TWMS J. App. Eng. Math. V.10, N.1, 2020, pp. 138-149

WAVELET PACKETS IN WEIGHTED SOBOLEV SPACE

RAJ KUMAR[1], MANISH CHAUHAN[2], §

ABSTRACT. We perform some splitting tricks over wavelets to construct basic wavelet packets in weighted Sobolev space. MRA based wavelet packet functions and their orthogonality at different levels in weighted Sobolev space are presented. Some examples of wavelet packets in weighted Sobolev space are given.

Keywords: Wavelets; wavelet packets; multiresolution analysis; weighted Sobolev space

AMS Subject Classification: 42C40; 40A30; 46E35

1. Introduction

There has been considerable focus upon wavelet packet analysis as an important generalization of wavelet analysis. Wavelet packet functions consist of a rich family of building block functions and are localised in time, but offer more flexibility than wavelets in representing different kinds of signals. The power of wavelet packets lies in the fact that we have much more freedom in selecting which basis functions are to be used to represent the given function. Decomposition of wavelet components by orthogonal wavelet packets were introduced by Coifman and Meyer [6] (see also [7], [20]). A detailed description of wavelet packets of $L^2(\mathbb{R})$ with dilation 2 is illustrated in [12]. Some good generalization of wavelet packets and efficient algorithms for finding best basis in wavelet packets apply to wavelet frame packets are given in Chen [13]. The concept of wavelet packet was subsequently generalized to $\mathbb{R}$ by taking tensor product version [6] and non-tensor product version for dyadic dilation by Shen [17]. Other remarkable generalizations are the biorthogonal wavelet packets [5], non-orthogonal version of wavelet packets [8], the biorthogonal, orthogonal and wavelet frame packet on $\mathbb{R}$ for the dyadic dilation by Long and Chen [14].

Wavelets and their properties in Sobolev space were explored by Bastin et al. [1, 3, 2], Dayong and Dengfeng [9], Walter [18, 19] and Pathak [15]. The wavelet packets and their orthogonal properties in Sobolev space $H^s(\mathbb{R})$ were introduced by Pathak and Manish [16]. Han and Shen [11], Ehler [10] introduced a new concept to simplify the construction of wavelet systems by constructing a pair of dual wavelet frames for a pair of Sobolev spaces.

[1] Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India.
e-mail: rajkmc@gmail.com; ORCID: http://orcid.org/0000-0003-0714-5045.
[2] Department of Mathematics, University of Delhi, Delhi-110007, India.
e-mail: manish17102021@gmail.com; ORCID: https://orcid.org/0000-0002-1406-4705.
§ Manuscript received: May 08, 2018; accepted: June 13, 2019.

An efficient and comparatively low complexity method in the weighted Sobolev space for the poor resolution text image enhancement has been presented in [4].

In this paper, we present wavelet packets as a generalisation of wavelets in weighted Sobolev space $W_2^1(\mathbb{R})$ [4].

Organization of the paper. Section 2, perform some splitting tricks over wavelets to construct basic wavelet packets in weighted Sobolev space. In section 3, MRA based wavelet packet functions and their orthogonality at different levels are presented. Further, Some examples of wavelet packets in weighted Sobolev space are given.

1.1. **Preliminaries.** Let us consider the weighted Sobolev space $W_2^1(\mathbb{R})$. The weighted Sobolev space is defined with a scalar product of functions as follows:

$$\langle f, g\rangle_W := (1-\beta)\int_{\mathbb{R}} f(x)g(x)dx + \beta\int_{\mathbb{R}} f^{'}(x)g^{'}(x)dx,$$

where $0 \leq \beta \leq 1$ is weight. The norms of the function and scalar product in a spectral domain are defined accordingly as follows:

$$\|f\|_W^2 := \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)|\hat{f}(\eta)|^2 d\eta,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ and the corresponding inner product is given by

$$\langle f, g\rangle_W := \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)\hat{f}(\eta)\overline{\hat{g}(\eta)}d\eta.$$

The Fourier transform $\hat{f}$, for $f \in L^1(\mathbb{R})$ is define to be

$$\hat{f}(\eta) := \int_{\mathbb{R}} e^{-ix.\eta} f(x)dx,$$

where $x.\eta$ is the Euclidean inner product of two vectors x and η in $\mathbb{R}$.

