TWMS J. App. Eng. Math. V.10, N.1, 2020, pp.11-23

SOME FIXED POINT RESULTS IN THE GENERALIZED CONVEX METRIC SPACES

KADRI DOGAN[1], FAIK GURSOY[2], VATAN KARAKAYA[3], §

Abstract. In this study, we introduce a new three step iteration process and show that the iteration process converges to the unique fixed point by two theorems under different conditions of contractive mappings on the generalized G- convex metric spaces. Also, we investigate data dependence result for this iterative process in the generalized G- convex metric spaces.

Keywords: Convex G-metric spaces, G-convergence, Fixed point iteration process

AMS Subject Classification: 47H09, 47H10

1. Introduction and preliminaries

By several mathematicians have been introduced different generalizations of the usual concept of a metric space . In 1963, Gahler [19], Ha et.al. [20], In 1992, Dhage [18], In 2006, Mustafa along with Sims proposed a new concept of generalized metric space called G-metric space [9]. Fixed point theory in these spaces was studied in [4], Banach contraction mapping being the main tool. Mustafa et al. studied many fixed point results for a self-mapping in G-metric space.[3]-[9] can be cited for reference. Takahashi [1] proposed the notion of convex structure in metric spaces and proved some fixed point results. Inspired by this Thangavelu et.al. [29] proposed the notion of convexity structure in D-metric space. They further extended this notion to get strong convex D-metric space, J-convex D-metric spaces, weak convex D-metric spaces and quasi convex D-Metric spaces. Recently, Modi and Bhatt [30] extend to G-metric space by providing different convex structures to D-metric space analogous to Thangavelu et.al. [29]. Therefore we propose a new iteration process and we prove that this fixed point iteration process converges to fixed point of contractive type mapping in the convex G-metric spaces.

An element x is said to be a fixed point of T if $Tx = x$.

[1]Department of Computer Engineering, Artvin Coruh University, Artvin, Turkey.
e-mail: dogankadri@hotmail.com; ORCID: https://orcid.org/0000-0002-6622-3122.

[2] Department of Mathematics, Adiyaman University, 02000, Adiyaman/ Turkey.
e-mail: faikgursoy02@hotmail.com; ORCID: https://orcid.org/0000-0002-7118-9088.

[3] Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey.
e-mail: vkkaya@yahoo.com; ORCID: https://orcid.org/0000-0003-4637-3139.

§ Manuscript received: January 11, 2018; accepted: January 22, 2019.
TWMS Journal of Applied and Engineering Mathematics, V10, No.1;

The iterative approximation of a fixed point for certain classes of mappings is one of the main tools in the fixed point theory. There are many studies conducted on this theory. Some of these [11]-[16].

Definition 1.1. [9] *Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
G1) $G(x, y, z) = 0$, if $x = y = z$, G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
G4) $G(x, y, z) = G(y, x, z) = G(z, y, x) = \cdots$ (symmetry in all three variables),
G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).
Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

The following useful properties of a G-metric are readily derived from the axioms.

Proposition 1.1. [9] *Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function*
1) if $G(x, y, z) = 0$, then $x = y = z$,
2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
3) $G(x, y, y) \leq 2G(y, x, x)$,
4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$,
7) $|G(x, y, z) - G(x, y, a)| \leq max\{G(a, z, z), G(z, a, a)\}$,
8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z)$,
9) $|G(x, y, z) - G(y, z, z)| \leq max\{G(x, z, z), G(z, x, x)\}$,
10) $|G(x, y, y) - G(y, x, x)| \leq max\{G(y, x, x), G(x, y, y)\}$.

Proposition 1.2. [9] *Let (X, G) be $G-$metric space, then for a sequence $(x_n) \subseteq X$ and point $x \in X$ the following are equivalent.*
1) (x_n) is $G-$convergent to x.
2) $d_G(x_n, x) \to 0$, as $n \to \infty$ (that is, (x_n) converges to x relative to the metric d_G).
3) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
4) $G(x_n, x, x) \to 0$, as $n \to \infty$.
5) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Definition 1.2. [1] *Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,*

$$d\left(u, W\left(x, y, \lambda\right)\right) \leq \lambda d\left(u, x\right) + \left(1 - \lambda\right) d\left(u, y\right)$$

If (X, d) is equipped with a Takahashi convex structure, then it is called a convex metric space indicated by (X, d, W). A Banach space , or any convex subset of it is a convex metric space with

$$W\left(x, y, \lambda\right) = \lambda x + \left(1 - \lambda\right) y.$$

Definition 1.3. *Let X be a convex metric space. A nonempty subset A of X is said to be convex if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.*

A Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0)$, then X is a convex metric space.

Definition 1.4. [2] *Let (X, d) be a metric space. A mapping $W : X \times X \times X \times [0, 1] \times [0, 1] \times [0, 1] \to X$ is said to be a generalized convex structure on X if for each $(x, y, z; a, b, c) \in$*

$X \times X \times X \times [0,1] \times [0,1] \times [0,1]$ *and* $u \in X$,

$$d(u, W(x, y, z; a, b, c)) \leq ad(u, x) + bd(u, y) + cd(u, z);$$

$a+b+c=1$. *The metric space* X *together with* W *is called a generalized convex metric space.*

Definition 1.5. [10] *Let* X *be a generalized convex metric space. A nonempty subset* A *of* X *is said to be generalized convex if* $W(x, y, z; a, b, c) \in A$ *whenever* $(x, y, z; a, b, c) \in A \times A \times A \times [0,1] \times [0,1] \times [0,1]$.

