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LONG TIME BEHAVIOR OF THE STRONGLY DAMPED WAVE EQUATION WITH p-LAPLACIAN IN $\mathbb{R}^n$

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ABSTRACT. In this paper, the initial value problem for the one dimensional strongly damped wave equation with p-Laplacian and localized damping in the whole space is concerned. Under the condition $2 < p < 4$, the existence of weak local attractors for this problem in $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$ is proved.

Keywords: wave equation, p-Laplacian, attractors.

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1. Introduction

This paper is devoted to the investigation of the long time behavior of the strongly damped wave equation including p-Laplacian and localized damping

$$u_{tt} - u_{txx} - u_{xx} - \frac{\partial}{\partial x}\left(|u_x|^{p-2} u_x\right) + a(x) u_t + f(u) = g(x), \text{ in } (0,\infty)\times\mathbb{R}, \quad (1)$$

with the initial data

$$u(0,\cdot) = u_0(\cdot) \quad , \quad u_t(0,\cdot) = u_1(\cdot), \text{ in } \mathbb{R}, \quad (2)$$

where

$$p > 2 , \ g \in L^2(\mathbb{R}) \quad (3)$$

and the functions $a(\cdot)$, $f(\cdot)$ satisfy the following conditions:

$$a \in L^1(\mathbb{R}), \, a(\cdot) \geq 0 \text{ a.e. in } \mathbb{R}, \quad (4)$$

$$a(\cdot) \geq a_0 > 0 \text{ a.e. in } \{x \in \mathbb{R} : |x| \geq r_0\}, \quad (5)$$

$$f \in C^1(\mathbb{R}) \text{ and } sf(s) \geq \lambda s^2, \ \forall s \in \mathbb{R}, \quad (6)$$

for some $r_0, \lambda > 0$.

The strongly damped wave equations have been drawing a great deal of attention of many mathematicians for many years. In particular, long term analysis of these types of equations has recently been an attractive research topic due to their presence in modeling many significant physical phenomena, such as motions of viscoelastic materials, heat conduction and so on. With particular reference to the fact that asymptotic behavior of

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evolution equations can be represented in terms of attractors, authors have been investigating the attractors for the strongly damped wave equations over the last years. In bounded domains, attractors for the equations involving linear Laplacian were studied in [1-11] and in [12-13] for the equations with the linear and nonlinear strong damping, respectively. For the equations with nonlinear Laplacian, it is referred to [14-16]. In [14], by applying the splitting method, the authors proved the existence of finite dimensional global attractor for the strongly damped wave equation including nonlinear Laplacian in the form $\frac{\partial}{\partial x}\sigma(u_x)$ under the conditions $\sigma \in C^1(\mathbb{R})$ and $\sigma'(\cdot) \geq r_0 > 0$. In [15], the existence of finite dimensional regular exponential and global attractors was obtained for the strongly damped wave equation with more general nonlinear Laplacian. Here, it is worth noting that the additional term $-\Delta u$ in the considered equation, together with nonlinear degenerate Laplacian, produces actually non-degenerate Laplacian, which yields some useful estimates for weak and strong solutions. For the strongly damped wave equations with degenerate Laplacian, it is referred to [16] in which the authors considered the strongly damped wave equation with p-Laplacian

$$u_{tt} - u_{txx} - \frac{\partial}{\partial x}\left(|u_x|^{p-2}\, u_x\right) + f\left(u\right) = g\left(x\right),$$

for $p > 2$, $g \in L^2(0,1)$ and

$$f \in C^1\left(\mathbb{R}\right) \quad \text{and} \quad \liminf_{|s|\to\infty} \frac{f\left(s\right)}{|s|^{p-2}\, s} > -\lambda^p,$$

where λ is the first eigenvalue of the Laplacian. In the paper, for $p > 2$, the existence of the weak local attractors in $W_0^{1,p}\left(0,1\right) \times L^2\left(0,1\right)$ was firstly proved. Then, under the restriction $2 < p < 4$, the authors obtained the boundedness of the attractors in $W^{1,\infty}\left(0,1\right) \times W^{1,\infty}\left(0,1\right)$ and thereby established the existence of a regular strong global attractor in $W_0^{1,p}\left(0,1\right) \times L^2\left(0,1\right)$.

In unbounded domains, the obstacle is that one can not apply the methods used in the study of long time behavior of wave equations in bounded domain because of the absence of Sobolev compact embedding theorems. In order to overcome this difficulty, many different approaches have been presented over the last years. As far as it is known, the first studies about the attractors for hyperbolic and hyperbolic-like equations were presented by Feireisl in [17] and [18] in which the existence of global attractors of weakly damped wave equations in $H^1\left(\mathbb{R}^n\right) \times L^2\left(\mathbb{R}^n\right)$ was proved using the property of finite propagation speed of wave equations. In those works, the author used the splitting method based on a decomposition of solution in asymptotically small and compact parts. It is referred to [19] and [20] in which the authors proved the existence of global attractors in $H^1\left(\mathbb{R}^n\right) \times L^2\left(\mathbb{R}^n\right)$ for the wave equations with the strong damping term $-\Delta u_t$ as well as weak damping term, under different conditions on nonlinearities. Here, while the strong damping term increases the dissipation, it also brings parabolicity to the equation, which means more regularization, but also an infinite propagation speed of initial disturbances. Therefore, the authors could not apply Feireisl's method. However, they overcame this situation by introducing further decomposition of the compact part of the solution. In the present paper, instead of using the splitting method, by establishing regular tail estimates for equation (1), the existence of weak local attractors in $\left(W^{1,p}(\mathbb{R}) \cap H^1\left(\mathbb{R}\right)\right) \times L^2\left(\mathbb{R}\right)$ for $2 < p < 4$ is shown.

