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MULTIPLICITY RESULTS TO A FOURTH-ORDER BOUNDARY VALUE PROBLEM FOR A STURM-LIOUVILLE TYPE EQUATION

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Abstract. We establish the existence of at least three distinct weak solutions for a fourth-order Sturm-Liouville type problem under appropriate hypotheses. Our main tools are based on variational methods and some critical points theorems. Moreover, when the energy functional is not coercive, an existence result of two distinct solutions is given. We give some examples to illustrate the obtained results.

Keywords: Fourth-order boundary value problem; Sturm-Liouville type problem; three solutions; variational methods.

AMS Subject Classification: 35J35, 35J60

1. Introduction

In this paper, we study the following two-point boundary-value problem of fourth-order Sturm-Liouville type

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda f(t,u) + \mu g(t,u) + h(u) & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $p, q, r \in L^\infty([0,1])$, with $p^- := \text{ess}\inf_{t\in[0,1]} p(t) > 0, \lambda \in]0,+\infty[$ and $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory function and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e,

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|,$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$.

Boundary value problems arise in several branches of physics as any physical differential equation will have them. Problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems. A large class of important boundary value problems is the Sturm-Liouville problem. To establish the existence and multiplicity of solutions to nonlinear differential problems is very important as well as the application of such results in the physical reality. For example, the deformations of

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an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by fourth-order boundary value problems. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for this kind of problems (see, for instance, [1, 2, 4, 6, 10, 15, 16, 17, 18] and references therein). In [7], the authors, employing a three critical point theorem due Bonanno and Marano [9, Theorem 2.6], established the existence of at least three distinct weak solutions to problem,

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda f(t,u) \quad \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{2}$$

Later Heidarkhani in [13], using critical point theory due Bonanno, proved the existence of at least one non-trivial solution for a class of two-point boundary-value problems for fourth-order Sturm-Liouville type equations.

The aim of this article is to prove the existence of at least three distinct weak solutions for (1). Moreover, when the energy functional is not coercive, an existence result of two distinct solutions is given. Our motivation comes from the papers [7, 8]. For basic notation and definitions, we refer the reader to [7, 11, 12, 14].

The rest of this paper is organized as follows. Section 2 contains some preliminary notations and our main tools. Section 3 contains our main results and their proofs.

2. Preliminaries and basic notations

Our main tools are the following critical point theorems. The first one due Averna and Bonanno [3, Theorem B], and the second one due Bonanno [5, Theorem1.1]. In the first one the coercivity of the functional $\Phi + \lambda\Psi$ is required.

Theorem 2.1 ([3, Theorem B]). *Let X be a reflexive Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put, for each $r > \inf_X \Phi$,*

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^{\omega}} \Psi}{r - \Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

where $\overline{\Phi^{-1}(]-\infty,r[)}^{\omega}$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology, and assume that

(i) *There is $r \in \mathbb{R}$ such that $\inf_X \Phi < r$ and $\varphi_1(r) < \varphi_2(r)$.*
Further, assume that:
(ii) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$.*

Then, for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$ the equation $\Phi'(u)(v) + \lambda\Psi'(u)(v) = 0$, has at least three solution in X.

Theorem 2.2 ([5, Theorem 1.1]). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functional. Assume that Φ is (strongly) continuous and satisfies $\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(j) $\inf_X \Phi < r_1 < r_2$;
(jj) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;
(jjj) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$,

where φ_1 is defined as in Theorem 2.1 and

$$\varphi_2^*(r_1, r_2) := \inf_{x \in \Phi^{-1}(]-\infty, r_1[)} \sup_{y \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)}.$$

Then, for each $\lambda \in]\frac{1}{\varphi_2^*(r_1,r_2)}, \min\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2}\}[$, the functional $\Phi + \lambda\Psi$ admits at least two critical points which lie in $\Phi^{-1}(]-\infty, r_1[)$ and $\Phi^{-1}(]r_1, r_2[)$ respectively.