Definition 1.6. [10] *Let* (X, G) *be a* $G-$*metric space. A mapping* $W : X \times X \times X \times [0,1] \times [0,1] \to X$ *is said to be a generalized convex structure on* X *if for each* $(x, y, z; a, b) \in X \times X \times X \times [0,1] \times [0,1]$, $a \geq b$ *and* $u, v \in X$,

$$G(u, v, W(x, y, z; a, b)) \leq (a-b)\, G(u, v, x) + (1-a)\, G(u, v, y) + bG(u, v, z);$$

Theorem 1.1. [4] *The G-metric space* X *together with* W *is called a generalized convex G-metric space. Let* (X, G) *be a complete* $G-$*metric space, and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*
$G(Tx, Ty, Tz) \leq a(Gx, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz)$
for all $x, y, z \in X$ *where* $0 \leq a, b, c, d < 1$, *then* T *has a unique fixed point (say* u, *i.e.,* $Tu = u$*), and* T *is G-continuous at* u.

Lemma 1.1. [17] *If* ρ *is a real number satisfying* $0 \leq \rho < 1$ *and* $(\epsilon_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers such that* $lim_{n\to\infty}\epsilon_n = 0$, *then for any sequence of positive numbers* $(\epsilon_n)_{n\in\mathbb{N}}$ *satisfying*

$$a_{n+1} \leq \rho a_n + \epsilon_n, n = 1, 2, ...,$$

one has

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 1.2. [31]*Let* $\{\psi_n\}$ *be a nonnegative sequence for which one supposes there exists* $n_0 \in \mathbb{N}$, *such that for all* $n \geq n_0$ *one has satisfied the following inequality:*

$$\psi_{n+1} \leq (1-\lambda_n)\, \psi_n + \lambda_n \phi_n$$

where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$ *and* $\phi_n \geq 0$, $\forall n \in \mathbb{N}$. *Then*

$$0 \leq \lim_{n\to\infty} \sup \psi_n \leq \lim_{n\to\infty} \sup \phi_n.$$

2. MAIN RESULTS

2.1. **convergenge analysis.**

We prove two theorems under different two conditions in the generalized convex G-metric spaces.

Theorem 2.1. *Let* K *be a nonempty closed convex subset of a* (X, G, W) *complete convex* $G-$*metric space with* W *convex structure and* $T : X \to X$ *be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz) \quad (1)$$

for all $x, y, z \in X$ *where* $0 \leq a, b, c, d < 1$ *and let* $\{x_n\}_{n\geq 0}$ *be the iterative scheme defined by*

$$\begin{cases} x_0 \in X, \forall n \in \mathbb{N}, \\ x_{n+1} = W(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n) \\ y_n = W(z_n, Tz_n, Tx_n : \alpha_n, \beta_n) \\ z_n = W(Tx_n, x_n, Tx_n : \theta_n, \theta_n) \end{cases} \quad (2)$$

such that $\lim_{n\to\infty} G(x_n, Tx_n, Tx_n) = 0$ *with* $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\theta_n\} \subset [0,1]$. *Then the sequence* $\{x_n\}_{n\geq 0}$ $G-$*convergence to unique fixed point* p *of* T.

Proof. Suppose that T satisfies condition (2), we have

$$\begin{aligned} G(x_{n+1}, p, p) &= G(W(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n), p, p) \\ &\leq (\gamma_n - \gamma_n) G(Ty_n, p, p) + (1-\gamma_n) G(Ty_n, p, p) + \gamma_n G(Ty_n, p, p) \\ &\leq aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + cG(p, p, Tp) + dG(p, p, Tp) \qquad (3) \\ &= aG(y_n, p, p) + bG(y_n, Ty_n, Ty_n) + (c+d) G(p, p, Tp) \end{aligned}$$

and

$$\begin{aligned} G(y_n, p, p) &= G(W(z_n, Tz_n, Tx_n : \alpha_n, \beta_n), p, p) \\ &\leq (\alpha_n - \beta_n) G(z_n, p, p) + (1-\alpha_n) G(Tz_n, p, p) + \beta_n G(Tx_n, p, p) \\ &\leq (\alpha_n - \beta_n + (1-\alpha_n) a) G(z_n, p, p) + \beta_n a G(x_n, p, p) \\ &\quad + (1-\alpha_n) bG(z_n, Tz_n, Tz_n) \qquad (4) \\ &\quad + \beta_n bG(x_n, Tx_n, Tx_n) + (1 - (\alpha_n - \beta_n))(c+d) G(p, Tp, Tp) \end{aligned}$$

and

$$\begin{aligned} G(z_n, p, p) &= G(W(Tx_n, x_n, Tx_n : \theta_n, \theta_n), p, p) \qquad (5) \\ &\leq (1 - \theta_n(1-a)) G(x_n, p, p) + \theta_n bG(x_n, Tx_n, Tx_n) \\ &\quad + \theta_n (c+d) G(p, Tp, Tp) \end{aligned}$$

Substituting (4) and (5) in (3), we obtain

$$\begin{aligned} G(x_{n+1}, p, p) &\leq a \begin{bmatrix} (\alpha_n - \beta_n + (1-\alpha_n) a) \begin{pmatrix} (1-\theta_n(1-a)) G(x_n, p, p) \\ +\theta_n bG(x_n, Tx_n, Tx_n) \\ +\theta_n (c+d) G(p, Tp, Tp) \end{pmatrix} \\ +\beta_n a G(x_n, p, p) + (1-\alpha_n) bG(z_n, Tz_n, Tz_n) \\ +\beta_n bG(x_n, Tx_n, Tx_n) + (1-(\alpha_n - \beta_n))(c+d) G(p, Tp, Tp) \end{bmatrix} \\ &\quad + bG(y_n, Ty_n, Ty_n) + (c+d) G(p, p, Tp) \\ &= a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) G(x_n, p, p) \\ &\quad + bG(y_n, Ty_n, Ty_n) + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ &\quad + a((\alpha_n - \beta_n + (1-\alpha_n) a)\theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \\ &\quad + \left[\left(a \begin{pmatrix} (\alpha_n - \beta_n + (1-\alpha_n) a)\theta_n (c+d) \\ +(1-(\alpha_n - \beta_n))(c+d) \end{pmatrix}\right) + (c+d)\right] G(p, Tp, Tp) \end{aligned}$$