This paper is structured as follows. In the next section, some definitions and the main result of the paper are stated. In the last section, the existence of the weak local attractors in $\left(W^{1,p}(\mathbb{R}) \cap H^1\left(\mathbb{R}\right)\right) \times L^2(\mathbb{R})$ for $2 < p < 4$ is established. Moreover, it is proved that these attractors attract the trajectories in the strong topology of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

2. The Statement of the Main Result

First of all, the definitions are given as follows:

Definition 2.1. *The function* $u \in L^1(0,T;W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))$ *satisfying* $u_t \in L^1(0,T;H^1(\mathbb{R})) \cap C([0,T];W^{-1,\frac{p}{p-1}}(\mathbb{R}) + H^{-1}(\mathbb{R}))$, $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$ *and the equation*

$$\frac{d}{dt}\int_{\mathbb{R}} u_t(t,x)v(x)dx + \int_{\mathbb{R}} u_{tx}(t,x)v'(x)dx + \int_{\mathbb{R}} u_x(t,x)v'(x)dx$$
$$+\int_{\mathbb{R}} |u_x(t,x)|^{p-2}u_x(t,x)v'(x)dx + \frac{d}{dt}\int_{\mathbb{R}} a(x)\,u(t,x)\,v(x)\,dx$$
$$+\int_{\mathbb{R}} f(u(t,x))v(x)dx = \int_{\mathbb{R}} g(x)v(x)dx,$$

in the sense of distributions on $(0,T)$, *for all* $v \in W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})$, *is called a weak solution to the problem (1)-(2) in* $[0,T]\times\mathbb{R}$.

Definition 2.2. *Let* $\{V(t)\}_{t\geq 0}$ *be an operator semigroup on a linear normed space* E *and* B *be a bounded subset of* E. *A set* $\mathcal{A}_B \subset E$ *is called a strong (weak) local attractor for* B *and the semigroup* $\{V(t)\}_{t\geq 0}$ *iff*

- $\mathcal{A}_B$ *is strongly (weakly) compact in* E;
- $\mathcal{A}_B$ *is invariant, i.e.* $V(t)\mathcal{A}_B = \mathcal{A}_B$, $\forall t \geq 0$;
- $\mathcal{A}_B$ *attracts the image of* B *in the strong (weak) topology, namely, for every neighborhood* $\mathcal{O}$ *of* $\mathcal{A}_B$ *in the strong (weak) topology of* E *there exists a* $T = T(\mathcal{O}) > 0$ *such that* $V(t)B \subset \mathcal{O}$ *for every* $t \geq T$.

The following well-posedness result can be obtained by applying the method of [15]:

Theorem 2.1. *Assume that the conditions (3)-(6) are satisfied. Then, for any* $T > 0$ *and* $u_0 \in W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, *the problem (1)-(2) admits a unique weak solution* $u(t,x)$ *which satisfies* $u \in L^\infty\left(0,T;W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)$, $u_t \in L^\infty\left(0,T;L^2(\mathbb{R})\right) \cap L^2\left(0,T;H^1(\mathbb{R})\right)$, $u_{tt} \in L^2\left(0,T;W^{-1,\frac{p}{p-1}}(\mathbb{R}) + H^{-1}(\mathbb{R})\right)$ *and the energy inequality*

$$E(u(t)) + \int_s^t \|u_{tx}(\tau)\|^2_{L^2(\mathbb{R})}d\tau + \int_s^t\int_{\mathbb{R}} a(x)\,|u_t(\tau,x)|^2\,dxd\tau \leq E(u(s))\,,\ \forall t\geq s\geq 0, \quad (7)$$

where $E(u(t)) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|u_x(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{p}\|u_x(t)\|^p_{L^p(\mathbb{R})} + \int_{\mathbb{R}} F(u(t,x))\,dx - \int_{\mathbb{R}} g(x)\,u(t,x)\,dx$ *and* $F(u) = \int_0^u f(z)dz$. *Moreover, if* $v \in L^\infty\left(0,T;W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right) \cap W^{1,\infty}\left(0,T;L^2(\mathbb{R})\right) \cap W^{1,2}\left(0,T;H^1(\mathbb{R})\right) \cap W^{2,2}\left(0,T;W^{-1,\frac{p}{p-1}}(\mathbb{R}) + H^{-1}(\mathbb{R})\right)$ *is also a weak solution to (1)-(2) with initial data* $(v_0,v_1) \in \left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$, *then*

$$\|u(t) - v(t)\|_{H^1(\mathbb{R})} + \|u_t(t) - v_t(t)\|_{H^{-1}(\mathbb{R})}$$
$$\leq C\left(T, \|(u_0,u_1)\|_{(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})}, \|(v_0,v_1)\|_{(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})}\right)$$
$$\times\left(\|u_0 - v_0\|_{H^1(\mathbb{R})} + \|u_1 - v_1\|_{L^2(\mathbb{R})}\right),\ \forall t\in[0,T] \quad (8)$$

where $C : \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function with respect to each variable.*

With the aid of Theorem 2.1, one can immediately derive that the problem (1)-(2) generates a weakly continuous semigroup $\{S(t)\}_{t\geq 0}$ in $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$, by the formula $S(t)(u_0, u_1) = (u(t), u_t(t))$, where $u(t,x)$ is the weak solution of the problem (1)-(2) with the initial data (u_0, u_1). By the weak continuity, it is meant that $\varphi_n \underset{n\to\infty}{\rightarrow} \varphi$ in $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$ implies $S(t)\varphi_n \underset{n\to\infty}{\overset{w}{\longrightarrow}} S(t)\varphi$ in $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$.

Now, the main result of this paper is as follows.