1.2. **Multiresolution analysis in $W_2^1(\mathbb{R})$.** The theory of multiresolution analysis in Sobolev spaces was developed by Bastin and Laubin [3]. In the present work we extend the theory over $W_2^1(\mathbb{R})$.

Definition 1.1. *A multiresolution analysis of $W_2^1(\mathbb{R})$ is a sequence $V_j,\ j \in \mathbb{Z}$, of closed linear subspaces of $W_2^1(\mathbb{R})$ such that*

- (a) $V_j \subset V_{j+1}$,
- (b) $\overline{\bigcup_{j=-\infty}^{j=\infty} V_j} = W_2^1(\mathbb{R})$,
- (c) $\bigcap_{j=-\infty}^{j=\infty} V_j = \{0\}$, *and*
- (d) *for every j, there is a function $\varphi^{(j)}$ such that the distributions $2^{j/2}\varphi^{(j)}(2^j x - k),\ k \in \mathbb{Z}$, form an orthonormal basis for V_j.*

Proposition 1.2. *If $\varphi^{(j)} \in W_2^1(\mathbb{R})$, $s \in \mathbb{R}$ and j is an integer, then distributions $\varphi_{j,k}^{(j)}(x) = 2^{j/2}\varphi^{(j)}(2^j x - k),\ k \in \mathbb{Z}$ are orthonormal in $W_2^1(\mathbb{R})$ iff*

$$\sum_{k\in\mathbb{Z}}\left(1-\beta+2^{2j}\beta|\eta+2k\pi|^2\right)|\hat{\varphi}^{(j)}(\eta+2k\pi)|^2 = 1 \quad (1)$$

almost everywhere. It follows that we have the bound

$$\left|\hat{\varphi}^{(j)}\left(2^{-j}\eta\right)\right| \leq \left(1-\beta+\beta|\eta|^2\right)^{-1/2}.$$

Proof. See Ref. [[3],p.482-483]. □

Proposition 1.3. *Let $\varphi^{(j)}$, $j \in \mathbb{Z}$, be a sequence of elements of $W_2^1(\mathbb{R})$ such that, for every j, the distributions $\varphi_{j,k}^{(j)}(x) = 2^{j/2}\varphi^{(j)}(2^j x - k)$, $k \in \mathbb{Z}$, are orthonormal in $W_2^1(\mathbb{R})$. If P_j is the orthogonal projection from $W_2^1(\mathbb{R})$ onto $V_j := \overline{> \varphi_{j,k}^{(j)} :\ k \in \mathbb{Z} <}$, then, for every $h \in W_2^1(\mathbb{R})$, we have*

$$\lim_{j\to+\infty}\left(\|P_j h\|_W^2 - \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)^2|\hat{h}(\eta)|^2|\hat{\varphi}^{(j)}(2^{-j}\eta)|^2 d\eta\right) = 0.$$

Moreover, if there are $A,\ \alpha > 0$ such that

$$\int_{\mathbb{R}}(1-\beta+\beta|\eta|)^{\alpha}|\hat{\varphi}^{(j)}(\eta)|^2 d\eta \leq A$$

for every $j \leq 0$, then $\bigcap_{j=-\infty}^{j=\infty} V_j = \{0\}$.

Proof. The proof is similar to proof of ref. [[3] Proposition 2.2, p.483-484]. □

1.3. **Construction of wavelets in Sobolev space $W_2^1(\mathbb{R})$.** The result of the previous subsection are the key for the construction of the wavelets.