Since $G(p, Tp, Tp) = 0$, we obtain

$$\begin{aligned} G(x_{n+1}, p, p) &\leq a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) G(x_n, p, p) \\ &\quad + bG(y_n, Ty_n, Ty_n) + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ &\quad + a((\alpha_n - \beta_n + (1-\alpha_n) a)\theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \end{aligned}$$

In order to satisfy the conditions of Lemma 1.1, we take δ, ε_n and κ_n as follows:

$$\begin{aligned} 0 &\leq \delta = a((\alpha_n - \beta_n + (1-\alpha_n) a)(1 - \theta_n(1-a)) + \beta_n a) < 1 \\ \varepsilon_n &= bG(y_n, Ty_n, Ty_n) + a((\alpha_n - \beta_n + (1-\alpha_n) a)\theta_n b + \beta_n b) G(x_n, Tx_n, Tx_n) \\ &\quad + a((1-\alpha_n) b) G(z_n, Tz_n, Tz_n) \\ \kappa_n &= G(x_n, p, p). \end{aligned}$$

Since

$$\lim_{n\to\infty} G\left(x_n, Tx_n, Tx_n\right) = \lim_{n\to\infty} G\left(y_n, Ty_n, Ty_n\right) = \lim_{n\to\infty} G\left(z_n, Tz_n, Tz_n\right) = 0$$

by Lemma 1.1, we have $\lim_{n\to\infty} G\left(x_n, p, p\right) = 0$. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a (X, G, W) complete convex $G-$metric space with W convex structure and $T : X \to X$ be a mapping satisfying the following condition:*

$$G\left(Tx, Ty, Tz\right) \leq a\left(Gx, y, z\right) + bG\left(x, Tx, Tx\right) + cG\left(y, Ty, Ty\right) + dG\left(z, Tz, Tz\right)$$

for all $x, y, z \in X$ where $0 \leq a, b \leq \frac{1}{4}$, $c, d \in [0, 1)$ and let $\{x_n\}_{n\geq 0}$ be defined by (2) with

i) $\{\theta_n\}_{n\geq 0} \subset \left[0, \frac{1}{4}\right)$,

ii) $\beta_n \leq (1-\alpha_n)\, a \leq \alpha_n$ and then the sequence $\{x_n\}_{n\geq 0}$ converges to unique fixed point p of T.

Proof. Suppose that T satisfies condition (2), we have

$$\begin{aligned} G\left(x_{n+1}, p, p\right) &= G\left(W\left(Ty_n, Ty_n, Ty_n : \gamma_n, \gamma_n\right), p, p\right) \\ &\leq (\gamma_n - \gamma_n)\, G\left(Ty_n, p, p\right) + (1-\gamma_n)\, G\left(Ty_n, p, p\right) + \gamma_n G\left(Ty_n, p, p\right) \quad (6) \\ &\leq aG\left(y_n, p, p\right) + bG\left(y_n, Ty_n, Ty_n\right) + cG\left(p, p, Tp\right) + dG\left(p, p, Tp\right) \\ &= aG\left(y_n, p, p\right) + bG\left(y_n, Ty_n, Ty_n\right) + (c+d)\, G\left(p, p, Tp\right) \end{aligned}$$

$$\begin{aligned} G\left(y_n, p, p\right) &= G\left(W\left(z_n, Tz_n, Tx_n : \alpha_n, \beta_n\right), p, p\right) \\ &\leq (\alpha_n - \beta_n)\, G\left(z_n, p, p\right) + (1-\alpha_n)\, G\left(Tz_n, p, p\right) + \beta_n G\left(Tx_n, p, p\right) \quad (7) \\ &\leq \left(\alpha_n - \beta_n + (1-\alpha_n)\, a\right) G\left(z_n, p, p\right) \\ &\quad + \beta_n aG\left(x_n, p, p\right) + (1-\alpha_n)\, bG\left(z_n, Tz_n, Tz_n\right) \\ &\quad + \beta_n bG\left(x_n, Tx_n, Tx_n\right) + \left(1 - (\alpha_n - \beta_n)\right)(c+d)\, G\left(p, Tp, Tp\right) \end{aligned}$$

$$\begin{aligned} G\left(z_n, p, p\right) &= G\left(W\left(Tx_n, x_n, Tx_n : \theta_n, \theta_n\right), p, p\right) \quad (8) \\ &\leq (\theta_n - \theta_n)\, G\left(Tx_n, p, p\right) + (1-\theta_n)\, G\left(x_n, p, p\right) + \theta_n G\left(Tx_n, p, p\right) \\ &\leq \left(1 - \theta_n\,(1-a)\right) G\left(x_n, p, p\right) + \theta_n bG\left(x_n, Tx_n, Tx_n\right) \\ &\quad + \theta_n\,(c+d)\, G\left(p, Tp, Tp\right) \end{aligned}$$

