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ANALYTICAL SOLUTION FOR THE CONFORMABLE FRACTIONAL TELEGRAPH EQUATION BY FOURIER METHOD

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Abstract. In this paper, the Fourier method is effectively implemented for solving a conformable fractional telegraph equation. We discuss and derive the analytical solution of the conformable fractional telegraph equation with nonhomogeneous Dirichlet boundary condition.

1. Introduction

The telegraph equation is better than the heat equation in modeling of physical phenomena, which has a parabolic behavior [3]. The one-dimensional telegraph equation can be written as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \left(\frac{R}{L} + \frac{G}{C}\right)\frac{\partial u(x,t)}{\partial t} + \frac{RG}{LC}u(x,t) = \frac{1}{LC}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (1.1)$$

where R and G are, respectively, the resistance and the conductance of resistor, C is the capacitance of capacitor, and L is the inductance of coil. Many concrete applications amount to replacing the time derivative in the telegraph equation with a fractional derivative. For example, in the works [4, 5], the authors have extensively studied the time-fractional telegraph equation with Caputo fractional derivative. For more details about the good effect of the fractional derivative, we refer to monographs [1, 2]. Recently, a new definition of fractional derivative, named "fractional conformable derivative", is introduced by Khalil et al. [6]. This novel fractional derivative is compatible with the classical derivative and it is excellent for studying nonregular solutions. The subject of the fractional conformable derivative has attracted the attention of many authors in domains such as mechanics, electronic, and anomalous diffusion. We are interested in studying in this paper the telegraph model (1.1) in framework of the time-fractional conformable derivative.

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Precisely, we will propose the following transformations:

$$\frac{\partial}{\partial t} \to \mathcal{D}_t^{(\alpha)} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \to \mathcal{D}_t^{(2\alpha)} = \mathcal{D}_t^{(\alpha)}\mathcal{D}_t^{(\alpha)}, \tag{1.2}$$

$$a = G/C, \quad b = R/L, \quad k^2 = 1/LC \tag{1.3}$$

where $\mathcal{D}_t^{(\alpha)}$ is the time-fractional conformable derivative operator [6]. Then, we get the fractional conformable telegraph model associated with the transformation (1.2) and (1.3) as follows:

$$\mathcal{D}_t^{(2\alpha)}u\left(x,t\right) + \left(a+b\right)\mathcal{D}_t^{(\alpha)}u\left(x,t\right) + abu\left(x,t\right) = k^2\frac{\partial^2 u\left(x,t\right)}{\partial x^2} + f\left(x,t\right), \tag{1.4}$$

where x and t are the space and time variables, $f\left(x,t\right)$ is a sufficiently smooth function.

2. Preliminaries on conformable fractional calculus

We start recalling some concepts on the conformable fractional calculus.

Definition 2.1. *([6]). Let $\varphi : [0,+\infty\,[\, \to \mathbb{R}$ be a function. Then, the conformable fractional derivative of function φ of order α at $t>0$ is defined by the following limit:*

$$\mathcal{D}_t^{(\alpha)}\left(\varphi\right)\left(t\right) = \lim_{\varepsilon\to 0}\frac{\varphi\left(t+\varepsilon t^{1-\alpha}\right) - \varphi\left(t\right)}{\varepsilon}, \tag{2.1}$$

when this limit exists and finished.

Definition 2.2. *([6]). Let $\alpha \in\,]\,0,1\,]$ and $\varphi : [0,+\infty\,[\, \to \mathbb{R}$ be real valued function. The conformable fractional integral of φ of order α from zero to t is defined by:*

$$\mathcal{I}_\alpha\varphi\left(t\right) := \int_0^t s^{\alpha-1}\varphi\left(s\right)ds, \quad t\geq 0, \tag{2.2}$$

Lemma. *Assume that φ is a continuous function on $]\,0,+\infty\,[$ and $0<\alpha\leq 1$. Then, for all $t>0$, we have $\mathcal{D}_t^{(\alpha)}\left[\mathcal{I}_\alpha\varphi\left(t\right)\right] = \varphi\left(t\right)$. According to [7], if φ is differentiable, then we have $\mathcal{I}_\alpha\left[\mathcal{D}_t^{(\alpha)}\left(\varphi\right)\left(t\right)\right] = \varphi\left(t\right) - \varphi\left(0\right)$.*

Definition 2.3. *([7]) Let $0<\alpha\leq 1$ and $\varphi : [0,+\infty\,[\, \to \mathbb{R}$ be real valued function. Then, the fractional Laplace transform of order α starting from zero of φ is defined by:*

$$\mathcal{L}_\alpha\left[\varphi\left(t\right)\right]\left(s\right) = \int_0^{+\infty} t^{\alpha-1}\varphi\left(t\right)e^{-s\frac{t^\alpha}{\alpha}}dt. \tag{2.3}$$