(1) By definition, V_j is the set of all $f \in W_2^1(\mathbb{R})$ such that

$$\hat{f}(\eta) = m\left(2^{-j}\eta\right)\hat{\varphi}^{(j)}\left(2^{-j}\eta\right),$$

where $m \in L^2_{loc}(\mathbb{R})$ is 2π-periodic. This follows immediately from the fact that the Fourier transform of $2^{j/2}\varphi^{(j)}(2^j x - k)$ is $2^{-j/2}e^{-i2^{-j}k.\eta}\hat{\varphi}^{(j)}(2^{-j}\eta)$.

(2) We have $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$ iff there are 2π-periodic functions $m_0^{(j)} \in L^2_{loc}(\mathbb{R})$ such that the following scale relation holds:

$$\hat{\varphi}^{(j)}(2\eta) = m_0^{(j+1)}(\eta)\,\hat{\varphi}^{(j+1)}(\eta).$$

Moreover, as $\varphi^{(j)}$ and $\varphi^{(j+1)}$ satisfy the hypothesis of Proposition 1.2, then these filters satisfy following condition

$$|m_0^{(j)}(\eta)|^2 + |m_0^{(j)}(\eta+\pi)|^2 = 1.$$

Let W_j be the orthogonal complement of V_j in V_{j+1}, for fix $j \in \mathbb{Z}$. We have the distribution $\psi_{j,k}^{(j)}(x) := 2^{j/2}\psi^{(j)}(2^j x - k) \in V_{j+1}$ if there is a 2π-periodic function $m_1^{(j+1)} \in L^2_{\text{loc}}(\mathbb{R})$ such that

$$\hat{\psi}^{(j)}(2^{-j}\eta) = m_1^{(j+1)}(2^{-j-1}\eta)\hat{\varphi}^{(j+1)}(2^{-j-1}\eta).$$

The distributions $\psi_{j,k}^{(j)} \in W_j$ are orthonormal if

$$|m_1^{(j+1)}(\eta)|^2 + |m_1^{(j+1)}(\eta+\pi)|^2 = 1,$$

and they are orthogonal to V_j if

$$m_1^{(j+1)}(\eta)\overline{m_0^{(j+1)}(\eta)} + m_1^{(j+1)}(\eta+\pi)\overline{m_0^{(j+1)}(\eta+\pi)} = 0.$$

It follows that we can define $\psi^{(j)}$ by the expression

$$\psi^{(j)}(2\eta) = e^{-i\eta}\overline{m_0^{(j+1)}(\eta+\pi)}N(\eta)\hat{\varphi}^{(j+1)}(\eta),$$

where $N \in L^2_{\text{loc}}(\mathbb{R})$ is $\pi-$periodic and has modulus 1.

2. **Some splitting tricks**

We consider sequences $\{\alpha_{k,\epsilon}^{(j)} : k \in \mathbb{Z}\}$, $\epsilon \in \{0,1\}$ in $l^2(\mathbb{Z})$, to define the functions $f_\epsilon \in W_2^1(\mathbb{R})$, by

$$f_\epsilon(x) = 2^{1/2}\sum_{k\in\mathbb{Z}}\alpha_{k,\epsilon}^{(j)}\varphi(2x-k), \tag{2}$$

where $\{\varphi_k(\cdot) := \varphi(\cdot - k) : k \in \mathbb{Z}\}$ are orthonormal basis in $W_2^1(\mathbb{R})$. By taking Fourier transform both side (2), we get

$$\hat{f}_\epsilon(\eta) = m_\epsilon^{(j)}(2^{-1}\eta)\hat{\varphi}(2^{-1}\eta),$$

where

$$m_\epsilon^{(j)}(\eta) = \sum_{k\in\mathbb{Z}}2^{-1/2}\alpha_{k,\epsilon}^{(j)}e^{-i\eta.k}, \quad \epsilon \in \{0,1\}.$$