Substituting (7) and (8) in (6), we have

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left[\begin{array}{c} \left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\theta_n bG\left(x_n,Tx_n,Tx_n\right)\\ +\theta_n\left(c+d\right)G\left(p,Tp,Tp\right)\end{array}\right)\\ +\beta_n aG\left(x_n,p,p\right)+\left(1-\alpha_n\right)bG\left(z_n,Tz_n,Tz_n\right)+\beta_n bG\left(x_n,Tx_n,Tx_n\right)\\ +\left(1-\left(\alpha_n-\beta_n\right)\right)\left(c+d\right)G\left(p,Tp,Tp\right)\end{array}\right]\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+\left(c+d\right)G\left(p,p,Tp\right)\\ &= a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right)\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+a\left(\left(1-\alpha_n\right)b\right)G\left(z_n,Tz_n,Tz_n\right)\\ &\quad +a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)G\left(x_n,Tx_n,Tx_n\right)\\ &\quad +\left[\left(a\left(\begin{array}{c}\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n\left(c+d\right)\\ +\left(1-\left(\alpha_n-\beta_n\right)\right)\left(c+d\right)\end{array}\right)\right)+\left(c+d\right)\right]G\left(p,Tp,Tp\right) \end{aligned}$$

Since $G\left(p,Tp,Tp\right)=0$

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right)\\ &\quad +bG\left(y_n,Ty_n,Ty_n\right)+a\left(\left(1-\alpha_n\right)b\right)G\left(z_n,Tz_n,Tz_n\right)\\ &\quad +a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)G\left(x_n,Tx_n,Tx_n\right) \end{aligned} \tag{9}$$

Continuing the process

$$G\left(x_n,Tx_n,Tx_n\right)\leq\left(\frac{1+2a}{1-2b}\right)G\left(x_n,p,p\right) \tag{10}$$

$$\begin{aligned} G\left(z_n,Tz_n,Tz_n\right) &\leq \left(\frac{1+2a}{1-2b}\right)G\left(z_n,p,p\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,Tx_n,Tx_n\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left[\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\right] \end{aligned} \tag{11}$$

$$\begin{aligned} G\left(y_n,Ty_n,Ty_n\right) &\leq \left(\frac{1+2a}{1-2b}\right)G\left(y_n,p,p\right)\\ &\leq \left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\\ \times\left(\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\end{array}\right]\right)\\ +\beta_n aG\left(x_n,p,p\right)+\beta_n b\left(\frac{1+2a}{1-2b}\right)G\left(x_n,p,p\right)\\ +\left(1-\alpha_n\right)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n bG\left(x_n,p,p\right)\end{array}\right] \end{array}\right) \end{aligned} \tag{12}$$

Substituting (10), (11) and (12) in (9), we obtain

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right) \\ &+b\left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\left(\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\right) \\ +\beta_n a G\left(x_n,p,p\right)+\beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +\left(1-\alpha_n\right)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\end{array}\right) \\ &+a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ &+a\left(\left(1-\alpha_n\right)b\right)\left(\frac{1+2a}{1-2b}\right)\left[\left(1-\theta_n\left(1-a\right)\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{aligned}$$

$$G\left(x_{n+1},p,p\right)\leq\left[\begin{array}{c} a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right) \\ +b\left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\left(\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\right) \\ +\beta_n a G\left(x_n,p,p\right)+\beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +\left(1-\alpha_n\right)b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c}\left(1-\theta_n\left(1-a\right)\right)\\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b\end{array}\right]\end{array}\right) \\ +a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ +a\left(\left(1-\alpha_n\right)b\right)\left(\frac{1+2a}{1-2b}\right)\left[\left(1-\theta_n\left(1-a\right)\right)+\left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{array}\right]G\left(x_n,p,p\right)$$

$$\begin{aligned} G\left(x_{n+1},p,p\right) &\leq a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\left(1-\theta_n\left(1-a\right)\right)+\beta_n a\right)G\left(x_n,p,p\right) \\ &+b\left(\frac{1+2a}{1-2b}\right)\left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right) \\ &+\left(\frac{1+2a}{1-2b}\right)^2\left[\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\right]\theta_n b^2 G\left(x_n,p,p\right) \\ &+b\left(\frac{1+2a}{1-2b}\right)\beta_n a G\left(x_n,p,p\right)+\left(\frac{1+2a}{1-2b}\right)^2\beta_n b^2 G\left(x_n,p,p\right) \\ &+\left(1-\alpha_n\right)b^2\left(\frac{1+2a}{1-2b}\right)^2\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right) \\ &+\left(\frac{1+2a}{1-2b}\right)^3\left(1-\alpha_n\right)\theta_n b^3 G\left(x_n,p,p\right) \\ &+a\left(\left(\alpha_n-\beta_n+\left(1-\alpha_n\right)a\right)\theta_n b+\beta_n b\right)\left(\frac{1+2a}{1-2b}\right)G\left(x_n,p,p\right) \\ &+a\left(\left(1-\alpha_n\right)b\right)\left(\frac{1+2a}{1-2b}\right)\left(1-\theta_n\left(1-a\right)\right)G\left(x_n,p,p\right) \\ &+a\left(\left(1-\alpha_n\right)b\right)\left(\frac{1+2a}{1-2b}\right)^2\theta_n b G\left(x_n,p,p\right) \end{aligned}$$