Theorem 2.2. *In addition to (3)-(6), assume that $p < 4$. Then, for every bounded subset B of $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$ the semigroup $\{S(t)\}_{t\geq 0}$, generated by the problem (1)-(2), has a weak local attractor $\mathcal{A}_B$ in $\left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R})$. Moreover, the weak local attractor $\mathcal{A}_B$ attracts the image of B in the strong topology of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.*

3. Existence of Weak Local Attractors

The aim of this section is to prove Theorem 2.2. To this end, we need the following lemmas.

Lemma 3.1. *Let the conditions (3)-(6) hold and B be a bounded set in $(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})) \times L^2(\mathbb{R})$. Then, for any $\varepsilon > 0$, there exist $T = T(B,\varepsilon) > 0$ and $R = R(B,\varepsilon) > 0$ such that*

$$\|S(t)\varphi\|_{H^1(\mathbb{R}\backslash(-r,r))\times L^2(\mathbb{R}\backslash(-r,r))} < \varepsilon, \qquad \forall t \geq T,\ \forall r \geq R,\ \forall \varphi \in B.$$

Proof. First of all, using (3)-(4) and (6) in (7), it is obtained that

$$\begin{cases} \|S(t)\varphi\|_{(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})} \leq C_B,\ \forall t \geq 0, \forall \varphi \in B, \\ \int\limits_0^t \|u_{tx}(\tau)\|^2_{L^2(\mathbb{R})}\, d\tau \leq C_B,\ \forall t \geq 0, \end{cases} \tag{9}$$

for some constant $C_B > 0$. Now, let $\varphi \in B$ and $(u(t), u_t(t)) = S(t)\varphi$. Define $\eta \in C^\infty(\mathbb{R})$, $0 \leq \eta(\cdot) \leq 1$, $\eta(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| \geq 2 \end{cases}$, and $\eta_r(x) = \eta\left(\frac{x}{r}\right)$. Multiplying the equation (1) by $\delta\eta_r^2 u$ and integrating the obtained equality over $\mathbb{R}$, by (6) and (9), it is found that

$$\frac{d}{dt}\left(\delta\int\limits_{\mathbb{R}} \eta_r^2(x)\, u_t(t,x)\, u(t,x)\, dx + \frac{\delta}{2}\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})} + \frac{\delta}{2}\int\limits_{\mathbb{R}} a(x)\,\eta_r^2(x)\,|u(t,x)|^2\, dx\right)$$
$$+\delta\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})} + \delta\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})} + \delta\lambda\|\eta_r u(t)\|^2_{L^2(\mathbb{R})} - \delta\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})}$$
$$\leq C_1\left(\frac{1}{r} + \frac{1}{r}\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R}\backslash(-r,r))}\right),\ \forall t \geq 0. \tag{10}$$

On the other hand, multiplying the equation (1) by $\eta_r^2 u_t$ and integrating the obtained equality over $\mathbb{R}$, by (5) and (9), it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{p}\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})}\right.$$
$$\left.+\int\limits_{\mathbb{R}} \eta_r^2(x)\, F(u(t,x))\, dx\right) + a_0\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})}$$
$$\leq C_2\left(\frac{1}{r} + \frac{1}{r}\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R}\backslash(-r,r))}\right),\ \forall r \geq r_0, \forall t \geq 0. \tag{11}$$

Now, summing (10) and (11), for sufficiently small δ, it is deduced that

$$\frac{d\Phi(t)}{dt}+C_3\left(\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})}+\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})}+\|\eta_r u(t)\|^2_{L^2(\mathbb{R})}+\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})}\right)$$

$$\leq C_4\left(\frac{1}{r}+\frac{1}{r}\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}+\|g\|^2_{L^2(\mathbb{R}\setminus(-r,r))}\right),\ \forall r\geq r_0,\forall t\geq 0, \tag{12}$$

where

$$\Phi(t):=\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})}+\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})}+\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})}+\int_{\mathbb{R}}\eta_r^2(x)F(u(t,x))\,dx$$

$$+\delta\int_{\mathbb{R}}\eta_r^2(x)u_t(t,x)u(t,x)\,dx+\frac{\delta}{2}\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})}+\frac{\delta}{2}\int_{\mathbb{R}}a(x)\eta_r(x)|u(t,x)|^2\,dx.$$

Furthermore, by (4), (6), (7) and (9), it is obtained that

$$C_5\left(\|\sqrt{\eta_r}u_t(t)\|^2_{L^2(\mathbb{R})}+\|\sqrt{\eta_r}u_x(t)\|^2_{L^2(\mathbb{R})}+\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})}+\|\sqrt{\eta_r}u(t)\|^2_{L^2(\mathbb{R})}\right)$$

$$\leq\Phi(t)\leq C_6\left(\|\eta_r u_t(t)\|^2_{L^2(\mathbb{R})}+\|\eta_r u_x(t)\|^2_{L^2(\mathbb{R})}\right.$$

$$\left.+\left\|\sqrt[p]{\eta_r^2}u_x(t)\right\|^p_{L^p(\mathbb{R})}+\|\eta_r u(t)\|^2_{L^2(\mathbb{R})}+\frac{1}{r}\right),\ \forall t\geq 0, \tag{13}$$

for some constants $C_5, C_6>0$. Considering (13) in (12), it follows that

$$\frac{d\Phi(t)}{dt}+C_7\Phi(t)\leq C_8\left(\frac{1}{r}+\frac{1}{r}\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}+\|g\|^2_{L^2(\mathbb{R}\setminus(-r,r))}\right),\ \forall r\geq r_0,\forall t\geq 0.$$

Therefore, together with (9) and (13), it is found that

$$\Phi(t)\leq C_9\left(e^{-C_7 t}+\frac{1}{r}+\|g\|^2_{L^2(\mathbb{R}\setminus(-r,r))}\right),\ \forall r\geq r_0,\forall t\geq 0.$$

Here, the constant $C_9>0$ is dependent on the set B and is independent of t and r as previous constants $C_i (i=\overline{1,8})$. Hence, taking into account the right side of the last inequality, together with (13), the proof of the lemma is completed. □