These function are 2π-periodic and are in $L^2(\mathbb{T})$; $\mathbb{T} = [-\pi,\pi]$, since the sequence $\{\alpha_{k,\epsilon}^{(j)}\}, \epsilon \in \{0,1\}$ are in $l^2(\mathbb{Z})$. Next we define the matrix

$$M^{(j)}(\eta) = \begin{pmatrix} m_0^{(j)}(\eta) & m_0^{(j)}(\eta+\pi) \\ m_1^{(j)}(\eta) & m_1^{(j)}(\eta+\pi) \end{pmatrix}, \ \eta \in \mathbb{R}.$$

Lemma 2.1. *Let $\{\varphi_k : k \in \mathbb{Z}\}$ be an orthonormal system in $W_2^1(\mathbb{R})$. Also f_ϵ defined by (2). Then $\{f_{\epsilon,k}(x) = f_\epsilon(x-k),\ 0 \le \epsilon \le 1,\ k \in \mathbb{Z}\}$ is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}} m_{\epsilon'}^{(j)}(\eta+k\pi)m_{\epsilon''}^{(j)}(\eta+k\pi) = \delta_{\epsilon',\epsilon''},\ 0 \le \epsilon',\epsilon'' \le 1. \tag{3}$$

Moreover, $\{f_{\epsilon,k}(x) = f_\epsilon(x-k),\ 0 \le \epsilon \le 1,\ k \in \mathbb{Z}\}$ is an orthonormal basis.

Proof. For $0 \le \epsilon',\epsilon'' \le 1,\ k \in \mathbb{Z}$ we have

$$\begin{aligned}
\langle f_{\epsilon',k}, f_{\epsilon'',k}\rangle_W &= \langle f_{\epsilon'}(\cdot), f_{\epsilon''}(\cdot - k)\rangle_W \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)\hat{f}_{\epsilon',0}(\eta)\overline{\hat{f}_{\epsilon'',k}(\eta)}d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)\hat{f}_{\epsilon'}(\eta)\overline{\hat{f}_{\epsilon''}(\eta)}e^{ik.\eta}d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)m_{\epsilon'}^{(j)}(2^{-1}\eta)\overline{m_{\epsilon''}^{(j)}(2^{-1}\eta)}|\hat{\varphi}(2^{-1}\eta)|^2e^{ik.\eta}d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\sum_{k'\in\mathbb{Z}}(1-\beta+\beta|\eta+2k'\pi|^2)m_{\epsilon'}^{(j)}(2^{-1}(\eta+2k'\pi)) \\
&\qquad \times \overline{m_{\epsilon''}^{(j)}(2^{-1}(\eta+2k'\pi))}|\hat{\varphi}(2^{-1}(\eta+2k'\pi))|^2e^{ik.\eta}d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\sum_{k'\in\mathbb{Z}}\sum_{\mu\in2\mathbb{Z}+1}(1-\beta+\beta|\eta+2\mu\pi+2^2k'\pi|^2)m_{\epsilon'}^{(j)}(2^{-1}\eta+\mu\pi+2k'\pi)
\end{aligned}$$

$$\times \overline{m_{\epsilon''}^{(j)}(2^{-1}\eta + \mu\pi + 2k'\pi)}|\hat{\varphi}(2^{-1}(\eta + \mu\pi + 2k'\pi))|^2 e^{ik.\eta} d\eta$$
$$= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{\mu\in 2\mathbb{Z}+1} m_{\epsilon'}^{(j)}(2^{-1}\eta + \mu\pi)\overline{m_{\epsilon''}^{(j)}(2^{-1}\eta + \mu\pi)} e^{ik.\eta} d\eta.$$