Since

$$0 \leq \left[\begin{array}{c} a\left(\left(\alpha_n - \beta_n + (1-\alpha_n)\, a\right)(1-\theta_n(1-a)) + \beta_n a\right) \\ +b\left(\frac{1+2a}{1-2b}\right)\left(\begin{array}{c} \left[(\alpha_n - \beta_n + (1-\alpha_n)\, a)\right]\left(\left[\begin{array}{c} (1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b \end{array} \right]\right) \\ +\beta_n a G(x_n,p,p) + \beta_n b\left(\frac{1+2a}{1-2b}\right) \\ +(1-\alpha_n)\, b\left(\frac{1+2a}{1-2b}\right)\left[\begin{array}{c} (1-\theta_n(1-a)) \\ +\left(\frac{1+2a}{1-2b}\right)\theta_n b \end{array} \right] \end{array} \right) \\ +a\left(\left(\alpha_n - \beta_n + (1-\alpha_n)\, a\right)\theta_n b + \beta_n b\right)\left(\frac{1+2a}{1-2b}\right) \\ +a\left((1-\alpha_n)\, b\right)\left(\frac{1+2a}{1-2b}\right)\left[(1-\theta_n(1-a)) + \left(\frac{1+2a}{1-2b}\right)\theta_n b\right] \end{array} \right] < 1$$

by Lemma 1.1, we have $\lim_{n\to\infty} G(x_n,p,p) = 0$. □

2.2. **Data dependency.** Let's prove that the iteration (2) is data-dependent.

Definition 2.1. *Let $T, \check{T}, \widetilde{T} : X \to X$ be three operators. We say that $\check{T}$ and $\widetilde{T}$ are the approximate operators of T if for all $x \in X$ and for a fixed $\epsilon > 0$, we have*

$$G\left(Tx, \check{T}x, \widetilde{T}x\right) = \max\left\{\left\|Tx - \check{T}x\right\|, \left\|\check{T}x - \widetilde{T}x\right\|, \left\|\widetilde{T}x - Tx\right\|\right\} \leq \epsilon.$$

Theorem 2.3. *Let K be a nonempty closed convex subset of a (X,G,W) complete convex $G-$metric space with W convex structure and $, \check{T}, \widetilde{T} : K \to X$ be the approximate operators of the $T : K \to X$ satisfying the mapping (1) and $\{x_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ and $\{k_n\}_{n\geq 0}$ three iteration schemes associated to $T, \check{T}$ and $\widetilde{T}$ defined by*

$$\left\{ \begin{array}{c} x_{n+1} = Ty_n \\ y_n = (\alpha_n - \beta_n)\, z_n + (1-\alpha_n)\, Tx_n + \beta_n T z_n \\ z_n = (1-\theta_n)\, x_n + \theta_n T x_n, \forall n \in \mathbb{N}, \end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{c} u_{n+1} = \check{T} v_n \\ v_n = (\alpha_n - \beta_n)\, w_n + (1-\alpha_n)\, \check{T} u_n + \beta_n \check{T} w_n \\ w_n = (1-\theta_n)\, u_n + \theta_n \check{T} u_n, \forall n \in \mathbb{N} \end{array} \right. \tag{14}$$

and

$$\left\{ \begin{array}{c} k_{n+1} = \widetilde{T} r_n \\ r_n = (\alpha_n - \beta_n)\, s_n + (1-\alpha_n)\, \widetilde{T} k_n + \beta_n \widetilde{T} s_n \\ s_n = (1-\theta_n)\, k_n + \theta_n \widetilde{T} k_n, \forall n \in \mathbb{N}, \end{array} \right. \tag{15}$$

respectively, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n \geq \beta_n$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Let $Tx^* = x^*$, $\check{T}k^* = k^*$ and $\widetilde{T}u^* = u^*$ with $\|Tx - Ty\| \leq a\|x-y\| + b\|x - Tx\| + c\|y - Ty\|$. Then for $25b > 0$, we have the following estimate:

$$G(x^*, k^*, u^*) \leq \frac{11\epsilon}{1-a}$$

Proof. Using iterative schemes (13), (14) and (15) yield the following inequalites:

$$G(x_{n+1}, k_{n+1}, u_{n+1}) = \max\{\|x_{n+1} - k_{n+1}\|, \|k_{n+1} - u_{n+1}\|, \|u_{n+1} - x_{n+1}\|\} \tag{16}$$

$$\|x_{n+1} - k_{n+1}\| = \left\|Ty_n - \check{T}r_n\right\| \leq \epsilon + a\|y_n - r_n\| + b\|y_n - Ty_n\| + c\|r_n - Tr_n\|, \tag{17}$$

$$
\begin{aligned}
\|y_n - r_n\| &\leq (\alpha_n - \beta_n)\|z_n - s_n\| + (1-\alpha_n)\left\|Tx_n - \check{T}k_n\right\| + \beta_n \left\|Tz_n - \check{T}s_n\right\| \\
&\leq (\alpha_n - \beta_n)\|z_n - s_n\| + (1-\alpha_n)\epsilon + \beta_n\epsilon + a(1-\alpha_n)\|x_n - k_n\| \qquad (18) \\
&\quad + b(1-\alpha_n)\|x_n - Tx_n\| + c(1-\alpha_n)\|k_n - Tk_n\| \\
&\quad + a\beta_n\|z_n - s_n\| + b\beta_n\|z_n - Tz_n\| + c\beta_n\|s_n - Ts_n\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_n - s_n\| &= (1-\theta_n)\|x_n - k_n\| + \theta_n\left\|Tx_n - \check{T}k_n\right\| \\
&\leq (1-\theta_n)\|x_n - k_n\| + \theta_n\epsilon + \theta_n a\|x_n - k_n\| \qquad (19) \\
&\quad + \theta_n b\|x_n - Tx_n\| + \theta_n c\|k_n - Tk_n\|.
\end{aligned}
$$