Lemma 3.2. *Assume that the conditions (3)-(6) hold and B is a bounded subset of $(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})$. Then, every sequence of the form $\{PS(t_k)\varphi_k\}_{k=1}^{\infty}$, where $\{\varphi_k\}_{k=1}^{\infty}\subset B$, $t_k\to\infty$, has a convergent subsequence in $H^1(\mathbb{R})$. Here, $P:(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})\to W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})$ is the projection operator defined as $P(\phi,\psi)=\phi$.*

Proof. Firstly, since the sequence $\{\varphi_k\}_{k=1}^{\infty}$ is bounded in $\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$, by (9), it is obtained that the sequence $\{S(\cdot)\varphi_k\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,\infty;\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)$

$\times L^2(\mathbb{R})$). Hence, by (1), (3)-(4), (6), (7) and (9), for any $T_0 \geq 1$, there exists a subsequence $\{k_m\}_{m=1}^{\infty}$ such that $t_{k_m} \geq T_0$ and

$$\begin{cases} S(t_{k_m} - T_0)\varphi_{k_m} \underset{m\to\infty}{\overset{w}{\to}} \varphi_0 \ \text{ in } \left(W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times L^2(\mathbb{R}), \\ u_m \underset{m\to\infty}{\overset{w^*}{\to}} u \quad \text{in } L^\infty\left(0,\infty; W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})\right), \\ u_{mt} \underset{m\to\infty}{\overset{w^*}{\to}} u_t \ \text{ in } L^\infty\left(0,\infty; L^2(\mathbb{R})\right), \\ u_{mt} \underset{m\to\infty}{\overset{w}{\to}} u_t \ \text{ in } L^2\left(0,\infty; H^1(\mathbb{R})\right), \\ u_{mtt} \underset{m\to\infty}{\overset{w}{\to}} u_{tt} \ \text{ in } L^2\left(0,\infty; H^{-2}(-r,r)\right), \forall r > 0, \\ u_m(t) \underset{m\to\infty}{\overset{w}{\to}} u(t) \quad \text{in } W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R}), \forall t \geq 0, \\ u_{mt}(t) \underset{m\to\infty}{\overset{w}{\to}} u_t(t) \ \text{ in } L^2(\mathbb{R}), \forall t \geq 0, \end{cases} \tag{14}$$

for some $\varphi_0 \in (W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})) \times L^2(\mathbb{R})$ and $u \in L^\infty(0,\infty; W^{1,p}(\mathbb{R}) \cap H^1(\mathbb{R})) \cap W^{1,\infty}(0,\infty; L^2(\mathbb{R})) \cap W^{1,2}(0,\infty; H^1(\mathbb{R})) \cap W^{2,2}(0,\infty; H^{-2}(-r,r))$, where $(u_m(t), u_{mt}(t)) = S(t + t_{k_m} - T_0)\varphi_{k_m}$.

Now, replacing u in the equation (1) with u_m and u_n, and then subtracting the obtained equations, the following equation is found:

$$u_{mtt}(t,x) - u_{ntt}(t,x) - (u_{mtxx}(t,x) - u_{ntxx}(t,x)) - (u_{mxx}(t,x) - u_{nxx}(t,x))$$
$$-\frac{\partial}{\partial x}(|u_{mx}(t,x)|^{p-2} u_{mx}(t,x) - |u_{nx}(t,x)|^{p-2} u_{nx}(t,x))$$
$$+a(x)(u_{mt}(t,x) - u_{nt}(t,x)) + f(u_m(t,x)) - f(u_n(t,x)) = 0.$$

Testing this equation with $2(1-\eta_r)t(u_m - u_n)$ in $(0,T) \times \mathbb{R}$, where η_r is the function defined in the previous lemma, and considering (4) and the inequality

$$\left(|a|^{p-2} a - |b|^{p-2} b\right)(a-b) \geq 0, \ a, b \in \mathbb{R},$$

it is deduced that

$$T\left\|\sqrt{1-\eta_r}\,(u_{mx}(T) - u_{nx}(T))\right\|^2_{L^2(\mathbb{R})}$$
$$\leq -2T\int_{\mathbb{R}}(1-\eta_r(x))(u_{mt}(T,x) - u_{nt}(T,x))(u_m(T,x) - u_n(T,x))\,dx$$
$$+2\int_0^T\int_{\mathbb{R}}(1-\eta_r(x))(u_{mt}(t,x) - u_{nt}(t,x))(u_m(t,x) - u_n(t,x))\,dxdt$$
$$+2\int_0^T t\left\|\sqrt{1-\eta_r}\,(u_{mt}(t) - u_{nt}(t))\right\|^2_{L^2(\mathbb{R})}dt$$
$$+\frac{2}{r}\int_0^T\int_{\mathbb{R}} t\eta'\left(\frac{x}{r}\right)(u_{mtx}(t,x) - u_{ntx}(t,x))(u_m(t,x) - u_n(t,x))\,dxdt$$
$$+\int_0^T\left\|\sqrt{1-\eta_r}\,(u_{mx}(t) - u_{nx}(t))\right\|^2_{L^2(\mathbb{R})}dt$$

$$+\frac{2}{r}\int_{0}^{T}\int_{\mathbb{R}} t\eta'\left(\frac{x}{r}\right)\left(u_{mx}(t,x)-u_{nx}(t,x)\right)\left(u_m(t,x)-u_n(t,x)\right)dxdt$$

$$-2\int_{0}^{T} t\left\|\sqrt{1-\eta_r}\left(u_{mx}(t)-u_{nx}(t)\right)\right\|_{L^2(\mathbb{R})}^2 dt$$