Therefore

$$\begin{aligned}
&\langle f_{\epsilon',k}, f_{\epsilon'',k}\rangle_W = \delta_{\epsilon',\epsilon''} \\
&\Leftrightarrow \sum_{\mu\in 2\mathbb{Z}+1} m_{\epsilon'}^{(j)}(2^{-1}\eta + \mu\pi)\overline{m_{\epsilon''}^{(j)}(2^{-1}\eta + \mu\pi)} = \delta_{\epsilon',\epsilon''} \qquad \text{for } a.e.\ \eta\in\mathbb{R}, \\
&\Leftrightarrow \sum_{\mu\in 2\mathbb{Z}+1} m_{\epsilon'}^{(j)}(\eta + \mu\pi)\overline{m_{\epsilon''}^{(j)}(\eta + \mu\pi)} = \delta_{\epsilon',\epsilon''} \qquad \text{for } a.e.\ \eta\in\mathbb{R}.
\end{aligned}$$

We have proved the first part of lemma.

Now assume that $\{f_{\epsilon,k}(x) = f_\epsilon(x-k),\ 0\le\epsilon\le 1,\ k\in\mathbb{Z}\}$ is an orthonormal system. We want to show that this is an orthonormal basis of $W_2^1(\mathbb{R})$. Let $f\in W_2^1(\mathbb{R})$ so there exist $\{\gamma_{p,\epsilon'}^{(j)} :\ p\in\mathbb{Z},\ 0\le\epsilon'\le 1\}\in l^2(\mathbb{Z})$ such that

$$f(x) = 2^{1/2}\sum_{k\in\mathbb{Z}} \gamma_{p,\epsilon'}^{(j)}\varphi(2x-p).$$

Assume that $f\perp f_{\epsilon,k}$ for all ϵ, k.

Claim $f=0$, for all ϵ, k such that $0\le\epsilon\le 1,\ k\in\mathbb{Z}$, we have

$$\begin{aligned}
0 &= \langle f_{\epsilon,k}, f\rangle_W \\
&= \langle f_{\epsilon,k}, 2^{1/2}\sum_{k\in\mathbb{Z}}\gamma_{p,\epsilon'}^{(j)}\varphi(2x-p)\rangle_W \\
&= \frac{2^{-1/2}}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2) m_\epsilon^{(j)}(2^{-1}\eta)|\hat{\varphi}^{(j)}(2^{-1}\eta)|^2 e^{-i\langle k,\eta\rangle}\sum_{p\in\mathbb{Z}}\overline{\gamma_{p,\epsilon'}^{(j)}} e^{ip.2^{-1}\eta} d\eta \\
&= \frac{2^{1/2}}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|2\eta|^2) m_\epsilon^{(j)}(\eta)|\hat{\varphi}^{(j)}(\eta)|^2 e^{-i\langle k,2\eta\rangle}\sum_{p\in\mathbb{Z}}\overline{\gamma_{p,\epsilon'}^{(j)}} e^{ip.\eta} d\eta \\
&= \frac{2^{1/2}}{2\pi}\int_{\mathbb{T}}\sum_{k'\in\mathbb{Z}}(1-\beta+\beta|2(\eta+2k'\pi)|^2) m_\epsilon^{(j)}(\eta+2k'\pi) \\
&\quad\times|\hat{\varphi}^{(j)}(\eta+2k'\pi)|^2 e^{-i\langle k,2\eta\rangle}\sum_{p\in\mathbb{Z}}\overline{\gamma_{p,\epsilon'}^{(j)}} e^{ip.\eta} d\eta \\
&= \frac{2^{1/2}}{2\pi}\int_{\mathbb{T}}\sum_{p\in\mathbb{Z}} m_\epsilon^{(j)}(\eta)\overline{\gamma_{p,\epsilon'}^{(j)}} e^{-i\langle k,2\eta\rangle} e^{ip.\eta} \\
&\quad\times\left\{\sum_{k'\in\mathbb{Z}}(1-\beta+\beta|2(\eta+2k'\pi)|^2)|\hat{\varphi}^{(j)}(\eta+2k'\pi)|^2\right\} d\eta \\
&= \frac{2^{1/2}}{2\pi}\int_{\mathbb{T}}\sum_{p\in\mathbb{Z}} m_\epsilon^{(j)}(\eta)\overline{\gamma_{p,\epsilon'}^{(j)}} e^{-i\langle k,2\eta\rangle} e^{ip.\eta} d\eta
\end{aligned}$$