Substituting (19) in (18) and (18) in (17), we have

$$
\begin{aligned}
\|x_{n+1} - k_{n+1}\| &\leq \epsilon_n + a\left[\begin{array}{c} (\alpha_n - \beta_n(1-a))\left[\begin{array}{c}(1-\theta_n(1-a))\|x_n - k_n\| \\ +\theta_n\epsilon_n + \theta_n b\|x_n - Tx_n\| + \theta_n c\|k_n - Tk_n\|\end{array}\right] \\ +(1-\alpha_n)\epsilon_n + \beta_n\epsilon_n + a(1-\alpha_n)\|x_n - k_n\| \\ +b(1-\alpha_n)\|x_n - Tx_n\| + c(1-\alpha_n)\|k_n - Tk_n\| \\ +b\beta_n\|z_n - Tz_n\| + c\beta_n\|s_n - Ts_n\| \end{array}\right] \\
&\quad + b\|y_n - Ty_n\| + c\|r_n - Tr_n\| \\
&\leq a\left[(\alpha_n - \beta_n(1-a))(1-\theta_n(1-a)) + a(1-\alpha_n)\right]\|x_n - k_n\| \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n b + b(1-\alpha_n)\right]\|x_n - Tx_n\| \qquad (20) \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n c + c(1-\alpha_n)\right]\|k_n - Tk_n\| \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n + (1-\alpha_n) + \beta_n + 1\right]\epsilon_n \\
&\quad + ab\beta_n\|z_n - Tz_n\| + ac\beta_n\|s_n - Ts_n\| + b\|y_n - Ty_n\| + c\|r_n - Tr_n\|.
\end{aligned}
$$

In the similar way, we obtain

$$
\begin{aligned}
\|k_{n+1} - u_{n+1}\| &\leq \epsilon_n + a\left[\begin{array}{c} (\alpha_n - \beta_n(1-a))\left[\begin{array}{c}(1-\theta_n(1-a)) \\ \times\|k_n - u_n\| + \theta_n\epsilon_n \\ +\theta_n b\left\|k_n - \check{T}k_n\right\| \\ +\theta_n c\left\|u_n - \check{T}u_n\right\|\end{array}\right] \\ +(1-\alpha_n)\epsilon_n + \beta_n\epsilon_n + a(1-\alpha_n)\|k_n - u_n\| \\ +b(1-\alpha_n)\left\|k_n - \check{T}k_n\right\| + c(1-\alpha_n)\left\|u_n - \check{T}u_n\right\| \\ +b\beta_n\left\|r_n - \check{T}r_n\right\| + c\beta_n\left\|w_n - \check{T}w_n\right\| \end{array}\right] \qquad (21) \\
&\quad + b\left\|r_n - \check{T}r_n\right\| + c\left\|v_n - \check{T}v_n\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq a\left[(\alpha_n - \beta_n(1-a))(1-\theta_n(1-a)) + a(1-\alpha_n)\right]\|k_n - u_n\| \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n b + b(1-\alpha_n)\right]\left\|k_n - \check{T}k_n\right\| \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n c + c(1-\alpha_n)\right]\left\|u_n - \check{T}u_n\right\| \\
&\quad + a\left[(\alpha_n - \beta_n(1-a))\theta_n + (1-\alpha_n) + \beta_n + 1\right]\epsilon \\
&\quad + ab\beta_n\left\|s_n - \check{T}s_n\right\| + ac\beta_n\left\|w_n - \check{T}w_n\right\| \\
&\quad + b\left\|r_n - \check{T}r_n\right\| + c\left\|v_n - \check{T}v_n\right\|
\end{aligned}
$$

and

$$\|u_{n+1}-x_{n+1}\| \leq \epsilon_n + a\left[\begin{array}{c} (\alpha_n-\beta_n(1-a))\left[\begin{array}{c}(1-\theta_n(1-a)) \\ \times\|u_n-x_n\|+\theta_n\epsilon_n \\ +\theta_n b\left\|u_n-\widetilde{T}u_n\right\| \\ +\theta_n c\left\|x_n-\widetilde{T}x_n\right\|\end{array}\right] \\ +(1-\alpha_n)\epsilon_n+\beta_n\epsilon_n+a(1-\alpha_n)\|u_n-x_n\| \\ +b(1-\alpha_n)\left\|u_n-\widetilde{T}u_n\right\|+c(1-\alpha_n)\left\|x_n-\widetilde{T}x_n\right\| \\ +b\beta_n\left\|w_n-\widetilde{T}w_n\right\|+c\beta_n\left\|z_n-\widetilde{T}z_n\right\| \end{array}\right] \tag{22}$$
$$+b\left\|v_n-\widetilde{T}v_n\right\|+c\left\|y_n-\widetilde{T}y_n\right\|$$

$$\begin{aligned} \leq\ & a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|u_n-x_n\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|x_n-\widetilde{T}x_n\right\| \\ & +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon_n \\ & +ab\beta_n\left\|w_n-\widetilde{T}w_n\right\|+ac\beta_n\left\|z_n-\widetilde{T}z_n\right\| \\ & +b\left\|v_n-\widetilde{T}v_n\right\|+c\left\|y_n-\widetilde{T}y_n\right\|. \end{aligned}$$