Suppose that

$$\min\{\frac{q^-}{\pi^2}, \frac{r^-}{\pi^4}, \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}\} > -p^-, \tag{3}$$

where $p^- := \mathrm{ess\ inf}_{t\in[0,1]}\, \mathrm{p(t)} > 0, \mathrm{q}^- := \mathrm{ess\ inf}_{t\in[0,1]}\, \mathrm{q(t)}, \mathrm{r}^- := \mathrm{ess\ inf}_{t\in[0,1]}\, \mathrm{r(t)}$. Moreover, set $\sigma := \min\{\frac{q^-}{\pi^2}, \frac{r^-}{\pi^4}, \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}, 0\}$, $\delta := \sqrt{p^- + \sigma}$, $\gamma := \Big(\|p\|_\infty + \frac{1}{\pi^2}\|q\|_\infty + \frac{1}{\pi^4}\|r\|_\infty\Big)^{\frac{1}{2}}$ and $k := 2\pi^2\delta^2(\frac{2048}{27}\gamma^2)^{-1}$. A simple computation shows that $\delta < \gamma$.

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm. We recall the following Poincaré type inequalities (see, for instance, [16, Lemma 2.3]):

$$\|u'\|^2_{L^2([0,1])} \leq \frac{1}{\pi^2}\|u''\|^2_{L^2([0,1])}, \tag{4}$$

$$\|u\|^2_{L^2([0,1])} \leq \frac{1}{\pi^4}\|u''\|^2_{L^2([0,1])}, \tag{5}$$

for all $u \in X$. Therefore, taking into account (3)-(5), the norm

$$\|u\|_X = \Big(\int_0^1 \big(p(x)|u''(x)|^2 + q(x)|u'(t)|^2 + r(x)|u(x)|^2\big)dx\Big)^{\frac{1}{2}},$$

is equivalent to the usual norm and, in particular, one has

$$\|u''\|_{L^2([0,1])} \leq \frac{1}{\delta}\|u\|_X. \tag{6}$$

We needs the following proposition in the proof of Theorems.

Proposition 2.1 ([7, proposition 2.1])**.** *Let $u \in X$. Then*

$$\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|_X. \tag{7}$$

We suppose that the Lipschitz constant L of function h satisfies $L < 4\pi^2\delta^2$. For each $(t,\xi) \in [0,1] \times \mathbb{R}$, put

$$F(t,\xi) = \int_0^\xi f(t,x)dx, \;\; G(t,\xi) = \int_0^\xi g(t,x)dx,$$

and $H(\xi) = \int_0^\xi h(x)dx$, for each $\xi \in \mathbb{R}$. Moreover, set

$$G^c := \int_0^1 \sup_{|\xi|<c} G(t,\xi)dt,$$

for every $c > 0$, and

$$G_d := \inf_{[0,1]\times[0,d]} G(t,\xi),$$

for every $d > 0$. If g is sign-changing, then $G^c \geq 0$ and $G_d \leq 0$.

We say that $u \in X$ is a weak solution of problem (1) if for every $v \in X$

$$\int_0^1 \big(p(t)u^{''}(t)v^{''}(t) + q(t)u^{'}(t)v^{'}(t) + r(t)u(t)v(t)\big)dt$$
$$-\lambda \int_0^1 f(t,u(t))v(t)dt - \mu \int_0^1 g(t,u(t))v(t)dt - \int_0^1 h(u(t))v(t)dt = 0.$$

3. Main results

In order to introduce our first result, fixing two positive constants c, d such that

$$\frac{(1+\frac{L}{4\pi^2\delta^2})d^2}{k(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c} F(t,\xi)dt)} < \frac{(1-\frac{L}{4\pi^2\delta^2})c^2}{\int_0^1 \max_{|\xi|<c} F(t,\xi)dt},$$

taking

$$\lambda \in \Lambda := \Big]\frac{2\pi^2\delta^2(1+\frac{L}{4\pi^2\delta^2})d^2}{k(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c} F(t,\xi)dt)}, \frac{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2}{\int_0^1 \max_{|\xi|<c} F(t,\xi)dt}\Big[,$$

and set $\delta_{\lambda,g}$ given by

$$\min\Big\{\frac{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2 - \lambda\int_0^1 \max_{|\xi|<c} F(t,\xi)dt}{G^c},$$
$$\frac{\frac{2\pi^2\delta^2}{k}(1+\frac{L}{4\pi^2\delta^2})d^2 - \lambda\big(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c} F(t,\xi)dt\big)}{G_d - G_c}\Big\}, \tag{8}$$

and

$$\bar{\delta}_{\lambda,g} := \min\Big\{\delta_{\lambda,g}, \frac{1}{\max\big\{0, \frac{1}{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2}\limsup_{|x|\to+\infty}\frac{\sup_{t\in[0,1]}G(t,x)}{x^2}\big\}}\Big\} \tag{9}$$

where we read $r/0 = +\infty$, so that, for instance, $\bar{\delta}_{\lambda,g} = +\infty$, when

$$\limsup_{|x|\to+\infty}\frac{\sup_{t\in[0,1]}G(t,x)}{x^2} \le 0,$$

and $G_d = G^c = 0$.

