y:\lambda)$$

and

$${}_HT_n^{(r)}(x,y:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_n^{(r)}\left(\frac{1}{2},0:\lambda\right)H_{n-k}\left(x-\frac{1}{2},y:\lambda\right).$$

Theorem 3.1. *There is the following relation between the fully degenerate Bernoulli polynomials, the fully degenerate Euler polynomials and the fully degenerate Tangent polynomials as*

$$\begin{aligned}&{}_H\mathfrak{B}_n(x,y:\lambda)2^{2n+1}\\ =&\sum_{q=0}^{n}\binom{n}{q}\ {}_HT_{n-q}(x,y:\lambda)\sum_{k=0}^{q}\binom{q}{k}\ {}_H\mathfrak{B}_{q-k}(x,y:\lambda)\\ &\cdot\ {}_H\mathfrak{E}_n(2x,14y:\lambda).\end{aligned} \tag{27}$$

Proof. From (56), (57) and (55), we write as

$$\sum_{n=0}^{\infty}\ {}_H\mathfrak{B}_n(x,y:\lambda)\frac{(4t)^n}{n!}=\left(\frac{4t}{(1+\lambda)^{\frac{4t}{\lambda}}-1}\right)(1+\lambda)^{\frac{4tx+y(4t)^2}{\lambda}}$$

$$\begin{aligned}&=\frac{1}{2}\frac{2e^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\frac{2te^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}\frac{2e^{\frac{2xt+14yt^2}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}+1}\\ &=\frac{1}{2}\sum_{n=0}^{\infty}\ {}_HT_n(x,y:\lambda)\frac{t^n}{n!}\sum_{p=0}^{\infty}\ {}_H\mathfrak{B}_p(x,y:\lambda)\frac{t^p}{p!}\sum_{q=0}^{\infty}\ {}_H\mathfrak{E}_q(2x,14y:\lambda)\frac{t^q}{q!}.\end{aligned}$$

By using the Cauchy product and comparing the coefficient of $\frac{t^n}{n!}$, we have (58). ☐

Theorem 3.2. *$n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} & {}_HT_n(x+2,y:\lambda) + {}_HT_n(x,y:\lambda) \\ = \; & \frac{2}{n+1}\left\{ {}_H\mathfrak{B}_{n+1}(x+1,y:\lambda) - {}_H\mathfrak{B}_{n+1}(x,y:\lambda)\right\}. \end{aligned} \tag{28}$$

Proof. By (55)

$$\frac{2t(1+\lambda)^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\left[(1+\lambda)^{\frac{2t}{\lambda}}+1\right] = \frac{2t(1+\lambda)^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}\left[(1+\lambda)^{\frac{t}{\lambda}}-1\right]$$

$$\frac{2t(1+\lambda)^{\frac{(x+2)t+yt^2}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1} + \frac{2t(1+\lambda)^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1} = \frac{2t(1+\lambda)^{\frac{(x+1)t+yt^2}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1} - \frac{2t(1+\lambda)^{\frac{xt+yt^2}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}$$

$$\begin{aligned} & t\sum_{n=0}^{\infty}\left\{ {}_HT_n(x+2,y:\lambda) + {}_HT_n(x,y:\lambda)\right\}\frac{t^n}{n!} \\ = \; & 2\sum_{n=0}^{\infty}\left\{ {}_H\mathfrak{B}_n(x+1,y:\lambda) - {}_H\mathfrak{B}_n(x,y:\lambda)\right\}\frac{t^n}{n!}. \end{aligned}$$

From the above equality we have (59). □

4. Poly-Tangent Polynomials

In this section, we define the poly-tangent numbers and polynomials and provide some of their relevant properties.

Definition 4.1. *We define the Hermite-based poly-tangent polynomials by*

$$\frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!}, \tag{29}$$

when $x=0$, ${}_H\mathcal{T}_n^{(k)} := {}_H\mathcal{T}_n^{(k)}(0,0)$ are called the Hermite-based poly-tangent numbers.