$$+\frac{2}{r}\int_{0}^{T}\int_{\mathbb{R}} t\eta'\left(\frac{x}{r}\right)\left(|u_{mx}(t,x)|^{p-2}u_{mx}(t,x)-|u_{nx}(t,x)|^{p-2}u_{nx}(t,x)\right)$$

$$\times\left(u_m(t,x)-u_n(t,x)\right)dxdt$$

$$+2\int_{0}^{T}\int_{\mathbb{R}} a(x)\left(1-\eta_r(x)\right)\left|\left(u_m(t,x)-u_n(t,x)\right)\right|^2 dxdt$$

$$-2\int_{0}^{T}\int_{\mathbb{R}} t\left(1-\eta_r(x)\right)\left(f\left(u_m(t,x)\right)-f\left(u_n(t,x)\right)\right)$$

$$\times\left(u_m(t,x)-u_n(t,x)\right)dxdt,\ \forall T\geq 0. \qquad (15)$$

For the fifth term on the right side of (15), it is obtained that

$$\int_{0}^{T}\left\|\sqrt{1-\eta_r}\left(u_{mx}(t)-u_{nx}(t)\right)\right\|_{L^2(\mathbb{R})}^2 dt\leq c_1+2\int_{1}^{T} t\left\|\sqrt{1-\eta_r}\left(u_{mx}(t)-u_{nx}(t)\right)\right\|_{L^2(\mathbb{R})}^2 dt$$

$$\leq c_1+2\int_{0}^{T} t\left\|\sqrt{1-\eta_r}\left(u_{mx}(t)-u_{nx}(t)\right)\right\|_{L^2(\mathbb{R})}^2 dt,\ \forall T\geq 1. \qquad (16)$$

Taking into account (16) in (15) and considering (4), (6), (9) and (14), it is found that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left\|u_{mx}(T)-u_{nx}(T)\right\|_{L^2(-r,r)}^2\leq\frac{c_1}{T},\quad\forall T\geq 1.$$

Thus, together with Lemma 3.1, taking $T=T_0$, for any $\varepsilon>0$, it is obtained that

$$\liminf_{k\to\infty}\liminf_{n\to\infty}\left\|PS(t_k)\varphi_k-PS(t_n)\varphi_n\right\|_{H^1(\mathbb{R})}<\frac{c_2}{\sqrt{T_0}}+\varepsilon,\quad\forall T_0\geq 1,$$

which, by taking limit as $T_0\to\infty$, yields

$$\liminf_{k\to\infty}\liminf_{n\to\infty}\left\|PS(t_k)\varphi_k-PS(t_n)\varphi_n\right\|_{H^1(\mathbb{R})}=0.$$

Moreover, it can be immediately seen that for every subsequence $\{k_m\}_{m=1}^{\infty}$, the following holds:

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\left\|PS(t_{k_m})\varphi_{k_m}-PS(t_{k_n})\varphi_{k_n}\right\|_{H^1(\mathbb{R})}=0. \qquad (17)$$

Now, it is concluded that the sequence $\{PS(t_k)\varphi_k\}_{k=1}^{\infty}$ has a convergent subsequence in $H^1(\mathbb{R})$. If the contrary is assumed, then, by the completeness of $H^1(\mathbb{R})$, there exist $\varepsilon_0>0$ and a subsequence $\{k_m\}_{m=1}^{\infty}$ such that

$$\left\|PS(t_{k_m})\varphi_{k_m}-PS(t_{k_n})\varphi_{k_n}\right\|_{H^1(\mathbb{R})}\geq\varepsilon_0,\ m\neq n,$$

which contradicts (17). Hence, the proof is completed. □

Lemma 3.3. *In addition to the conditions (3)-(6), assume that* $p < 4$*. If* u *is a weak solution to problem (1)-(2), then there exists a constant* $c > 0$ *such that*

$$\sup_{t\geq 1}\|u_t(t)\|_{H^{1-\alpha}(\mathbb{R})} \leq c, \quad \alpha \in \left[\frac{p-2}{p}, \frac{1}{2}\right).$$

Proof. By taking derivative of the equation (1) according to t and denoting $v := u_t$, it is found that

$$v_{tt} + \left(\widetilde{\Lambda} v_t - v_t\right) + \left(\widetilde{\Lambda} v - v\right) - (p-1)\frac{\partial}{\partial x}\left(|u_x|^{p-2} u_{tx}\right) + a(x) v_t + f'(u) u_t = 0, \quad (18)$$

where $\widetilde{\Lambda} : H^2(-2n, 2n) \cap H_0^1(-2n, 2n) \to L^2(-2n, 2n)$, $\widetilde{\Lambda}\varphi = -\frac{\partial^2\varphi}{\partial x^2} + \varphi$. Testing (18) with $t^2\widetilde{\Lambda}^{-\alpha} v_t$ in $\mathbb{R}$, where $\alpha \in \left[\frac{p-2}{p}, \frac{1}{2}\right)$, by the self adjointness of the operator $\widetilde{\Lambda}$, it is deduced that

$$\frac{d}{dt}\left(\frac{t^2}{2}\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} + \frac{t^2}{2}\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v(t)\right\|^2_{L^2(\mathbb{R})}\right) + t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})}$$
$$= t\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} + t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} + t\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v(t)\right\|^2_{L^2(\mathbb{R})}$$
$$+t^2\left\langle \widetilde{\Lambda}^{-\frac{\alpha}{2}} v(t), \widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\rangle - (p-1)t^2\left\langle |u_x(t)|^{p-2} u_{tx}(t), \frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha} v_t(t)\right)\right\rangle$$
$$-t^2\left\langle a v_t(t), \widetilde{\Lambda}^{-\alpha} v_t(t)\right\rangle - t^2\left\langle f'(u(t)) u_t(t), \widetilde{\Lambda}^{-\alpha} v_t(t)\right\rangle, \quad \forall t \geq 0. \quad (19)$$