Theorem 3.1. *Assume that there exist two positive constants c, d, with $c < \frac{32}{3\sqrt{3}\pi}d$, such that*

(A$_1$) $F(t,\xi) \ge 0$ *for all* $(t,\xi) \in ([0,\frac{3}{8}] \cup [\frac{5}{8},1]) \times [0,d]$,

(A$_2$) $\frac{\int_0^1 \max_{|\xi|\le c} F(t,\xi)dt}{c^2} < k\frac{(1-\frac{L}{4\pi^2\delta^2})}{(1+\frac{L}{4\pi^2\delta^2})}\frac{\big(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|\le c} F(t,\xi)dt\big)}{d^2}$,

(A$_3$) $\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,1]}F(t,\xi)}{\xi^2} < 0$.

Then, for each $\lambda \in \Lambda$, and for every L^2-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|x|\to+\infty}\frac{\sup_{t\in[0,1]}G(t,x)}{x^2} < +\infty,$$

there exists $\bar{\delta}_{\lambda,g} > 0$ given by (9) such that, for each $\mu \in [0, \bar{\delta}_{\lambda,g}[$, the problem (1) admits at least three distinct weak solution in X.

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$, for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_0^1 H(u(x))dx, \quad \Psi(u) = -\big(\int_0^1 F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^1 G(t,u(t))dt\big).$$

It is well known that these functionals are well-defined and satisfy the regularity assumptions required in Theorem 2.1. In particular, one has

$$\Phi^{'}(u)(v) = \int_0^1 \big(p(t)u^{''}(t)v^{''}(t) + q(t)u^{'}(t)v^{'}(t) + r(t)u(t)v(t)\big)dt - \int_0^1 h(u(t))v(t)dt,$$

and

$$\Psi^{'}(u)(v) = -\int_0^1 f(t,u(t))v(t)dt - \frac{\mu}{\lambda}\int_0^1 g(t,u(t))v(t)dt,$$

for each u, $v \in X$. Then each critical point for the functional $\Phi + \lambda\Psi$ is a weak solution for problem (1). Let us consider φ_1 and φ_2 given in Theorem 2.1. We can observe $\inf_X \Phi = \Phi(0) = 0$ and that, for each $r > 0$, $0 \in \Phi^{-1}(]-\infty, r[)$ and $\overline{\Phi^{-1}(]-\infty, r[)}^{\omega} = \Phi^{-1}(]-\infty, r[)$. Put $r := 2\pi^2\delta^2(1 - \frac{L}{4\pi^2\delta^2})c^2$. The inequality

$$\frac{1}{2}(1 - \frac{L}{4\pi^2\delta^2})\|u\|^2 \leq \Phi(u), \tag{10}$$

for each $u \in X$ in conjunction with (3) yields

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) &\subseteq \left\{u \in X; \frac{1}{2}(1 - \frac{L}{4\pi^2\delta^2})\|u\|^2 \leq r\right\} \\ &\subseteq \{u \in X;\ |u(t)| \leq c,\ \text{for each } t \in [0,1]\}, \end{aligned}$$

which follows

$$\begin{aligned} \varphi_1(r) &\leq \frac{\Psi(0) - \inf_{\Phi^{-1}(]-\infty,r[)}\Psi}{r - \Phi(0)}, \\ &\leq \sup_{\Phi(u)\leq r} \frac{\int_0^1 F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^1 G(t,u(t))dt}{r}, \\ &\leq \frac{\int_0^1 \max_{|\xi|<c} F(t,\xi)dt + \frac{\mu}{\lambda}G^c}{r}. \end{aligned} \tag{11}$$