$$= \frac{2^{1/2}}{2\pi}\int_{2^{-1}\mathbb{T}} \sum_{\mu\in 2\mathbb{Z}+1}\sum_{p\in\mathbb{Z}} m_{\epsilon}^{(j)}(\eta+\mu\pi)\overline{\gamma_{p,\epsilon'}^{(j)}}e^{-i\langle k,2(\eta+\mu\pi)\rangle}e^{ip.(\eta+\mu\pi)}d\eta$$

$$= \frac{2^{1/2}}{2\pi}\int_{2^{-1}\mathbb{T}} \left\{\sum_{\mu\in 2\mathbb{Z}+1}\sum_{p\in\mathbb{Z}} m_{\epsilon}^{(j)}(\eta+\mu\pi)\overline{\gamma_{p,\epsilon'}^{(j)}}e^{i\langle p,(\eta+\mu\pi)\rangle}\right\}e^{-ik.2\eta}d\eta.$$

Since $\left\{\frac{2^{1/2}}{2\pi}e^{ik.2\cdot}\right\}:\ k\in\mathbb{Z}$ is an orthonormal basis for $L^2(2^{-1}\mathbb{T})$, then the above equation is given by

$$\sum_{\mu\in 2\mathbb{Z}+1}\sum_{p\in\mathbb{Z}} m_{\epsilon}^{(j)}(\eta+\mu\pi)\overline{\gamma_{p,\epsilon'}^{(j)}}e^{ip.(\eta+\mu\pi)} = 0,\ a.e.$$

For $0\le\epsilon'\le 1$, define

$$B_{\epsilon'}^{(j)}(\eta) = \sum_{p\in\mathbb{Z}}\gamma_{p,\epsilon'}^{(j)}e^{ip.\eta}.$$

So we have

$$\sum_{\mu\in 2\mathbb{Z}+1}\overline{B_{\epsilon'}^{(j)}(\eta+\mu\pi)}m_{\epsilon}^{(j)}(\eta+\mu\pi) = 0. \tag{4}$$

Equation (4) says that the vector

$$\left\{B_{\epsilon'}^{(j)}(\eta+\mu\pi):\ 0\le\epsilon'\le 1,\ \mu\in 2\mathbb{Z}+1\right\}$$

is zero, because these are orthogonal to each member of

$$\left\{m_{\epsilon}^{(j)}(\eta+\mu\pi):\ 0\le\epsilon'\le 1,\ \mu\in 2\mathbb{Z}+1\right\}.$$

Therefore, $f=0$. □

With the help of above splitting lemma we can define wavelet packets in $W_2^1(\mathbb{R})$.

Let $\mathfrak{p}_0^{(j)} = \varphi^{(j)}$, $\mathfrak{p}_1^{(j)} = \psi^{(j)}$. We define the basis Wavelet packets associated with scaling function $\varphi^{(j)}$ recursively as follows:

$$\mathfrak{p}_{2n}^{(j)}(t) = 2\sum_{k\in\mathbb{Z}}\alpha_{k,0}^{(j+1)}\mathfrak{p}_n^{(j+1)}(2t-k),$$
$$\mathfrak{p}_{2n+1}^{(j)}(t) = 2\sum_{k\in\mathbb{Z}}\alpha_{k,1}^{(j+1)}\mathfrak{p}_n^{(j+1)}(2t-k). \tag{5}$$