First of all, for the first term on the right side of (19), by interpolation and Young inequality, for any $\varepsilon > 0$, it is obtained that

$$t\left\|\Lambda^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} + C(\varepsilon)\left\|\widetilde{\Lambda}^{-\frac{1+\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})}, \forall t \geq 0. \quad (20)$$

For the second term on the right side of (20), from (1) and using (3)-(4), (6) and (9), it follows that

$$\left\|\widetilde{\Lambda}^{-\frac{1+\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} \leq C_1\left(1 + \|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a(x)|u_t(t,x)|^2 dx\right), \ \forall t \geq 0,$$

which, together with (20), yields

$$t\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})}$$
$$+\widetilde{C}(\varepsilon)\left(1 + \|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a(x)|u_t(t,x)|^2 dx\right), \ \forall t \geq 0. \quad (21)$$

Similarly, it is found that

$$t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})}$$
$$+\widetilde{C}(\varepsilon) t^2\left(1 + \|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a(x)|u_t(t,x)|^2 dx\right), \ \forall t \geq 0. \quad (22)$$

Regarding the fourth term on the right side of (19), by (9), it is obtained that

$$t^2\left\langle \widetilde{\Lambda}^{-\frac{\alpha}{2}} v(t), \widetilde{\Lambda}^{-\frac{\alpha}{2}} v_t(t)\right\rangle \leq C_2\left(C(\varepsilon) t^2 + \varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}} v_t(t)\right\|^2_{L^2(\mathbb{R})}\right), \ \forall t \geq 0. \quad (23)$$

Considering the fifth term on the right side of (19), by (9) and the embedding $H^{\frac{p-2}{p}}(\mathbb{R}) \hookrightarrow L^{\frac{2p}{4-p}}(\mathbb{R})$, it follows that

$$-(p-1)\,t^2\left\langle |u_x(t)|^{p-2}u_{tx}(t),\frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha}v_t(t)\right)\right\rangle$$
$$\leq C_3t^2\|u_{tx}(t)\|_{L^2(\mathbb{R})}\left\|\frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha}v_t(t)\right)\right\|_{H^{\frac{p-2}{p}}(\mathbb{R})}$$
$$\leq C_4\left(C(\varepsilon)\,t^2\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}+\varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}\right),\ \forall t\geq 0. \tag{24}$$

Now, let us estimate the sixth term on the right side of (19). Since, by the interpolation and Young inequality, for any $\mu>0$, it is obtained that

$$\|v_t(t)\|^2_{H^{1-2\alpha}(\mathbb{R})}\leq C_5\left(C(\mu)\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}+\mu\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}\right),\ \forall t\geq 0,$$

applying Young inequality, for any $\beta>0$, by (4) and (22), it is found that

$$-t^2\left\langle av_t(t),\widetilde{\Lambda}^{-\alpha}v_t(t)\right\rangle\leq C_6t^2\|v_t(t)\|_{L^\infty(\mathbb{R})}\left\|\widetilde{\Lambda}^{-\alpha}v_t(t)\right\|_{L^\infty(\mathbb{R})}$$
$$\leq C_7\left(\beta t^2\|v_t(t)\|^2_{H^{1-\alpha}(\mathbb{R})}+C(\beta)\|v_t(t)\|^2_{H^{1-2\alpha}(\mathbb{R})}\right)$$
$$\leq C_8\left[\left(\beta+\mu C(\beta)+\varepsilon\widehat{C}(\mu,\beta)\right)t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}\right.$$
$$\left.+\widehat{\widehat{C}}(\mu,\beta,\varepsilon)\,t^2\left(1+\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}+\int_{\mathbb{R}}a(x)\,|u_t(t,x)|^2\,dx\right)\right],\ \forall t\geq 0. \tag{25}$$

For the last term on the right side of (19), by (6) and (9), it follows that

$$-t^2\left\langle f'(u(t))\,u_t(t),\widetilde{\Lambda}^{-\alpha}v_t(t)\right\rangle\leq C_9\left(C(\varepsilon)\,t^2+\varepsilon t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}\right),\ \forall t\geq 0. \tag{26}$$

If we use (21)-(26) in (19), it is obtained that

$$\frac{d}{dt}\left(\frac{t^2}{2}\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}+\frac{t^2}{2}\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right)+t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}$$
$$\leq C_{10}\left[\left(\beta+\varepsilon+\mu C(\beta)+\varepsilon\widehat{C}(\mu,\beta)\right)t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}\right.$$
$$+\widetilde{\widetilde{C}}(\mu,\beta,\varepsilon)\left(\left(1+t+t^2\right)\left(1+\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}\right)\right.$$
$$\left.\left.+\left(1+t^2\right)\int_{\mathbb{R}}a(x)\,|u_t(t,x)|^2\,dx\right)\right],\ \forall t\geq 0. \tag{27}$$

Now, testing (8) with $\delta t^2\widetilde{\Lambda}^{-\alpha}v$ in $\mathbb{R}$, by (9), it is obtained that

$$\frac{d}{dt}\left(\delta t^2\left\langle\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t),\widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\rangle+\frac{\delta}{2}t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right)+\delta t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}$$
$$=\delta t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})}+\delta t\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}+\delta t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}$$
$$+\delta\left(t^2+2t\right)\left\langle\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t),\widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\rangle$$

$$-\delta(p-1)t^2\left\langle |u_x(t)|^{p-2}u_{tx}(t), \frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha}v(t)\right)\right\rangle$$
$$-\delta t^2\left\langle av_t(t), \widetilde{\Lambda}^{-\alpha}v(t)\right\rangle - \delta t^2\left\langle f'(u(t))u_t(t), \widetilde{\Lambda}^{-\alpha}v(t)\right\rangle, \ \forall t\geq 0. \quad (28)$$