Since $\mu < \delta_{\lambda,g}$ one has, $\mu < \frac{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2 - \lambda\int_0^1 \max_{|\xi|<c} F(t,\xi)dt}{G^c}$, this means

$$\varphi_1(r) < \frac{1}{\lambda}. \tag{12}$$

Now, let $\bar{y} \in X$ be defined by

$$\bar{y}(t) = \begin{cases} -\frac{64d}{9}(t^2 - \frac{3}{4}t) & t \in [0, \frac{3}{8}], \\ d & t \in]\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(t^2 - \frac{5}{4}t + \frac{1}{4}) & t \in]\frac{5}{8}, 1]. \end{cases}$$

It is easy to verify that

$$\frac{2048}{27}\delta^2(1 - \frac{L}{4\pi^2\delta^2})d^2 \leq \Phi(\bar{y}) \leq \frac{2048}{27}\gamma^2(1 + \frac{L}{4\pi^2\delta^2})d^2. \tag{13}$$

Taking into account $c < \frac{32}{3\sqrt{3}\pi}d$, we observe that $r < \Phi(\bar{y})$. On the other hand, in view of (A_1), since $0 \leq \bar{y}(t) \leq d$ for each $t \in [0,1]$ we have

$$-\Psi(\bar{y}) \geq \int_{\frac{3}{8}}^{\frac{5}{8}} F(t, \bar{y}(t))dt + \frac{\mu}{\lambda} \inf_{[0,1]\times[0,d]} G(x,t) \geq \int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt + \frac{\mu}{\lambda} G_d,$$

so, thanks to (13) we get

$$\begin{aligned}\varphi_2(r) &\geq \inf_{x \in \Phi^{-1}(]-\infty, r[)} \frac{\Psi(x) - \Psi(\bar{y})}{\Phi(\bar{y})}, \\ &\geq \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt + \frac{\mu}{\lambda} G_d - \int_0^1 \max_{|\xi|<c} F(t,\xi)dt - \frac{\mu}{\lambda} G^c}{\frac{2\pi^2\delta^2}{k}(1 + \frac{L}{4\pi^2\delta^2})d^2}.\end{aligned}$$

Furthermore, $\mu < \frac{\frac{2\pi^2\delta^2}{k}(1+\frac{L}{4\pi^2\delta^2})d^2 - \lambda\left(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c} F(t,\xi)dt\right)}{G_d - G_c}$, this means

$$\varphi_2(r) > \frac{1}{\lambda}. \tag{14}$$

Hence from (12) and (14), hypothesis (i) of Theorem 2.1 is fulfilled. Finally, since $\mu < \bar{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,x)}{x^2} < l,$$

and

$$\mu l < \frac{4\pi^2\delta^2 - L}{2}.$$

Therefore, there exists a function $h_\mu \in L^1([0,1])$ such that

$$G(t,\xi) < |\xi|^2 + h_\mu(t), \tag{15}$$

for every $t \in [0,1]$ and every $\xi \in \mathbb{R}$. Now, fix

$$0 < \epsilon < \frac{4\pi^2\delta^2 - L}{2\lambda} - \frac{\mu l}{\lambda}.$$

due to A_3, there exists a function $h_\epsilon \in L^1([0,1])$ such that

$$F(t,\xi) < \epsilon|\xi|^2 + h_\epsilon(t), \tag{16}$$

for every $t \in [0,1]$ and every $\xi \in \mathbb{R}$. Fixed $u \in X$, from (15) and (16), one has

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 H(u(x))dx - \lambda \int_0^1 F(t,u(t))dt - \mu \int_0^1 G(t,u(t))dt, \\ &\geq \frac{1}{2}\|u\|^2 - \frac{L}{8\pi^2\delta^2}\|u\|^2 - \lambda\epsilon \int_0^1 u^2(x)dx - \lambda\|h_\epsilon\|_{L^1([0,1])} \\ &- \mu l \int_0^1 u^2(x)dx - \mu\|h_\mu\|_{L^1([0,1])}, \\ &\geq \frac{1}{2}\big(1 - \frac{L}{4\pi^2\delta^2} - \frac{\lambda\epsilon}{2\pi^2\delta^2} - \frac{\mu l}{2\pi^2\delta^2}\big)\|u\|^2 - \lambda\|h_\epsilon\|_{L^1([0,1])} - \mu\|h_\mu\|_{L^1([0,1])}.\end{aligned}$$