Substituting (20), (21) and (22) in (16), we have

$$G(x_{n+1},k_{n+1},u_{n+1}) \leq \max\left\{\begin{array}{c} a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|x_n-k_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\|x_n-Tx_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|k_n-\check{T}k_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\|z_n-Tz_n\| \\ +ac\beta_n\left\|s_n-\check{T}s_n\right\|+b\|y_n-Ty_n\|+c\left\|r_n-\check{T}r_n\right\| \\ ,\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|k_n-u_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|k_n-\check{T}k_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\left\|s_n-\check{T}s_n\right\| \\ +ac\beta_n\left\|w_n-\widetilde{T}w_n\right\|+b\left\|r_n-\check{T}r_n\right\|+c\left\|v_n-\widetilde{T}v_n\right\| \\ ,a\left[(\alpha_n-\beta_n(1-a))(1-\theta_n(1-a))+a(1-\alpha_n)\right]\|u_n-x_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n b+b(1-\alpha_n)\right]\left\|u_n-\widetilde{T}u_n\right\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n c+c(1-\alpha_n)\right]\|x_n-Tx_n\| \\ +a\left[(\alpha_n-\beta_n(1-a))\theta_n+(1-\alpha_n)+\beta_n+1\right]\epsilon+ab\beta_n\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\beta_n\|z_n-Tz_n\|+b\left\|v_n-\widetilde{T}v_n\right\|+c\|y_n-Ty_n\|. \end{array}\right\} \tag{23}$$

Let $1-\beta_n \leq \beta_n$, then the inequality (23) is rearranged as follows:

$$G(x_{n+1},k_{n+1},u_{n+1}) \leq \max \left\{ \begin{array}{c} [1-\beta_n(1-a)]\|x_n-k_n\| + a2\beta_n\|x_n-Tx_n\| + a2\beta_n\left\|k_n-\check{T}k_n\right\| \\ +11\beta_n\epsilon + ab\beta_n\|z_n-Tz_n\| + ac\beta_n\left\|s_n-\check{T}s_n\right\| + b2\beta_n\|y_n-Ty_n\| \\ +c2\beta_n\left\|r_n-\check{T}r_n\right\|, \\ [1-\beta_n(1-a)]\|k_n-u_n\| + a2\beta_n\left\|k_n-\check{T}k_n\right\| \\ +a2\beta_n\left\|u_n-\widetilde{T}u_n\right\| + 11\beta_n\epsilon + ab\beta_n\left\|s_n-\check{T}s_n\right\| \\ +ac\beta_n\left\|w_n-\widetilde{T}w_n\right\| + b2\beta_n\left\|r_n-\check{T}r_n\right\| + b2\beta_n\left\|v_n-\widetilde{T}v_n\right\| \\ ,[1-\beta_n(1-a)]\|u_n-x_n\| + a2\beta_n\left\|u_n-\widetilde{T}u_n\right\| \\ +a2\beta_n\|x_n-Tx_n\| + 11\beta_n\epsilon + ab\beta_n\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\beta_n\|z_n-Tz_n\| + b2\beta_n\left\|v_n-\widetilde{T}v_n\right\| + b2\beta_n\|y_n-Ty_n\|. \end{array} \right\} \quad (24)$$

If simplifications are made in the (24), we arrive at

$$\|k_{n+1}-u_{n+1}\| \leq [1-\beta_n(1-a)]\|k_n-u_n\| + \beta_n(1-a)\frac{\begin{array}{c} a2\left\|k_n-\check{T}k_n\right\| + a2\left\|u_n-\widetilde{T}u_n\right\| + 11\epsilon + ab\left\|s_n-\check{T}s_n\right\| \\ +ac\left\|w_n-\widetilde{T}w_n\right\| + b2\left\|r_n-\check{T}r_n\right\| + b2\left\|v_n-\widetilde{T}v_n\right\| \end{array}}{(1-a)}$$

$$\|u_{n+1}-x_{n+1}\| \leq [1-\beta_n(1-a)]\|u_n-x_n\| + \beta_n(1-a)\frac{\begin{array}{c} a2\left\|u_n-\widetilde{T}u_n\right\| + a2\|x_n-Tx_n\| + 11\epsilon + ab\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\|z_n-Tz_n\| + b2\left\|v_n-\widetilde{T}v_n\right\| + b2\|y_n-Ty_n\| \end{array}}{(1-a)}.$$

Define

$$\lambda_n := \beta_n(1-a)$$

$$\phi_n := \frac{\begin{array}{c} a2\left\|u_n-\widetilde{T}u_n\right\| + a2\|x_n-Tx_n\| + 11\epsilon + ab\left\|w_n-\widetilde{T}w_n\right\| \\ +ac\|z_n-Tz_n\| + b2\left\|v_n-\widetilde{T}v_n\right\| + b2\|y_n-Ty_n\| \end{array}}{(1-a)}$$

By Lemma 1.2, we obtain

$$\|x^*-k^*\| \leq \frac{11\epsilon}{1-a}, \|k^*-u^*\| \leq \frac{11\epsilon}{1-a} \text{ and } \|u^*-x^*\| \leq \frac{11\epsilon}{1-a}.$$

Then

$$G(x^*,k^*,u^*) = \max\{\|x^*-k^*\|, \|k^*-u^*\|, \|u^*-x^*\|\} \leq \frac{11\epsilon}{1-a}.$$

□

References

[1] Takahashi, W., A convexity in metric space and nonexpansive mappings I, Kodai Math. Sem. Rep. 22 (1970), 142–149.

[2] Tian, Y-X., Convergence of an Ishikawa type iterative scheme for asymptotically quasi-nonexpansive mappings, Comput. Math. Appl. 49 (2005) 1905–1912.

[3] Mustafa, Z., Sims, B., "Some remarks concerning D-metric spaces," in International Conference on Fixed Point Theory and Applications, pp. 189–198, Yokohama, Yokohama, Japan, 2004.9.

[4] Mustafa, Z., Obiedat, H., Awawdeh, F., Some of fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory Appl., 2008(2008), Article ID 189870,page 12.

[5] Mustafa, Z., A new structure for generalized metric spaces with applications to FIxed point theory, Ph.D. thesis, University of Newcastle, Newcastle, UK, 2005.