For the fourth term on the right side of (28), by (9), it follows that

$$\delta\left(t^2+2t\right)\left\langle \widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t), \widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\rangle \leq \delta t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + C_{11}(t+2)^2, \ \forall t\geq 0. \quad (29)$$

Taking into account the fifth term on the right side of (28), by (9), it is found that

$$-\delta(p-1)t^2\left\langle |u_x(t)|^{p-2}u_{tx}(t), \frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha}v(t)\right)\right\rangle$$
$$\leq C_{12}t^2\|u_{tx}(t)\|_{L^2(\mathbb{R})}\left\|\frac{\partial}{\partial x}\left(\widetilde{\Lambda}^{-\alpha}v(t)\right)\right\|_{H^{\frac{p-2}{p}}(\mathbb{R})}$$
$$\leq C_{13}\left(t^2\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + t^2\|u_t(t)\|^2_{H^1(\mathbb{R})}\right)$$
$$\leq C_{14}\left(t^2\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + t^2\right), \ \forall t\geq 0. \quad (30)$$

For the sixth term on the right side of (28), by (4), (9), it is obtained that

$$-\delta t^2\left\langle av_t(t), \widetilde{\Lambda}^{-\alpha}v(t)\right\rangle$$
$$\leq \delta t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + C_{15}\left(t^2\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + t^2\right), \ \forall t\geq 0. \quad (31)$$

For the last term on the right side of (28), by (6) and (9), it is found that

$$-\delta t^2\left\langle f'(u(t))u_t(t), \widetilde{\Lambda}^{-\alpha}v(t)\right\rangle \leq C_{16}t^2, \ \forall t\geq 0. \quad (32)$$

Hence, using (29)-(32) in (28), we have

$$\frac{d}{dt}\left(\delta t^2\left\langle \widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t), \widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\rangle + \frac{\delta}{2}t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right) + \delta t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}$$
$$\leq \delta C_{17}t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + C_{18}\left(t+t^2\right)\|u_{tx}(t)\|^2_{L^2(\mathbb{R})} + C_{17}\left(1+t+t^2\right), \forall t\geq 0,$$

and summing this equation with (27), for sufficiently small ε, μ, β and δ, it follows that

$$\frac{d\Phi(t)}{dt} + C_{19}\left(t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right)$$
$$\leq C_{20}\left(1+t+t^2\right)\left(1+\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}\right) + C_{21}\left(1+t^2\right)\int_{\mathbb{R}} a(x)|u_t(t,x)|^2\,dx, \ \forall t\geq 0, \quad (33)$$

where

$$\Phi(t) := \delta t^2\left\langle \widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t), \widetilde{\Lambda}^{-\frac{\alpha}{2}}v(t)\right\rangle + \frac{\delta}{2}t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}$$
$$+\frac{t^2}{2}\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + \frac{t^2}{2}\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}.$$

Furthermore, there exist constants $C_{22}, C_{23} > 0$ such that

$$C_{22}\left(t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right)$$
$$\leq \Phi(t) \leq C_{23}\left(t^2\left\|\widetilde{\Lambda}^{-\frac{\alpha}{2}}v_t(t)\right\|^2_{L^2(\mathbb{R})} + t^2\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\right), \ \forall t\geq 0. \quad (34)$$

Considering (34) in (33) gives that

$$\frac{d\Phi(t)}{dt}+C_{24}\Phi(t)$$
$$\leq C_{25}\left(1+t+t^2\right)\left(1+\|u_{tx}(t)\|^2_{L^2(\mathbb{R})}\right)+C_{26}\left(1+t^2\right)\int_{\mathbb{R}} a(x)\,|u_t(t,x)|^2\,dx, \forall t\geq 0,$$

which, together with (3)-(6) and (7), yields

$$\Phi(t)\leq C_{27}\left(1+t+t^2\right),\ \forall t\geq 0.$$

Hence, by (34), it is obtained that

$$\left\|\widetilde{\Lambda}^{\frac{1-\alpha}{2}}v(t)\right\|^2_{L^2(\mathbb{R})}\leq C_{28},\ \forall t\geq 1,$$

which gives the result. □

Now, the weak ω-limit set of the trajectories emanating from a set $B\subset(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})$ is defined as follows:

$$\omega_w(B):=\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}S(t)B}^{w}.$$

where the bar over a set means weak closure in $(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})$. It can be easily shown that $\varphi\in\omega_w(B)$ if and only if there exist sequences $\{t_k\}_{k=1}^\infty$, $t_k\to\infty$ and $\{\varphi_k\}_{k=1}^\infty\subset B$ such that $S(t_k)\varphi_k\to\varphi$ weakly in $(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R}))\times L^2(\mathbb{R})$. Moreover, the following invariance property of the set $\omega_w(B)$ is stated:

Lemma 3.4. *For any bounded set $B\subset\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$, the set $\omega_w(B)$ is invariant.*

Proof. Let $\psi\in\omega_w(B)$ and $z=S(t)\psi$ for $t\geq 0$. Then, by the definition of $\omega_w(B)$, there exist the sequences $\{t_k\}_{k=1}^\infty$, $t_k\to\infty$ and $\{\psi_k\}_{k=1}^\infty\subset B$ such that $S(t_k)\psi_k\to\psi$ weakly in $\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$. Moreover, by Lemma 3.1, Lemma 3.2 and Lemma 3.3, there exists a subsequence $\{k_m\}_{m=1}^\infty$ such that $S(t_{k_m})\psi_{k_m}\underset{m\to\infty}{\longrightarrow}\psi$ in $H^1(\mathbb{R})\times H^{-1}(\mathbb{R})$. Then, denoting $\tau_{k_m}:=t+t_{k_m}$, by (8), it is obtained that

$$S(\tau_{k_m})\psi_{k_m}=S(t)S(t_{k_m})\psi_{k_m}\underset{m\to\infty}{\overset{w}{\longrightarrow}}S(t)\psi=z\text{ in }\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R}),$$

which implies $z\in\omega_w(B)$. Therefore, it follows that $S(t)\omega_w(B)\subset\omega_w(B)$.