So, the functional $\Phi + \lambda\Psi$ is coercive, and the condition (ii) of Theorem 2.1 is verified. This completes the proof. □

Now we point out the following consequence of Theorem 3.1:

Theorem 3.2. *Let $\alpha : [0,1] \to \mathbb{R}$ be a nonnegative, non-zero and essentially bounded function, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$. Assume that there exist two positive constants c, d, with $c < \frac{32}{3\sqrt{3}\pi} d$, such that*

(A_4) $F(\xi) \geq 0$ *for all* $\xi \in [0, d]$,

(A_5) $\frac{\|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)}{c^2} < k \frac{(1-\frac{L}{4\pi^2\delta^2})}{(1+\frac{L}{4\pi^2\delta^2})} \frac{\left(F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt - \|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)\right)}{d^2}$,

(A_6) $\limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} < 0$.

Then, for each λ, in

$$\left]\frac{2\pi^2\delta^2(1+\frac{L}{4\pi^2\delta^2})d^2}{k\left(F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt - \|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)\right)}, \frac{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2}{\|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)}\right[,$$

and for every L^2-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,x)}{x^2} < +\infty,$$

there exists $\delta_{\lambda,g} > 0$ given by

$$\min\left\{\frac{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2 - \lambda\|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)}{G^c},\right.$$

$$\left.\frac{\frac{2\pi^2\delta^2}{k}(1+\frac{L}{4\pi^2\delta^2})d^2 - \lambda\left(F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt - \|\alpha\|_1 \max_{|\xi|\leq c} F(\xi)\right)}{G_d - G_c}\right\},$$

such that, for each μ in

$$\left[0, \min\left\{\delta_{\lambda,g}, \frac{1}{\max\left\{0, \frac{1}{2\pi^2\delta^2(1-\frac{L}{4\pi^2\delta^2})c^2} \limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,x)}{x^2}\right\}}\right\}\right[,$$

the problem

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda\alpha(t)f(u) + \mu g(x,u) + h(u) & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{17}$$

admits at least three distinct weak solution in X.

Proof. It is enough to apply Theorem 3.1, to the function $\alpha(t)f(u)$ instead of $f(t,u)$. □

Example 3.1. *Consider the problem*

$$\begin{cases} (3u''(t))'' - ((t^2-\pi^2)u'(t))' + (t^2-\pi^4)u(t) = \lambda f(u) + 2\mu u + \sin u & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{18}$$

where

$$f(x) = \begin{cases} 10 & x \leq 1, \\ 10(800x - 799), & 1 < x \leq 2, \\ 8010 & x > 2. \end{cases}$$

Note that $p^- = 3$, $q^- = -\pi^2$, *and* $r^- = -\pi^4$, *we have* $\sigma = -2$, *and so* $\delta = 1$. *Also we have* $\gamma = \sqrt{5}$. *Our aim is to apply Theorem 3.2. Let* $c = 1$, *and* $d = 2$. (A_4) *is clearly true. One has,* $\frac{\max_{|\xi|\leq c} F(\xi)}{c^2} = 10$, *and*

$$k\frac{(1-\frac{L}{4\pi^2\delta^2})}{(1+\frac{L}{4\pi^2\delta^2})}\frac{\left(F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} dt - \max_{|\xi|\leq c} F(\xi)\right)}{d^2} = \frac{27\pi^2}{5120}\frac{(1-\frac{1}{4\pi^2})}{(1+\frac{1}{4\pi^2})}\frac{\left(\frac{4020}{4}-10\right)}{4} > 12,$$

then, (A_5) *is verified. And since,* $\lim_{|\xi|\to+\infty}\frac{F(\xi)}{\xi^2} = 0, (\mathrm{A}_6)$ *is satisfied too. Finally,* $G^1 = 1$, $G_2 = 0$, $\limsup_{|x|\to+\infty}\frac{\sup_{t\in[0,1]} G(t,x)}{x^2} = 1$, *and*

$$\frac{2\pi^2(1+\frac{1}{4\pi^2})4}{\frac{27\pi^2}{5120}(\frac{4020}{4}-10)} < 1.57,$$

$$\frac{2\pi^2(1-\frac{1}{4\pi^2})}{10} > 1.92,$$

so for every $\lambda \in \left]1.57, 1.92\right[$, *and for every* $\mu \in \left[0, 19.2\right[$ *the problem* (18) *admits at least three distinct weak solution in* X.

