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ON HARDY TYPE INEQUALITIES VIA K-FRACTIONAL INTEGRALS

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Abstract. In this study, we will give the k-fractional integral inequalities to take advantage of the some results of Hardy type inequalities and some special cases.

Keywords: Hölder's inequality, k-fractional integrals, Hardy inequality.

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1. Introduction

The classical Hardy inequality (see [4]) states that for $f \geq 0$ and integrable over any finite interval $(0, x)$ and f^p is integrable and convergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^\infty \left(\frac{1}{x} \left(\int_0^x f(t)\, dt \right) dx \right)^p \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx,$$

unless $f = 0$. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This inequality has been proved by Hardy in 1925 and plays an important role in analysis and its applications, see ([1], [4]-[9], [12]-[16]) and the references therein.

Now, we give some motivating results to our work. Firstly, the following generalization is accomplished by N. Levinson in [9] :

$$\int_a^b \left(\frac{F(x)}{x} \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_a^b f^p(t)\, dt,$$

where $f > 0$ on $[a, b] \subseteq [0, \infty)$, $p > 1$, and $F(x) = \int_0^x f(t)\, dt$.

Then, in [15] W.T. Sulaiman presented the following like Hardy İnequality:

$$p \int_a^b \left(\frac{F(x)}{x} \right)^p dx \leq (b-a)^p \int_a^b \left(\frac{f(x)}{x} \right)^p dx - \int_a^b \left(1 - \frac{a}{x} \right)^p f^p(x)\, dx. \quad (1)$$

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Lately, in [14] B. Sroysang established the following generalized result:

$$p\int_a^b \frac{F^p(x)}{x^q}dx \leq (b-a)^p \int_a^b \frac{f^p(x)}{x^q}dx - \int_a^b \frac{(x-a)^p}{x^q} f^p(x)\,dx. \quad (2)$$

The significant integral results given in the paper by S.Wu et al. [16] is other motivation for us. As our results, some inequalities of this reference be able to make a deduction as some special cases. We also generalise some results obtained by the authors of [7].

2. Preliminaries

In this section, we will give some necessary definitions and mathematical preliminaries of k-fractional calculus theory which are used further in this paper.

In [2] Diaz and Pariguan have defined k -gamma function Γ_k, k -beta function B_k and the Pochhammer k -symbol $(x)_{n,k}$ that is generalization of the classical gamma, beta functions and the classical Pochhammer symbol. Γ_k is given by formula

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad k>0.$$

It has shown that Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, clearly given by

$$\Gamma_k(\alpha) := \int_0^\infty e^{-\frac{t^k}{k}} t^{\alpha-1} dt.$$

Obviously, $\Gamma_k(x+k) = x\Gamma_k(x)$, $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x)$ and $\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma(\frac{x}{k})$. Later, in [10] Mubeen and Habibullah have introduced the k-fractional integral of Riemann-Liouville type as follows:

$$J^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad \alpha>0,\ x>0,\ k>0.$$

Furthermore, in [11] Romero and et al. give the following definition.

Definition 2.1. *Let α be a real non negative number. Let f be piece wise continuous on $I' = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then k-Riemann Liouville fractional integral of f order α*

$$J_a^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x>a, k>0. \quad (3)$$

Note that when $k=1$ in the above integral, then it reduces to the classical Riemann–Liouville fractional integral. Also, for the expression (3), when $f(x) = (x-a)^\mu$,we get:

$$J_a^{\alpha,k}(x-a)^\mu = \frac{\Gamma_k(\mu k+k)}{\Gamma_k(\alpha+\mu k+k)}(x-a)^{\mu+\frac{\alpha}{k}}, x\in[a,b],$$

and for $x=b$, we have

$$J_a^{\alpha,k} f(b) = \frac{1}{k\Gamma_k(\alpha)} \int_a^b (b-t)^{\frac{\alpha}{k}-1} f(t)dt.$$

Besides, we have the folllowing properties for $\alpha>0, \beta>0, k>0$:

$$J_a^{\alpha,k} J_a^{\beta,k} f(x) = J_a^{\alpha+\beta,k} f(x),$$

$$J_a^{\alpha,k} J_a^{\beta,k} f(x) = J_a^{\beta,k} J_a^{\alpha,k} f(x).$$

For some recent results connected with k -gamma function, k -beta function and k-fractional integral inequalities see ([2], [3], [8], [10], [11],[13]) and the references therein.