On the other hand, if $\psi\in\omega_w(B)$, then there exist $\{t_k\}_{k=1}^\infty$, $t_k\to\infty$ and $\{\psi_k\}_{k=1}^\infty\subset B$ such that $S(t_k)\psi_k\to\psi$ weakly in $\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$. Now, define $\varphi_k=S(t_k-t)\psi_k$, for $t_k\geq t\geq 0$. By (9), there exists a subsequence $\{k_m\}_{m=1}^\infty$ such that $\varphi_{k_m}\to\varphi$ weakly in $\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$ for some $\varphi\in\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$, which gives that $\varphi\in\omega_w(B)$. Furthermore, using Lemma 3.1, Lemma 3.2 and Lemma 3.3, and passing to a subsequence, it follows that $\varphi_{k_{m_n}}\underset{n\to\infty}{\longrightarrow}\varphi$ in $H^1(\mathbb{R})\times H^{-1}(\mathbb{R})$. Since

$$S\left(t_{k_{m_n}}\right)\psi_{k_{m_n}}=S(t)S\left(t_{k_{m_n}}-t\right)\psi_{k_{m_n}}=S(t)\varphi_{k_{m_n}}$$

applying (8), it is observed that $S\left(t_{k_{m_n}}\right)\psi_{k_{m_n}}\underset{n\to\infty}{\overset{w}{\longrightarrow}}S(t)\varphi$ in $\left(W^{1,p}(\mathbb{R})\cap H^1(\mathbb{R})\right)\times L^2(\mathbb{R})$. Hence, $\psi=S(t)\varphi$ and so $\omega_w(B)\subset S(t)\omega_w(B)$. □

Thus, by applying Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, the proof of Theorem 2.2 is obtained.

References

[1] Kalantarov, V., (1986), Attractors for some nonlinear problems of mathematical physics, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 152, pp. 50–54.

[2] Ghidaglia, J.M. and Marzocchi, A., (1991), Longtime behaviour of strongly damped wave equations, global attractors and their dimension, SIAM J. Math. Anal., 22, pp. 879–895.

[3] Zhou, S., (1999), Global attractor for strongly damped nonlinear wave equations, Funct. Diff. Eqns., 6, pp. 451–470.

[4] Carvalho, A.N. and Cholewa, J.W., (2002), Attractors for strongly damped wave equations with critical nonlinearities, Pacific J. Math., 207, pp. 287–310.

[5] Pata, V. and Squassina, M., (2005), On the strongly damped wave equation, Commun. Math. Phys., 253, pp. 511–533.

[6] Pata, V. and Zelik, S., (2006), Smooth attractors for strongly damped wave equations, Nonlinearity, 19, pp. 1495–1506.

[7] Yang, M. and Sun, C., (2009), Attractors for strongly damped wave equations, Nonlinear Anal.: Real World Applications, 10, pp. 1097-1100.

[8] Dell'Oro, F. and Pata, V., (2011), Long-term analysis of strongly damped nonlinear wave equations, Nonlinearity, 24, pp. 3413–3435.

[9] Dell'Oro, F. and Pata, V., (2012), Strongly damped wave equations with critical nonlinearities, Nonlinear Anal., 75, pp. 5723–5735.

[10] Khanmamedov, A.Kh., (2011), Global attractors for strongly damped wave equations with displacement dependent damping and nonlinear source term of critical exponent, Discrete Contin. Dyn. Syst. Ser. A, 31, pp. 119–138.

[11] Khanmamedov, A.Kh., (2014), Strongly damped wave equation with exponential nonlinearities, J. Math. Anal. Appl., 419, pp. 663–687.

[12] Khanmamedov, A.Kh., (2008), On the existence of a global attractor for the wave equation with nonlinear strong damping perturbed by nonmonotone term, Nonlinear Anal., 69, pp. 3372-3385.

[13] Chueshov, I. and Lasiecka, I., (2008), Long time behavior of second order evolution equations with nonlinear damping. Mem. Amer. Math. Soc., 195.

[14] Chen, F., Guo, B. and Wang, P., (1998), Long time behavior of strongly damped nonlinear wave equations, J. Diff. Equations, 147, pp. 231-241.

[15] Kalantarov, V. and Zelik, S., (2009), Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, J. Diff. Equations, 247, pp. 1120-1155.

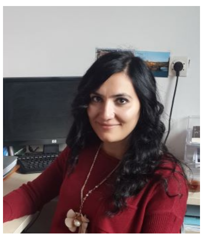
[16] Khanmamedov, A.Kh., Şen, Z., (2017), Attractors for the Strongly Damped Wave Equation with p-Laplacian, Mathematical Methods in the Applied Sciences, 40, pp. 4436-4447.

[17] Feireisl, E., (1994), Attractors for semilinear damped wave equations on $\mathbb{R}^3$, Nonlinear Anal., 23, pp. 187–195.

[18] Feireisl, E., (1995), Asymptotic behavior and attractors for a semilinear damped wave equation with supercritical exponent, Proc. Roy. Soc. Edinburgh, 125, pp. 1051–1062.

[19] Belleri, V. and Pata, V., (2001), Attractors for semilinear strongly damped wave equations on $\mathbb{R}^3$, Discrete Contin. Dyn. Syst, 7, pp. 719–735.

[20] Conti, M., Pata, V. and Squassina, M., (2005), Strongly damped wave equations on $\mathbb{R}^3$ with critical nonlinearities, Commun. Appl. Anal., 9, pp. 61–176.



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