Here is another multiplicity result in which assumption (A_3) is not required.

Theorem 3.3. *Let* $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ *be a* L^2*- Carathéodory function and assume that there exist three positive constants* c_1, c_2, *and* d, *with* $c_1 < \frac{32}{3\sqrt{3}\pi}d < \frac{\delta}{\gamma}c_2$, *such that*

(B_1) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in ([0,\frac{3}{8}]\cup[\frac{5}{8},1])\times[0,d]$,

(B_2) $\frac{\int_0^1 \max_{|\xi|\leq c_1} F(t,\xi)dt}{c_1^2} < k\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|\leq c_1} F(t,\xi)dt}{d^2}$,

(B_3) $\frac{\int_0^1 \max_{|\xi|\leq c_2} F(t,\xi)dt}{c_2^2} < k\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|\leq c_1} F(t,\xi)dt}{d^2}$.

Then, for every

$$\lambda \in \left]\frac{2\pi^2\delta^2}{k}\frac{d^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c_1} F(t,\xi)dt},\right.$$
$$\left. 2\pi^2\delta^2\min\{\frac{c_1^2}{\int_0^1 \max_{|\xi|<c_1} F(t,\xi)dt}, \frac{c_2^2}{\int_0^1 \max_{|\xi|<c_2} F(t,\xi)dt}\}\right[$$

the problem

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda f(x,u) \quad \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{19}$$

has admits at least two solution $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\|u_{1,\lambda}\|_\infty < c_1$ *and* $\|u_{2,\lambda}\|_\infty < c_2$.

Proof. Our aim is to apply Theorem 2.2 to our problem. Let us introduce two functionals $\Phi, \Psi : X \to \mathbb{R}$, for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2,\ \Psi(u) = -\int_0^1 F(t,u(t))dt.$$

It is known that these functionals are well-defined and satisfy the regularity assumptions required in Theorem 2.2, and each critical point for the functional $\Phi+\lambda\Psi$ is a weak solution for problem (19). Put $r_1 = 2\pi^2\delta^2 c_1^2$, and $r_2 = 2\pi^2\delta^2 c_2^2$. Obviously $\inf_X \Phi = \Phi(0) = 0$, and $\inf_X \Phi < r_1 < r_2$. Let $v \in X$ be defined by

$$v(t) = \begin{cases} -\frac{64d}{9}(t^2 - \frac{3}{4}t) & t \in [0, \frac{3}{8}], \\ d & t \in]\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(t^2 - \frac{5}{4}t + \frac{1}{4}) & t \in]\frac{5}{8}, 1]. \end{cases}$$

We know that

$$\frac{2048}{27}\delta^2 d^2 \leq \Phi(v) \leq \frac{2048}{27}\gamma^2 d^2, \tag{20}$$

then from $c_1 < \frac{32}{3\sqrt{3}\pi}d < \frac{\delta}{\gamma}c_2$, we have $r_1 < \Phi(v) < r_2$. Similar to the proof of the Theorem 3.1, we have $\Phi^{-1}(]-\infty, r_i[) \subseteq \{u \in X;\ |u(t)| \leq c_i, \text{ for each } t \in [0,1]\}$, $i = 1, 2$. Consequently

$$\sup_{\Phi(u) \leq r_i} \int_0^1 F(t, u(t))dt \leq \int_0^1 \max_{|\xi| \leq c_i} F(t, \xi)dt,\ i = 1, 2. \tag{21}$$

Due to (B_1), one has that

$$-\Psi(v) = \int_0^1 F(t, v(t))dt \geq \int_{\frac{3}{8}}^{\frac{5}{8}} F(t, v(t))dt = \int_{\frac{3}{8}}^{\frac{5}{8}} F(t, d)dt,$$

so, thanks to (20) we get

$$\begin{aligned} \varphi_2^*(r_1, r_2) &\geq \inf_{x \in \Phi^{-1}(]-\infty, r_1[)} \frac{\Psi(x) - \Psi(v)}{\Phi(v)} \\ &\geq \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t, d)dt - \int_0^1 \max_{|\xi|<c_1} F(t, \xi)dt}{\frac{2\pi^2\delta^2}{k}d^2}. \end{aligned} \tag{22}$$