In this paper, we establish several new inequalities of Hardy's type inequalities via k-fractional integral. Now, we give our main results.

3. Main Results

We start with the following Theorem:

Theorem 3.1. *Let η be a non negative real number and let $f > 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty)$. If $\frac{x-a+\eta}{g(x)}$ is non-increasing, then for all $p > 1$, $\frac{\alpha}{k} \geq 1$, the k-fractional integral inequality*

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)} \right)^p dx \\
\leq\ & \frac{\Gamma_k^{p-1}\left(k - \frac{k}{p}\right)}{\Gamma_k^{p-1}\left(\alpha + k - \frac{k}{p}\right)\left(\frac{\alpha}{k}(p-1) - p + \frac{1}{p}\right)} \\
&\times \left\{ (b-a)^{\frac{\alpha}{k}(p-1)-p+\frac{1}{p}} \left(J_a^{\alpha,k}\left[\frac{f(b)}{g(b)} (b-a+\eta)^p (b-a)^{\frac{p-1}{p}} \right] \right) \right. \\
&\left. - J_a^{\alpha,k}\left[\frac{f(b)}{g(b)} (b-a+\eta)^p (b-a)^{\frac{\alpha}{k}(p-1)-p+1} \right] \right\}
\end{aligned}$$

is valid.

Proof. We have

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)} \right)^p dx \\
=\ & \int_a^b g^{-p}(x) \left[\int_a^x \frac{1}{k\Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1} f(t) (t-a)^{\frac{p-1}{p^2}} (t-a)^{\frac{1-p}{p^2}} dt \right]^p dx.
\end{aligned}$$

Thanks to Hölder inequality, we find that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)} \right)^p dx \\
\leq\ & \frac{1}{k^p \Gamma_k^p(\alpha)} \int_a^b g^{-p}(x) \\
&\times \left\{ \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt \right]^{\frac{1}{p}} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} (t-a)^{\left(\frac{1-p}{p^2}\right)\left(\frac{p}{p-1}\right)} dt \right]^{1-\frac{1}{p}} \right\}^p dx.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{1}{k^p \Gamma_k^p(\alpha)} \int_a^b g^{-p}(x) \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt\right] \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} (t-a)^{\frac{-1}{p}} dt\right]^{p-1} dx \\
=\ & \frac{1}{k\Gamma_k(\alpha)} \int_a^b g^{-p}(x) \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt\right] \left[J_a^{\alpha,k} (x-a)^{\frac{-1}{p}}\right]^{p-1} dx.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma_k^{p-1}\left(k-\frac{k}{p}\right)}{k\Gamma_k(\alpha)\Gamma_k^{p-1}\left(\alpha+k-\frac{k}{p}\right)} \\
& \times \left\{\int_a^b g^{-p}(x) (x-a)^{\left(\frac{\alpha}{k}-\frac{1}{p}\right)(p-1)} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt\right]\right\} dx.
\end{aligned}$$

This is to say that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma_k^{p-1}\left(k-\frac{k}{p}\right)}{k\Gamma_k(\alpha)\Gamma_k^{p-1}\left(\alpha+k-\frac{k}{p}\right)} \\
& \times \left\{\int_a^b \left(\frac{x-a}{g(x)}\right)^p (x-a)^{\frac{\alpha}{k}(p-1)-p-1+\frac{1}{p}} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt\right]\right\} dx.
\end{aligned}$$

Since $\frac{x-a+\eta}{g(x)}$ is non increasing and with the change of integration order, then we can write