For $k=1$ and $L_{i_k}(z) = -\log(1-z)$, from (60)

$$\frac{2L_{i_1}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2} = \frac{2e^{xt+yt^2}}{e^{2t}+1} = \sum_{n=0}^{\infty} {}_H\mathcal{T}_n(x,y)\frac{t^n}{n!}. \tag{30}$$

By (61), we get

$${}_H\mathcal{T}_n^{(1)}(x,y) = {}_HT_n(x,y).$$

Theorem 4.1. *n, $k \in \mathbb{Z}_+$, we have*

$${}_H\mathcal{T}_n^{(k)}(x,y) = \frac{1}{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j} {}_H\mathcal{T}_{n+1}(x-j,y). \tag{31}$$

Proof. Consider

$$\sum_{n=0}^{\infty} {}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}\frac{\left(1-e^{-t}\right)^{m+1}}{(m+1)^k}\frac{e^{xt+yt^2}}{t\left(e^{2t}+1\right)}$$

$$
\begin{aligned}
&= 2\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{e^{-tj+xt+yt^2}}{t\left(e^{2t}+1\right)}\\
&= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{1}{t}\frac{2}{e^{2t}+1}e^{t(x-j)+yt^2}\\
&= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{n=0}^{\infty}\left({}_H\mathcal{T}_n(x-j,y)\right)\frac{t^{n-1}}{n!}
\end{aligned}
$$

$$
=\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{n=-1}^{\infty}\frac{{}_H\mathcal{T}_{n+1}(x-j,y)}{n+1}\frac{t^n}{n!}.
$$

Comparing the coefficients both sides, we have (62). □

Theorem 4.2. *There is the following relation between poly-tangent polynomials and the Stirling numbers of the second kind and the Hermite-based Bernoulli polynomials as*

$$
{}_H\mathcal{T}_n^{(k)}(x,y)=\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\ {}_H\mathfrak{B}_i^{(r)}(x,y)\ \ {}_H\mathcal{T}_{n-l-i}^{(r)}. \tag{32}
$$

Proof. From (60), we write as

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!} &= \frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2}\\
&= \frac{\left(e^t-1\right)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt+yt^2}\frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(e^t-1\right)^r}{r!}\left(\sum_{n=0}^{\infty}{}_H\mathfrak{B}_n^{(r)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{q=0}^{\infty}{}_H\mathcal{T}_q^{(r)}\frac{t^q}{q!}\right)\frac{r!}{t^r}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}{}_H\mathfrak{B}_i^{(r)}(x,y)\ \ {}_H\mathcal{T}_{n-l-i}^{(r)}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain (63). □

Theorem 4.3. *There is the following relation between the poly-tangent polynomials, the poly-Genocchi numbers and the Hermite-based tangent polynomials*

$$
{}_H\mathcal{T}_n^{(k)}(x,y)=\frac{1}{2}\sum_{p=0}^{n}\binom{n}{p}G_{n-p}^{(k)}\left\{\ {}_H\mathcal{T}_n(x+1,y)+\ {}_H\mathcal{T}_n(x,y)\right\}. \tag{33}
$$

Proof. From (60) and (49)

$$
\sum_{n=0}^{\infty}{}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!}=\frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2}
$$

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{2L_{i_k}\left(1-e^{-t}\right)}{e^t+1}\right)\frac{2\left(e^t+1\right)e^{xt+yt^2}}{t\left(e^{2t}+1\right)} \\
&= \frac{1}{2}\frac{2L_{i_k}\left(1-e^{-t}\right)}{e^t+1}\left(\frac{2e^{(x+1)t+yt^2}}{t\left(e^{2t}+1\right)}+\frac{2e^{xt+yt^2}}{t\left(e^{2t}+1\right)}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}\left\{\sum_{p=0}^{\infty}\ {}_H\mathcal{T}_p\left(x+1,y\right)+\ {}_H\mathcal{T}_p\left(x,y\right)\frac{t^p}{p!}\right\}.
\end{aligned}$$

Comparing the coefficients of both sides, we have (64). □

The Bernoulli polynomials $B_n^{(r)}(x)$ of order α, the Euler polynomials $E_n^{(r)}(x;\lambda)$ of order α and the Genocchi polynomials $G_n^{(r)}(x;\lambda)$ of order α are defined as respectively:

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\ |t|<2\pi, \tag{34}$$

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},\ |t|<\pi \tag{35}$$

and

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}G_n^{(r)}(x)\frac{t^n}{n!},\ |t|<\pi. \tag{36}$$

When $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$, $E_n^{(r)}(0)=E_n^{(r)}$ and $G_n^{(r)}(0)=G_n^{(r)}$ are called Bernoulli numbers of order r, Euler numbers of order r and Genocchi numbers of order r, respectively.