[6] Mustafa, Z., Shatanawi, W. and Bataineh, M., Fixed point theorems on uncomplete G-metric spaces, J. Math. Stat. 4(4)(2008), 196-201.

[7] Mustafa, Z., Shatanawi, W., Bataineh, M., Existence of fixed point result in G-metric spaces, Int. J. Math. Math. Sci. 2009(2009), page 10, Article ID 283028.

[8] Mustafa, Z. and Sims, B., Fixed point theorems for contractive mappings in complete G-metric space, Fixed Point Theory Appl. 2009(2009), page 10, Article ID 917175.

[9] Mustafa, Z. and Sims, B., "A new approach to generalized metric spaces," Journal of Nonlinear and Convex Analysis, vol. 7, no. 2, pp. 289–297, 2006.

[10] Rafik, A., Fixed Points of Ciric Quasi-contractive Operators in Generalized Convex Metric Spaces,General Mathematics Vol. 14, No. 3 (2006), 79–90.

[11] Dogan, K., Karakaya, V., On the Convergence and Stability Results for a New General Iterative Process. The Scientific World Journal, 2014.

[12] Karakaya, V., Dogan, K., Gursoy, F., Erturk, M., Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces. In Abstract and Applied Analysis (Vol. 2013), 2013.

[13] Gursoy, F. and Karakaya, V., A Picard-S hybrid type iteration method for solving a differential equation with retarded argument. arXiv preprint arXiv:1403.2546, 2014.

[14] Mann, W. R., Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506-510.

[15] Ishikawa, S., Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147-150.

[16] Noor, M. A., New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217-229.

[17] Weng, X., Fixed point iteration for local strictly pseudo-contractivemapping, Proceedings of the American Mathematical Society, 113(1991), 727-731.

[18] Gahler, S., "2-metrische Raume und ihre topologische Struktur," Mathematische Nachrichten, vol. 26, pp. 115–148, 1963.

[19] Gahler, S., "Zur geometric 2-metriche raume," Revue Roumaine de Math´ematiques Pures et Appliquees, vol. 40, pp. 664–669, 1966.

[20] Ha, K.S., Cho, Y.J., White, A., "Strictly convex and strictly 2-convex 2-normed spaces,"Mathematica Japonica, vol. 33, no. 3, pp. 375–384, 1988.

[21] Dhage, B. C., "Generalized metric space and mapping with fixed point," Bulletin of the Calcutta Mathematical Society, vol. 84, pp. 329–336, 1992. 5.

[22] Dhage, B. C., "Generalized metric spaces and topological structure.I,"Analele Stiintificeale Universitatii Al. I. Cuza din Iasi. Serie Noua. Matematica, vol. 46, no. 1, pp. 3–24, 2000. 6.

[23] Dhage, B. C., "On generalized metric spaces and topological structure. II," Pure and Applied Mathematika Sciences, vol. 40, no. 1-2, pp. 37–41, 1994. 7.

[24] Dhage, B. C., "On continuity of mappings in D-metric spaces," Bulletin of the Calcutta Mathematical Society, vol. 86, no. 6, pp. 503–508, 1994. 8.

[25] Reich, S., "Some remarks concerning contraction mappings," Canadian Mathematical Bulletin, vol. 14, pp. 121–124, 1971. 11

[26] Bianchini, R. M. T., "Su un problema di S. Reich riguardante la teoria dei punti fissi," Bollettino dell'Unione Matematica Italiana, vol. 5, no. 4, pp. 103–108, 1972. 12.

[27] ´Ciric, Lj. B., "A generalization of Banach's contraction principle," Proceedings of the American Mathematical Society, vol. 45, pp. 267–273, 1974. 13.

[28] Chatterjea, S. K., "Fixed-point theorems," Doklady Bolgarsko Akademii Nauk. Comptes Rendus de l Academie Bulgare des Sciences, vol. 25, pp. 727–730, 1972.

[29] Thangavelu P., Shyamala Malini S., Jeyanthi, P., Convexity in D-Metric Spaces and its applications to fixed point theorems, International Journal of Statistika and Mathematika, Vol. 2, Issue 3, (2012), 05-12.

[30] Modi, G., Bhatt, B., "Fixed Point Results for Weakly Compatible Mappings in Convex G-Metric Space", International Journal Of Mathematics And Statistics Invention, Volume 2 Issue 11, 2014, pp 34-38.

[31] S oltuz SM, Grosan T., Data dependence for Ishikawa iteration when dealing with contractive like operators., Fixed Point Theory A., 2008, 2008, 1-7.

Kadri DOĞAN received a Ph.D. degree in mathematical engineering at the Yildiz Technical University under the supervision of Professor Vatan Karakaya. He is still working as an assistant professor in the Department of Computer Engineering at the Artvin Coruh University, Artvin-Turkey. Kadri's current research interests include Fixed Point Theory and Applications, the geometry of Banach spaces and Optimization.

Faik GRSOYserves as associate professor for the Department of Mathematics at the Adiyaman University, Adiyaman, Turkey. He received a Ph.D. degree in mathematics at the Yildiz Technical University under the co-supervision of Professor Vatan Karakaya and Professor Billy E. Rhoades. Faiks current research interests include Fixed Point Theory, Optimization, and Intuitionistic Fuzzy Normed Spaces.

Vatan KARAKAYAwas born in Malatya-Turkey, in 1971. He received Ph. D. degree in Analysis and Functions Spaces. His research interests include summability, sequence and series, the geometry of Banach spaces, the operator theory and spectral properties, intuitionistic fuzzy spaces, and fixed point theory. At the present, he is Professor of Mathematical Engineering at Yildiz Technical University in Istanbul-Turkey.