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma_k^{p-1}\left(k-\frac{k}{p}\right)}{k\Gamma_k(\alpha)\Gamma_k^{p-1}\left(\alpha+k-\frac{k}{p}\right)} \\
& \times \left\{\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\frac{\alpha}{k}-1} f^p(t) (t-a)^{\frac{p-1}{p}} \left[\int_t^b (x-a)^{\frac{\alpha}{k}(p-1)-1+\frac{1}{p}-p} dx\right] dt\right\}.
\end{aligned}$$

Therefore,

$$\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx$$

$$\leq \frac{\Gamma_k^{p-1}\left(k-\frac{k}{p}\right)}{k\Gamma_k(\alpha)\Gamma_k^{p-1}\left(\alpha+k-\frac{k}{p}\right)\left(\frac{\alpha}{k}(p-1)+\frac{1}{p}-p\right)}$$

$$\times\left\{\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\frac{\alpha}{k}-1} f^p(t)(t-a)^{\frac{p-1}{p}}\left[(b-a)^{\frac{\alpha}{k}(p-1)+\frac{1}{p}-p}-(t-a)^{\frac{\alpha}{k}(p-1)+\frac{1}{p}-p}\right]dt\right\}.$$

Consequently,

$$\int_a^b \left(\frac{J_a^{\alpha,k} f(x)}{g(x)}\right)^p dx$$

$$\leq \frac{\Gamma_k^{p-1}\left(k-\frac{k}{p}\right)}{k\Gamma_k(\alpha)\Gamma_k^{p-1}\left(\alpha+k-\frac{k}{p}\right)\left(\frac{\alpha}{k}(p-1)+\frac{1}{p}-p\right)}$$

$$\times\left[(b-a)^{\frac{\alpha}{k}(p-1)+\frac{1}{p}-p}\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\frac{\alpha}{k}-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right.$$

$$\left.-\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\frac{\alpha}{k}-1} f^p(t)(t-a)^{\frac{\alpha}{k}(p-1)+1-p}dt\right].$$

Finally by rearranging the above inequality, we get the desired result. □

Remark 3.1. *Taking $\alpha=1$ and $k=1$ in Theorem 3.1,we obtain Theorem 3.1 of* [16].

Theorem 3.2. *Let $f>0$ and $g>0$ on $[a,b]\subseteq[0,\infty)$ such that g is non-decreasing, then for all $p>1, q>0$, $\frac{\alpha}{k}\geq 1$, we have*

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)}dx \tag{4}$$

$$\leq \frac{1}{\Gamma_k^{p-1}(\alpha+k)\left(\frac{\alpha}{k}(p-1)+1\right)}$$

$$\times\left\{(b-a)^{\frac{\alpha}{k}(p-1)+1}J_a^{\alpha,k}\left(\frac{f^p(b)}{g^q(b)}\right)-J_a^{\alpha,k}\left[\frac{f^p(b)}{g^q(b)}(b-a)^{\frac{\alpha}{k}(p-1)+1}\right]\right\}.$$

Proof. We have,

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)}dx = \int_a^b g^{-q}(x)\left[\int_a^x \frac{1}{k\Gamma_k(\alpha)}(x-t)^{\frac{\alpha}{k}-1} f(t)\,dt\right]^p dx$$

and then,

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)}dx \leq \int_a^b g^{-q}(x)\left[\left(J_a^{\alpha,k} f^p(x)\right)^{\frac{1}{p}}\left(J_a^{\alpha,k}(1)\right)^{1-\frac{1}{p}}\right]^p dx.$$

Accordingly,

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx$$
$$\leq \int_a^b g^{-q}(x) \left\{ \left[\frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt \right] \left[\frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1}\, dt \right]^{p-1} \right\} dx.$$

So, we obtain

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx$$
$$\leq \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \int_a^b g^{-q}(x)\,(x-a)^{\frac{\alpha}{k}(p-1)} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt \right] dx.$$

Since g is non-decreasing and with the change of integration order, we have

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx$$
$$\leq \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \int_a^b g^{-q}(t) f^p(t)\,(b-t)^{\frac{\alpha}{k}-1}\, dt \int_t^b (x-a)^{\frac{\alpha}{k}(p-1)}\, dx.$$