The familiar tangent polynomials $T_n^{(r)}(x)$ of order r are defined by the generating functions ([12]-[15], [17])

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}T_n^{(r)}\left(x\right)\frac{t^n}{n!},\ |2t|<\pi. \tag{37}$$

When $x=0$, $T_n^{(r)}(0)=T_n^{(r)}$ are called the tangent numbers.

2-variable Hermite-Kampéde Fériet polynomials are defined in ([5], [11]) as

$$\sum_{n=0}^{\infty}H_n\left(x,y\right)\frac{t^n}{n!}=e^{xt+yt^2}. \tag{38}$$

Khan *et al.* in [5] defined and studied on Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials as

$$\sum_{n=0}^{\infty}\ {}_H\mathcal{B}_n(x,y)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt+yt^2},\ |t|<2\pi \tag{39}$$

and

$$\sum_{n=0}^{\infty}\ {}_H\mathcal{E}_n(x,y)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt+yt^2},\ |t|<\pi, \tag{40}$$

respectively.

Carlitz in [1] defined degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}\,(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\mathfrak{B}_n(x\mid\lambda)\frac{t^n}{n!}. \tag{41}$$

When $x=0$, $\mathfrak{B}_n(\lambda)=\mathfrak{B}_n(0\mid\lambda)$ are called the degenerate Bernoulli numbers.

From (41), we can easily derive the following equation

$$\mathfrak{B}_n(x\mid\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{n-l}(\lambda)\,(x\mid\lambda)_l\,,\;n\geq 0,$$

where $(x\mid\lambda)_n=x\,(x-\lambda)\cdots(x-\lambda\,(n-1))$ and $(x\mid\lambda)_n=1$.

Dolgy *et. al.* [2] defined the modified degenerate Bernoulli polynomials, which are different from Carlitz's degenerate Bernoulli polynomials as

$$\frac{t}{(1+\lambda)^{t/\lambda}-1}\,(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{42}$$

When $x=0$, $\mathfrak{B}_{n,\lambda}=\mathfrak{B}_{n,\lambda}(0)$ are called the modified degenerate Bernoulli numbers. From (42) we note that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{t}{(1+\lambda)^{t/\lambda}-1}\,(1+\lambda)^{xt/\lambda}\\ &= \frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.\end{aligned} \tag{43}$$

Thus, by (43)

$$\lim_{\lambda\to 0}\mathfrak{B}_{n,\lambda}(x)=B_n(x).$$

H.-In Known *et. al.* [8] defined the modified degenerate Euler polynomials as

$$\frac{2}{(1+\lambda)^{t/\lambda}+1}\,(1+\lambda)^{xt/\lambda}=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda}(x)\frac{t^n}{n!} \tag{44}$$

and T. Kim *et. al.* in [6] defined the modified degenerate Genocchi polynomials as

$$\frac{2t}{(1+\lambda)^{t/\lambda}+1}\,(1+\lambda)^{tx/\lambda}=\sum_{n=0}^{\infty}\mathfrak{G}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{45}$$

From (44) and (45), we get

$$\lim_{\lambda\to 0}\mathfrak{E}_{n,\lambda}(x)=E_n(x),\;\lim_{\lambda\to 0}\mathfrak{G}_{n,\lambda}(x)=G_n(x).$$

For $k\in\mathbb{Z}$, $k>1$, then k-th polylogarithm is defined by Kaneko [4] as

$$L_{i_k}(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^k}. \tag{46}$$

Thus this function is convergent for $|z|<1$, when $k=1$

$$L_{i_1}(z)=-\log(1-z). \tag{47}$$

Kim *et. al.* in [7] defined the poly-Bernoulli polynomials and the poly-Genocchi polynomials as

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{(k)}(x)\frac{t^n}{n!}=\frac{L_{i_k}(1-e^{-t})}{1-e^{-t}}e^{xt} \tag{48}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{G}_n^{(k)}(x)\frac{t^n}{n!} = \frac{2L_{i_k}(1-e^{-t})}{e^t+1}e^{xt}, \tag{49}$$

respectively.