Moreover, as we saw in Theorem 3.1,

$$\varphi_1(r_1) \leq \frac{\int_0^1 \max_{|\xi|<c_1} F(t, \xi)dt}{2\pi^2\delta^2 c_1^2}, \tag{23}$$

$$\varphi_1(r_2) \leq \frac{\int_0^1 \max_{|\xi|<c_2} F(t, \xi)dt}{2\pi^2\delta^2 c_2^2}. \tag{24}$$

At this point, combining (22)-(24), assumption (B_2) and (B_3) we obtain

$$\varphi_1(r_1) \leq \varphi_2^*(r_1, r_2), \tag{25}$$

$$\varphi_1(r_2) \leq \varphi_2^*(r_1, r_2). \tag{26}$$

Therefore all assumption of Theorem 2.2 are satisfied. Hence, since one has

$$\begin{aligned} \frac{1}{\varphi_2^*(r_1, r_2)} &\leq \frac{\frac{2\pi^2\delta^2}{k}d^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t, d)dt - \int_0^1 \max_{|\xi|<c_1} F(t, \xi)dt} \\ &\leq 2\pi^2\delta^2 \min\{\frac{c_1^2}{\int_0^1 \max_{|\xi|<c_1} F(t, \xi)dt}, \frac{c_2^2}{\int_0^1 \max_{|\xi|<c_2} F(t, \xi)dt}\} \\ &\leq \min\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_2(r_2)}\}, \end{aligned}$$

for each $\lambda \in \Big]\frac{2\pi^2\delta^2}{k}\frac{d^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|<c_1} F(t,\xi)dt}$,
$2\pi^2\delta^2 min\{\frac{c_1^2}{\int_0^1 \max_{|\xi|<c_1} F(t,\xi)dt}, \frac{c_2^2}{\int_0^1 \max_{|\xi|<c_2} F(t,\xi)dt}\}\Big[$ problem (19) admits at least two solution $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|^2 < 4\pi^2\delta^2 c_1^2 < \|u_{2,\lambda}\|^2 < 4\pi^2\delta^2 c_2^2$, and we can complete the proof. □

Now, we deduce the following straightforward consequence of Theorem 3.3.

Theorem 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a L^2- Carathéodory function and assume that there exist three positive constants e_1, e_2, and d, with $e_1 < \frac{32}{3\sqrt{3}\pi}d < \frac{\delta}{\gamma}e_2$, such that*

(B_4) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in ([0,\frac{3}{8}] \cup [\frac{5}{8},1]) \times [0,d]$,

(B_5) $\frac{\int_0^1 \max_{|\xi|\leq e_1} F(t,\xi)dt}{e_1^2} < \frac{k}{k+1}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2}$,

(B_6) $\frac{\int_0^1 \max_{|\xi|\leq e_2} F(t,\xi)dt}{e_2^2} < \frac{k}{k+1}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2}$.

Then, for every

$$\lambda \in \Big]\frac{2\pi^2(k+1)\delta^2}{k}\frac{d^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}, 2\pi^2\delta^2 \min\{\frac{e_1^2}{\int_0^1 \max_{|\xi|<e_1} F(t,\xi)dt}, \frac{e_2^2}{\int_0^1 \max_{|\xi|<e_2} F(t,\xi)dt}\}\Big[$$

the problem

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda f(x,u), & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{27}$$

has admits at least two solution $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|_\infty < e_1$ and $\|u_{2,\lambda}\|_\infty < e_2$.

Proof. Put $c_1 = e_1$ and $c_2 = e_2$. Taking into account that $c_1 = e_1 < d$, from (B_5) we get

$$\begin{aligned} k\frac{\big(\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt - \int_0^1 \max_{|\xi|\leq c_1} F(t,\xi)dt\big)}{d^2} &> k\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2} - \frac{\int_0^1 \max_{|\xi|\leq c_1} F(t,\xi)dt}{c_1^2} \\ &> k\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2} - k\frac{k}{k+1}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2} \\ &= \frac{k}{k+1}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(t,d)dt}{d^2}. \end{aligned}$$

Hence, using (B_5), (B_6) hypotheses (B_2), (B_3) of Theorem 3.3 are fulfilled. □

Now, we present a consequence of Theorem 3.4.