Theorem. *([7]) Let $0<\alpha\leq 1$ and $\varphi : [0,+\infty\,[\, \to \mathbb{R}$ be differentiable real valued function. Then, we have:*

$$\mathcal{L}_\alpha\left[\mathcal{D}_t^{(\alpha)}\varphi\left(t\right)\right]\left(s\right) = s\mathcal{L}_\alpha\left[\varphi\left(t\right)\right]\left(s\right) - \varphi\left(0\right). \tag{2.4}$$

We introduce the following theorem, which is used further in this paper.

Proposition. *Let* λ *and* μ *be positive constants with* $4\mu > \lambda^2$ *and* $g : [0, +\infty[\to \mathbb{R}$ *is a continuous function. For all* $0 < \alpha \leq 1$*, The initial value problem*

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} y(t) + \lambda \mathcal{D}_t^{(\alpha)} y(t) + \mu y(t) = g(t), \\ y(0) = y_0, \quad \mathcal{D}_t^{(\alpha)} y(0) = y_\alpha. \end{cases} \tag{2.5}$$

admits a unique solution given by

$$\begin{aligned} y(t) = &\left[y_0 \cos\left(\sqrt{4\mu - \lambda^2} \frac{t^\alpha}{2\alpha} \right) + \frac{\lambda y_0 + 2y_\alpha}{\sqrt{4\mu - \lambda^2}} \sin\left(\sqrt{4\mu - \lambda^2} \frac{t^\alpha}{2\alpha} \right) \right] e^{-\frac{\lambda t^\alpha}{2\alpha}} \\ &+ \frac{2}{\sqrt{4\mu - \lambda^2}} \int_0^t e^{-\frac{\lambda \tau^\alpha}{2\alpha}} \sin\left(\sqrt{4\mu - \lambda^2} \frac{\tau^\alpha}{2\alpha} \right) g(t - \tau)\, d\tau. \end{aligned} \tag{2.6}$$

3. Nonhomogeneous conformable fractional telegraph equation with Dirichlet boundary condition

We determine the solution of conformable fractional telegraph equation (1.4) with the intial conditions

$$u(x, 0) = \phi(x), \quad \mathcal{D}_t^{(\alpha)} u(x, 0) = \psi(x), \quad 0 \leq x \leq \ell, \tag{3.1}$$

and the nonhomogeneous Dirichlet boundary conditions

$$u(0, t) = \mu_1(t), \quad u(\ell, t) = \mu_2(t), \quad t > 0, \tag{3.2}$$

where $\mu_1(t)$ and $\mu_2(t)$ are nonzero smooth functions with order-one continuous derivative, using the method of separating variables, in which $\phi(x)$, $\psi(x)$ are continuous functions satisfying

$$\phi(0) = \mu_1(0) \quad \text{and} \quad \phi(\ell) = \mu_2(0). \tag{3.3}$$

Other hand, we assume that

$$\frac{|a - b|}{2k} < \frac{\pi}{\ell}. \tag{3.4}$$

In order to solve the problem with nonhomogeneous boundary, we firstly transform the nonhomogeneous boundary into a homogeneous condition. Let

$$u(x, t) = W_1(x, t) + V_1(x, t),$$

where $W_1(x, t)$ is a new unknown function and

$$V_1(x, t) = \mu_1(t) + \frac{[\mu_2(t) - \mu_1(t)]\, x}{\ell} \tag{3.5}$$

which satisfies the boundary conditions

$$V_1(0, t) = \mu_1(t) \quad \text{and} \quad V_1(\ell, t) = \mu_2(t). \tag{3.6}$$

The function $W_1(x, t)$ then satisfies the problem with homogeneous boundary conditions:

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} W_1(x, t) + (a + b)\, \mathcal{D}_t^{(\alpha)} W_1(x, t) + abW_1(x, t) = k^2 \frac{\partial^2 W_1(x,t)}{\partial x^2} + \tilde{f}(x, t), \\ W_1(x, 0) = \phi_1(x), \quad \mathcal{D}_t^{(\alpha)} W_1(x, 0) = \psi_1(x), \\ W_1(0, t) = W_1(\ell, t) = 0, \end{cases} \tag{3.7}$$

in which

$$\begin{aligned} \tilde{f}(x) &= -\mathcal{D}_t^{(2\alpha)} V_1(x,t) - (a+b)\,\mathcal{D}_t^{(\alpha)} V_1(x,t) - abV_1(x,t) + f(x,t)\,, \\ \phi_1(x) &= \phi(x) - \mu_1(0) - \frac{\mu_2(0) - \mu_1(0)}{\ell} x, \\ \psi_1(x) &= \psi(x) - \mathcal{D}_t^{(\alpha)} \mu_1(0) - \frac{\mathcal{D}_t^{(\alpha)} \mu_2(0) - \mathcal{D}_t^{(\alpha)} \mu_1(0)}{\ell} x. \end{aligned} \tag{3.8}$$

We solve the corresponding homogeneous equation (3.7) ($\tilde{f}(x,t)$ being replaced by 0) with the boundary conditions by the method of separation of variables.