For $k=1$, by use (47) in (48) and (49), we get

$$\mathfrak{B}_n^{(1)}(x) = (-1)^{n+1}B_n(x),\ \mathfrak{G}_n^{(1)}(x) = G_n(x).$$

Hamahata [3] defined poly-Euler polynomials by

$$\sum_{n=0}^{\infty} \mathfrak{E}_n^{(k)}(x)\frac{t^n}{n!} = \frac{2L_{i_k}(1-e^{-t})}{t\left(e^t+1\right)}e^{xt}.$$

For $k=1$, we get $\mathfrak{E}_n^{(1)}(x) = E_n(x)$.

From (37), we obtain the following equalities easily

$$T_n^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}x^{n-k},$$

$$\begin{aligned} T_n^{(r)}(x+y) &= \sum_{l=0}^{k}\binom{k}{l}T_k^{(r)}(x)y^{k-l}, \\ T_n^{(r_1+r_2)}(x+y) &= \sum_{k=0}^{n}\binom{n}{k}T_k^{(r_1)}(x)T_{n-k}^{(r_2)}(y) \end{aligned}$$

and

$$T_n^{(r)}\left(2\left(x+1\right)\right) = 2T_n^{(r-1)}\left(2x\right).$$

5. Hermite Based Tangent Polynomials

Khan *et. al.* in [5] and Ozarslan [11] introduced and investigated the Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials. They proved some identities and relations for these polynomials.

By this motivation, we define Hermite-based Tangent polynomials of order r as

$$\sum_{n=0}^{\infty} {}_HT_n^{(r)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right)^r e^{xt+yt^2}. \tag{50}$$

Theorem 5.1. *Let $r_1,\ r_2 \in \mathbb{Z}_+$. We have*

$$\begin{aligned} {}_HT_n^{(r)}(x,y) &= \sum_{k=0}^{n}\binom{n}{k}T_k^{(r)}(0,0)\,H_{n-k}(x,y), \\ {}_HT_n^{(r)}(x+u,y+v) &= \sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r)}(x,y)\,H_{n-k}(u,v) \end{aligned}$$

and

$${}_HT_n^{(r_1+r_2)}(x+u,y+v) = \sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r_1)}(x,y)\ \ {}_HT_{n-k}^{(r_2)}(u,v)\,.$$

Theorem 5.2. *Let $r \in \mathbb{Z}_+$. Then we obtain*

$${}_HT_n^{(r)}\left(2\left(x+u\right),2\left(y+v\right)\right) = \sum_{m=0}^{n}\binom{n}{m}\ {}_HT_{n-m}^{(r)}(x,y)\sum_{p=0}^{m}\binom{m}{p}H_p(x,y)H_{m-p}(x,y).$$

Theorem 5.3. *There is the following implicit relation for the Hermite-based Tangent polynomials as*

$$_{H}T_{n+m}^{(r)}\left(u,v\right)=\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}\left(v-y\right)^{p+q}\ {}_{H}T_{n+m-p-q}^{(r)}\left(x,y\right). \tag{51}$$

Proof. From (50), we replace t by $t+u$ and rewrite the generating function as

$$\frac{2e^{y(t+u)^{2}}}{e^{2t}+1}=e^{-x(t+u)}\sum_{n=0}^{\infty}T_{n+m}^{(r)}\left(x,y\right)\frac{t^{n}}{n!}\frac{u^{m}}{m!}.$$

Replacing x by v in the above equation to the above equation, we get

$$\sum_{n,m=0}^{\infty}\ {}_{H}T_{n+m}^{(r)}\left(v,y\right)\frac{t^{n}}{n!}\frac{u^{m}}{m!}=e^{(t+u)(v-x)}\sum_{n,m=0}^{\infty}\ {}_{H}T_{n+m}^{(r)}\left(x,y\right)\frac{t^{n}}{n!}\frac{u^{m}}{m!}$$

which on using formula [19, Srivastava p. 52]

$$\sum_{N=0}^{\infty}f\left(N\right)\frac{\left(x+y\right)^{N}}{N!}=\sum_{n,m=0}^{\infty}f\left(n+m\right)\frac{x^{n}}{n!}\frac{y^{m}}{m!}. \tag{52}$$

The right hand side on (52) becomes

$$\begin{aligned}&\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(v-x\right)^{p+q}\frac{t^{p}}{p!}\frac{u^{q}}{q!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\ {}_{H}T_{n+m}^{(r)}\left(x,y\right)\frac{t^{n}}{n!}\frac{u^{m}}{m!}\\ =&\sum_{n,m=0}^{\infty}\ {}_{H}T_{n+m}^{(r)}\left(v,y\right)\frac{t^{n}}{n!}\frac{u^{m}}{m!}.\end{aligned}$$