Hence,

$$\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx$$
$$\leq \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)\left(\frac{\alpha}{k}(p-1)+1\right)}$$
$$\times \left\{ \int_a^b g^{-q}(t) f^p(t)\,(b-t)^{\frac{\alpha}{k}-1} \left[(b-a)^{\frac{\alpha}{k}(p-1)+1} - (t-a)^{\frac{\alpha}{k}(p-1)+1} \right] dt \right\}.$$

Finally by rearranging the above inequality, we get the desired result. □

Remark 3.2. *(i) Putting $\alpha = 1$, $k = 1$ in Theorem 3.2, we obtain the first part of Theorem 3.5 in* [16].

(ii) Taking $\alpha = 1$, $k = 1$ and $g(x) = x$ in Theorem 3.2, we obtain Sroysang inequality (2).

(iii) Putting $\alpha = 1$, $k = 1$, $g(x) = x$ and $p = q$ in Theorem 3.2,we obtain Sulaiman inequality (1).

Now, we give the last main result with the following theorem.

Theorem 3.3. *Let $f \geq 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty)$ such that g is non-decreasing. Then, for all $0 < p < 1, q > 0, \frac{\alpha}{k} \geq 1$, we have*

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{g^{-q}(b)}{\left(\frac{\alpha}{k}(p-1)+1\right)\Gamma_k^{p-1}(\alpha+k)} \\
&\times \left[\frac{(-1)^{\frac{\alpha}{k}(p-1)+1}}{\Gamma_k(\alpha)}\Gamma_k(\alpha p+k) J_b^{\alpha p+k,k} f^p(a) - (b-a)^{\frac{\alpha}{k}(p-1)+1} J_b^{\alpha,k} f^p(a)\right].
\end{aligned}$$

Proof. Thanks to the weighted reverse Hölder inequality, we have

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{1}{k^p \Gamma_k^p(\alpha)} \int_a^b g^{-q}(x) \left\{ \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt\right]^{\frac{1}{p}} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} dt\right]^{1-\frac{1}{p}} \right\}^p dx \\
=\ & \frac{1}{k\Gamma_k(\alpha)} \int_a^b g^{-q}(x) \left\{ \left(\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt\right) \left(J_a^{\alpha,k}(1)\right)^{p-1} \right\} dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \int_a^b g^{-q}(x)(x-a)^{\frac{\alpha}{k}(p-1)} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt\right] dx.
\end{aligned}$$

Since g is non-decreasing and with the change of integration order, we obtain

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \int_a^b g^{-q}(b)(x-a)^{\frac{\alpha}{k}(p-1)} \left[\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)\, dt\right] dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{1}{k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \int_a^b g^{-q}(b)(a-t)^{\frac{\alpha}{k}-1} f^p(t) \left[\int_t^b (x-a)^{\frac{\alpha}{k}(p-1)} dx\right] dt \\
=\ & \frac{1}{\left(\frac{\alpha}{k}(p-1)+1\right) k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \\
&\times \left\{ \int_a^b g^{-q}(b)(a-t)^{\frac{\alpha}{k}-1} f^p(t) \left[(t-a)^{\frac{\alpha}{k}(p-1)+1} - (b-a)^{\frac{\alpha}{k}(p-1)+1}\right] dt \right\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_a^b \frac{\left(J_a^{\alpha,k} f(x)\right)^p}{g^q(x)} dx \\
\geq\ & \frac{1}{\left(\frac{\alpha}{k}(p-1)+1\right) k\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)} \\
&\times \left[(b-a)^{\frac{\alpha}{k}(p-1)+1} \int_a^b g^{-q}(b)(a-t)^{\frac{\alpha}{k}-1} f^p(t)\,dt \right.\\
&\left. - \int_a^b g^{-q}(b)(a-t)^{\frac{\alpha}{k}-1} f^p(t)(t-a)^{\frac{\alpha}{k}(p-1)+1}\,dt\right].
\end{aligned}$$

Finally by rearranging the above inequality, we get the desired result. □

Remark 3.3. *Taking $k = 1$ in the above theorems, we get generalizations of the results in the paper* [7].

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