Theorem 3.5. *Let $\alpha : [0,1] \to \mathbb{R}$ be a nonnegative, non-zero and essentially bounded function, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$. Assume that there exist three positive constants e_1, e_2, and d, with $e_1 < \frac{32}{3\sqrt{3}\pi}d < \frac{\delta}{\gamma}e_2$, such that*

(B_7) $F(\xi) \geq 0$ *for all* $\xi \in [0,d]$,

(B_8) $\frac{\|\alpha\|_1 \max_{|\xi|\leq e_1} F(\xi)}{e_1^2} < \frac{k}{k+1}\frac{F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt}{d^2}$,

(B_9) $\frac{\|\alpha\|_1 \max_{|\xi|\leq e_2} F(\xi)}{e_2^2} < \frac{k}{k+1} \frac{F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt}{d^2}$.

Then, for every

$$\lambda \in \left]\frac{2\pi^2(k+1)\delta^2}{k}\frac{d^2}{F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt}, \frac{2\pi^2\delta^2}{\|\alpha\|_1}\min\{\frac{e_1^2}{\max_{|\xi|\leq e_1} F(\xi)}, \frac{e_2^2}{\max_{|\xi|\leq e_2} F(\xi)}\}\right[$$

the problem

$$\begin{cases} (p(t)u''(t))'' - (q(t)u'(t))' + r(t)u(t) = \lambda\alpha(t)f(u), & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{28}$$

has admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|_\infty < e_1$, and $\|u_{2,\lambda}\|_\infty < e_2$.

Example 3.2. *Consider the problem*

$$\begin{cases} u^{i\nu} - \pi^2 u' + (t-\pi)u = \lambda t f(u) & \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{29}$$

where

$$f(x) = \begin{cases} 1 & x \leq 1, \\ 800x - 799 & 1 < x \leq 2, \\ -800x + 2401 & 2 < x \leq 3, \\ 1 & 3 < x \leq 20, \\ f^*(x) & x > 20, \end{cases}$$

and $f^* : (20, +\infty) \to \mathbb{R}$ is an arbitrary function such that f be a continuous function. By choosing, for instance, $e_1 = 1$, $d = 2$, and $e_2 = 20$, all assumptions of Theorem 3.5 are satisfied. In fact, Note that $\min\{1, -\frac{1}{\pi^2}, 1 - \frac{1}{\pi^3}\} > -1$, *and one has,* $\delta = (1 - \frac{1}{\pi^3})^{\frac{1}{2}}$, $\gamma = (2 + \frac{1}{\pi^3})^{\frac{1}{2}}$. *Moreover,* $F(2)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt = \frac{402}{8}$,

$$\frac{k}{k+1} = \frac{27\pi^2(1-\frac{1}{\pi^3})}{1024(2+\frac{1}{\pi^3}) + 27\pi^2(1-\frac{1}{\pi^3})} > \frac{1}{10},$$

$$\frac{\|\alpha\|_1 \max_{|\xi|\leq e_1} F(\xi)}{e_1^2} = \frac{1}{2}, \text{ and } \frac{\|\alpha\|_1 \max_{|\xi|\leq e_2} F(\xi)}{e_2^2} = \frac{820}{800},$$

so, our claim is proved. Since $\frac{2\pi^2(k+1)\delta^2}{k}\frac{d^2}{F(d)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(t)dt} = \frac{2(k+1)(\pi^3-1)}{k\pi}\frac{32}{402} \leq 15$,

$$\frac{2\pi^2\delta^2}{\|\alpha\|_1}\frac{e_1^2}{\max_{|\xi|\leq e_1} F(\xi)} = 4(\pi^2 - \frac{1}{\pi}) > 38, \text{ and } \frac{2\pi^2\delta^2}{\|\alpha\|_1}\frac{e_2^2}{\max_{|\xi|\leq e_2} F(\xi)} = \frac{1600}{820}(\pi^2 - \frac{1}{\pi}) > 18,$$

owing to Theorem 3.5 for each $\lambda \in]15, 18[$ the problem (29) admits at least two distinct weak solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|_\infty < 1$, and $\|u_{2,\lambda}\|_\infty < 20$.

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