By using Cauchy product and comparing the coefficients of both sides, we have (51). □

Theorem 5.4. *There is the following relation between the Hermite-based Tangent polynomials and the Hermite-based Bernoulli polynomials as*

$$_{H}\mathfrak{B}_{n}^{(r)}\left(\frac{x+u}{4},\frac{y+v}{16}\right)=2^{r-n-k}\sum_{k=0}^{n}\binom{n}{k}\ {}_{H}T_{k}^{(r)}\left(x,y\right)\ {}_{H}\mathfrak{B}_{n-k}^{(r)}\left(\frac{u}{2},\frac{v}{4}\right). \tag{53}$$

Proof. From (50), we obtain

$$\sum_{n=0}^{\infty}\ {}_{H}\mathfrak{B}_{n}^{(r)}\left(\frac{x+u}{4},\frac{y+v}{16}\right)\frac{(4t)^{n}}{n!}=\left(\frac{2\times 4t}{e^{4t}-1}\right)^{(r)}e^{(x+u)t+(y+v)t^{2}}$$

$$\begin{aligned}&=\left(\frac{2}{e^{2t}+1}\right)^{(r)}e^{xt+yt^{2}}2^{r}\left(\frac{2t}{e^{2t}-1}\right)^{(r)}e^{ut+vt^{2}}\\ &=\sum_{n=0}^{\infty}\ {}_{H}T_{n}^{(r)}\left(x,y\right)\frac{t^{n}}{n!}2^{r}\sum_{q=0}^{\infty}\ {}_{H}\mathfrak{B}_{q}^{(r)}\left(\frac{u}{2},\frac{v}{4}\right)\frac{(2t)^{n}}{n!}.\end{aligned}$$

By using Cauchy product and comparing the coefficients of both sides. We get (53). □

6. Modified Degenerate Hermite-Based Tangent Polynomials

Dolgy *et. al.* [2] introduced and investigated the modified degenerate Bernoulli polynomials. Known *et. al.* [8] defined and investigated the modified degenerate Euler polynomials. They proved some properties for these polynomials.

By these motivations, we define 2-variable fully degenerate Hermite polynomials and the fully degenerate Hermite-based Tangent polynomials of order r

$$\sum_{n=0}^{\infty} H_n(x,y:\lambda)\frac{t^n}{n!}=(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{54}$$

and

$$\sum_{n=0}^{\infty} {}_HT_n^{(r)}(x,y:\lambda)\frac{t^n}{n!}=\left(\frac{2}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\right)^r(1+\lambda)^{\frac{xt+yt^2}{\lambda}}, \tag{55}$$

respectively.

From (54) and (55), we get

$$\lim_{\lambda\longrightarrow 0} H_n(x,y:\lambda)=H_n(x,y),\ \lim_{\lambda\longrightarrow 0} {}_HT_n^{(r)}(x,y:\lambda)= {}_HT_n^{(r)}(x,y).$$

Similiary, we define the fully Hermite-based Bernoulli poynomials and the fully Hermite-based Euler polynomials as

$$\sum_{n=0}^{\infty} {}_H\mathfrak{B}_n(x,y:\lambda)\frac{t^n}{n!}=\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{56}$$

and

$$\sum_{n=0}^{\infty} {}_H\mathfrak{E}_n(x,y:\lambda)\frac{t^n}{n!}=\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{xt+yt^2}{\lambda}} \tag{57}$$

respectively.

From (55), we obtain the following relations easily

$${}_HT_n^{(r_1+r_2)}(x+u,y+v:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r_1)}(x,y:\lambda)\ {}_HT_{n-k}^{(r_2)}(u,v:\lambda),$$

$${}_HT_n^{(r)}(x,y:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_k^{(r)}(0,0:\lambda)\,H_{n-k}(x,y:\lambda),$$

$${}_HT_n^{(r)}(x+2,y:\lambda)+\ {}_HT_n^{(r)}(x,y:\lambda)=2\ {}_HT_n^{(r-1)}(x,y:\lambda)$$

for $r=1$,

$${}_HT_n(x+2,y:\lambda)+\ {}_HT_n(x,y:\lambda)=2H_n(x,y:\lambda)$$

and

$${}_HT_n^{(r)}(x,y:\lambda)=\sum_{k=0}^{n}\binom{n}{k}\ {}_HT_n^{(r)}\left(\frac{1}{2},0:\lambda\right)H_{n-k}\left(x-\frac{1}{2},y:\lambda\right).$$