If we let $W_1(x,t) = X(x)\,Y(t)$ and substitute for $W_1(x,t)$ in (3.7), we obtain an ordinary linear differential equation for $X(x)$:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(\ell) = 0, \end{cases} \tag{3.9}$$

and a fractional ordinary linear differential equation with the conformable derivative for $Y(t)$,

$$\mathcal{D}_t^{(2\alpha)} Y(t) + (a+b)\,\mathcal{D}_t^{(\alpha)} Y(t) + \left(ab + \lambda k^2\right) Y(t) = 0, \tag{3.10}$$

where the parameter λ is a positive constant.

The Sturm-Liouville problem given by (3.9) has eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{\ell^2}, \quad n \in \mathbb{N}^*, \tag{3.11}$$

and corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{N}^*. \tag{3.12}$$

Now we seek a solution of the nonhomogeneous problem (3.7) in the following form

$$W_1(x,t) = \sum_{n=1}^{+\infty} B_n(t) \sin\left(\frac{n\pi x}{\ell}\right). \tag{3.13}$$

We assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$:

$$\tilde{f}(x,t) = \sum_{n=1}^{+\infty} \tilde{f}_n(t) \sin\left(\frac{n\pi x}{\ell}\right), \text{ where } \tilde{f}_n(t) = \frac{2}{\ell} \int_0^\ell \tilde{f}(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{3.14}$$

Substituting (3.13), (3.14) into (3.7) yields

$$\mathcal{D}_t^{(2\alpha)} B_n(t) + (a+b)\,\mathcal{D}_t^{(\alpha)} B_n(t) + \left(ab + \lambda_n k^2\right) B_n(t) = \tilde{f}_n(t)\,. \tag{3.15}$$

Since $W_1(x,t)$ satisfies the initial conditions in (3.7), we must have

$$\begin{cases} \sum_{n=0}^{+\infty} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \phi_1(x)\,, \quad 0 < x < \ell, \\ \sum_{n=0}^{+\infty} \mathcal{D}_t^{(\alpha)} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \psi_1(x)\,, \quad 0 < x < \ell, \end{cases} \tag{3.16}$$

which yields

$$\begin{cases} B_n(0) = \frac{2}{\ell} \int_0^\ell \phi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*, \\ \mathcal{D}_t^{(\alpha)} B_n(0) = \frac{2}{\ell} \int_0^\ell \psi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*. \end{cases} \tag{3.17}$$

For each value of n, (3.15) and (3.17) make up a fractional initial value problem.

According to Proposition, we obtain the solution of problem (1.4), (3.1) and (3.2) as

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{+\infty} \Bigg[& B_n(0)\, e^{-\frac{(a+b)t^\alpha}{2\alpha}} \cos\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{t^\alpha}{2\alpha}\right) \\
& + \frac{(a+b)\,B_n(0) + 2\mathcal{D}_t^{(\alpha)} B_n(0)}{\sqrt{4k^2\lambda_n - (a-b)^2}}\, e^{-\frac{(a+b)t^\alpha}{2\alpha}} \sin\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{t^\alpha}{2\alpha}\right) \\
& + \frac{2}{\sqrt{4k^2\lambda_n - (a-b)^2}} \int_0^t e^{-\frac{(a+b)\tau^\alpha}{2\alpha}} \sin\left(\sqrt{4k^2\lambda_n - (a-b)^2}\,\frac{\tau^\alpha}{2\alpha}\right) \tilde{f}_n(t-\tau)\, d\tau \Bigg] \sin\left(\frac{n\pi x}{\ell}\right) \\
& + \mu_1(t) + \frac{(\mu_2(t) - \mu_1(t))\,x}{\ell}.
\end{aligned}
\tag{3.18}
$$

4. Conclusion

We have derived the analytical solution of the nonhomogeneous conformable fractional telegraph equation with Dirichlet boundary condition using Fourier method. The solution is given in the form of a series of functions with the use of the Fourier sine series and the conformable Lapalce transform.

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