Theorem 6.1. *There is the following relation between the fully degenerate Bernoulli polynomials, the fully degenerate Euler polynomials and the fully degenerate Tangent polynomials as*

$$\begin{aligned}&{}_H\mathfrak{B}_n(x,y:\lambda)\,2^{2n+1}\\ =&\sum_{q=0}^{n}\binom{n}{q}\ {}_HT_{n-q}(x,y:\lambda)\sum_{k=0}^{q}\binom{q}{k}\ {}_H\mathfrak{B}_{q-k}(x,y:\lambda)\\ &\cdot\ {}_H\mathfrak{E}_n(2x,14y:\lambda).\end{aligned} \tag{58}$$

Proof. From (56), (57) and (55), we write as

$$\sum_{n=0}^{\infty} {}_{H}\mathfrak{B}_{n}(x,y:\lambda)\frac{(4t)^{n}}{n!} = \left(\frac{4t}{(1+\lambda)^{\frac{4t}{\lambda}}-1}\right)(1+\lambda)^{\frac{4tx+y(4t)^{2}}{\lambda}}$$

$$= \frac{1}{2}\frac{2e^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\frac{2te^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}\frac{2e^{\frac{2xt+14yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} {}_{H}T_{n}(x,y:\lambda)\frac{t^{n}}{n!}\sum_{p=0}^{\infty} {}_{H}\mathfrak{B}_{p}(x,y:\lambda)\frac{t^{p}}{p!}\sum_{q=0}^{\infty} {}_{H}\mathfrak{E}_{q}(2x,14y:\lambda)\frac{t^{q}}{q!}.$$

By using the Cauchy product and comparing the coefficient of $\frac{t^n}{n!}$, we have (58). □

Theorem 6.2. *$n \in \mathbb{Z}_{+}$, we have*

$$\begin{aligned} &{}_{H}T_{n}(x+2,y:\lambda) + {}_{H}T_{n}(x,y:\lambda) \\ = &\frac{2}{n+1}\left\{ {}_{H}\mathfrak{B}_{n+1}(x+1,y:\lambda) - {}_{H}\mathfrak{B}_{n+1}(x,y:\lambda)\right\}. \end{aligned} \tag{59}$$

Proof. By (55)

$$\frac{2t(1+\lambda)^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1}\left[(1+\lambda)^{\frac{2t}{\lambda}}+1\right] = \frac{2t(1+\lambda)^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}\left[(1+\lambda)^{\frac{t}{\lambda}}-1\right]$$

$$\frac{2t(1+\lambda)^{\frac{(x+2)t+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1} + \frac{2t(1+\lambda)^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{2t}{\lambda}}+1} = \frac{2t(1+\lambda)^{\frac{(x+1)t+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1} - \frac{2t(1+\lambda)^{\frac{xt+yt^{2}}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1}$$

$$\begin{aligned} &t\sum_{n=0}^{\infty}\left\{ {}_{H}T_{n}(x+2,y:\lambda) + {}_{H}T_{n}(x,y:\lambda)\right\}\frac{t^{n}}{n!} \\ = \ &2\sum_{n=0}^{\infty}\left\{ {}_{H}\mathfrak{B}_{n}(x+1,y:\lambda) - {}_{H}\mathfrak{B}_{n}(x,y:\lambda)\right\}\frac{t^{n}}{n!}. \end{aligned}$$

From the above equality we have (59). □

7. Poly-Tangent Polynomials

In this section, we define the poly-tangent numbers and polynomials and provide some of their relevant properties.

Definition 7.1. *We define the Hermite-based poly-tangent polynomials by*

$$\frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^{2}} = \sum_{n=0}^{\infty} {}_{H}\mathcal{T}_{n}^{(k)}(x,y)\frac{t^{n}}{n!}, \tag{60}$$

when $x=0$, ${}_{H}\mathcal{T}_{n}^{(k)} := {}_{H}\mathcal{T}_{n}^{(k)}(0,0)$ are called the Hermite-based poly-tangent numbers.

For $k=1$ and $L_{i_k}(z) = -\log(1-z)$, from (60)

$$\frac{2L_{i_1}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^{2}} = \frac{2e^{xt+yt^{2}}}{e^{2t}+1} = \sum_{n=0}^{\infty} {}_{H}\mathcal{T}_{n}(x,y)\frac{t^{n}}{n!}. \tag{61}$$

By (61), we get

$${}_{H}\mathcal{T}_{n}^{(1)}(x,y) = {}_{H}T_{n}(x,y).$$

Theorem 7.1. *n, $k \in \mathbb{Z}_+$, we have*

$$ {}_H\mathcal{T}_n^{(k)}(x,y) = \frac{1}{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\ {}_H\mathcal{T}_{n+1}(x-j,y). \tag{62} $$

Proof. Consider

$$ \sum_{n=0}^{\infty} {}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}\frac{\left(1-e^{-t}\right)^{m+1}}{(m+1)^k}\frac{e^{xt+yt^2}}{t\left(e^{2t}+1\right)} $$

$$ \begin{aligned} &= 2\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{e^{-tj+xt+yt^2}}{t\left(e^{2t}+1\right)} \\ &= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{1}{t}\frac{2}{e^{2t}+1}e^{t(x-j)+yt^2} \\ &= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{n=0}^{\infty}\left({}_H\mathcal{T}_n(x-j,y)\right)\frac{t^{n-1}}{n!} \\ &= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{n=-1}^{\infty}\frac{{}_H\mathcal{T}_{n+1}(x-j,y)}{n+1}\frac{t^n}{n!}. \end{aligned} $$

Comparing the coefficients both sides, we have (62). □

Theorem 7.2. *There is the following relation between poly-tangent polynomials and the Stirling numbers of the second kind and the Hermite-based Bernoulli polynomials as*

$$ {}_H\mathcal{T}_n^{(k)}(x,y) = \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\ {}_H\mathfrak{B}_i^{(r)}(x,y)\ \ {}_H\mathcal{T}_{n-l-i}^{(r)}. \tag{63} $$

Proof. From (60), we write as

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{T}_n^{(k)}(x,y)\frac{t^n}{n!} &= \frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2} \\ &= \frac{\left(e^t-1\right)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt+yt^2}\frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)} \end{aligned} $$

$$ \begin{aligned} &= \frac{\left(e^t-1\right)^r}{r!}\left(\sum_{n=0}^{\infty} {}_H\mathfrak{B}_n^{(r)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{q=0}^{\infty} {}_H\mathcal{T}_q^{(r)}\frac{t^q}{q!}\right)\frac{r!}{t^r} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\ {}_H\mathfrak{B}_i^{(r)}(x,y)\ \ {}_H\mathcal{T}_{n-l-i}^{(r)}\right)\frac{t^n}{n!}. \end{aligned} $$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain (63). □

Theorem 7.3. *There is the following relation between the poly-tangent polynomials, the poly-Genocchi numbers and the Hermite-based tangent polynomials*

$$ {}_H\mathcal{T}_n^{(k)}(x,y) = \frac{1}{2}\sum_{p=0}^{n}\binom{n}{p}G_{n-p}^{(k)}\left\{\ {}_H\mathcal{T}_n(x+1,y) +\ {}_H\mathcal{T}_n(x,y)\right\}. \tag{64} $$

Proof. From (60) and (49)

$$\sum_{n=0}^{\infty} {}_{H}\mathcal{T}_{n}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2L_{i_k}\left(1-e^{-t}\right)}{t\left(e^{2t}+1\right)}e^{xt+yt^2}$$

$$= \frac{1}{2}\left(\frac{2L_{i_k}\left(1-e^{-t}\right)}{e^t+1}\right)\frac{2\left(e^t+1\right)e^{xt+yt^2}}{t\left(e^{2t}+1\right)}$$

$$= \frac{1}{2}\frac{2L_{i_k}\left(1-e^{-t}\right)}{e^t+1}\left(\frac{2e^{(x+1)t+yt^2}}{t\left(e^{2t}+1\right)}+\frac{2e^{xt+yt^2}}{t\left(e^{2t}+1\right)}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}\left\{\sum_{p=0}^{\infty} {}_{H}\mathcal{T}_{p}(x+1,y)+ {}_{H}\mathcal{T}_{p}(x,y)\frac{t^p}{p!}\right\}.$$

Comparing the coefficients of both sides, we have (64). □

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