The Fourier transform equivalence of the of the scaling relation for the Wavelet packets is given by

$$\hat{\mathfrak{p}}_{2n}^{(j)}(\eta) = m_0^{(j+1)}(\eta/2)\hat{\mathfrak{p}}_n^{(j+1)}(\eta/2),$$
$$\hat{\mathfrak{p}}_{2n+1}^{(j)}(\eta) = m_1^{(j+1)}(\eta/2)\hat{\mathfrak{p}}_n^{(j+1)}(\eta/2).$$

We consider

$$m_0^{(j)}(\eta) = \sum_{k\in\mathbb{Z}}\alpha_{k,0}^{(j)}e^{i\eta.k},$$
$$m_1^{(j)}(\eta) = \sum_{k\in\mathbb{Z}}\alpha_{k,1}^{(j)}e^{i\eta.k}.$$

These function are 2π-periodic and are in $L^2(\mathbb{T})$, since the sequence $\{\alpha_{k,\epsilon}^{(j)}\},\epsilon\in\{0,1\}$ are in $l^2(\mathbb{Z})$.

Proposition 2.2. *Let* n *be any non-negative integer. Then the Fourier transform of basis wavelet packet defined by* (5) *is given by*

$$\hat{\mathfrak{p}}_n^{(j)}(\eta) = \prod_{q=1}^{\infty} m_{\epsilon_q}^{(j+q)}(2^{-q}\eta)(1-\beta+\beta|2^j\eta|)^{-1/2}.$$

Proof. The proof is similar to proof of ref. [[16] Proposition 2.2]. □

3. WAVELET PACKET FUNCTIONS GENERATED BY MRA IN THE SOBOLEV SPACE $W_2^1(\mathbb{R})$

Suppose that $\varphi^{(j)}(t)$ generates an orthonormal multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ with associated wavelet function $\psi^{(j)}(t)$ in $W_2^1(\mathbb{R})$. The wavelet packet functions are defined by $\varphi^{(j)} = \mathfrak{p}_0^{(j)}$, $\mathfrak{p}_1^{(j)} = \psi^{(j)}$ and for $n = 1, 2, 3, \ldots$ we define

$$\mathfrak{p}_{2n}^{(j)}(t) = 2^{1/2}\sum_{k\in\mathbb{Z}} \alpha_{k,0}^{(j+1)}\mathfrak{p}_n^{(j+1)}(2t-k),$$
$$\mathfrak{p}_{2n+1}^{(j)}(t) = 2^{1/2}\sum_{k\in\mathbb{Z}} \alpha_{k,1}^{(j+1)}\mathfrak{p}_n^{(j+1)}(2t-k).$$

Let $\mathfrak{p}_n^{(j)}(t)$ be a wavelet function associated with scaling function $\varphi^{(j)}(t)$. Here n is a non-negative integer. For integers j, we define

$$\mathfrak{p}_{j,k,n}^{(j)}(t) = 2^{j/2}\mathfrak{p}_n^{(j)}(2^jt-k), \quad k\in\mathbb{Z}. \tag{6}$$

Theorem 3.1. *If* $\mathfrak{p}_{j,k,n}^{(j+1)}(t) \in W_2^1(\mathbb{R})$ *and* j, k *are integers, then the distributions* $\{2^{(j+1)/2}\mathfrak{p}_{[n/2]}^{(j+1)}(2^{(j+1)}t-k)\}, k\in\mathbb{Z}$, *are orthonormal in* $W_2^1(\mathbb{R})$ *iff*

$$\sum_{r\in\mathbb{Z}} |\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\eta+2\pi r)|^2(1-\beta+2^{2(j+1)}\beta|(\eta+2\pi r)|^2) = 1.$$

Proof. Since $\mathfrak{p}_{j,k,n}^{(j+1)}(t)\in W_2^1(\mathbb{R})$, the series

$$M(\eta) = \sum_{r\in\mathbb{Z}} |\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\eta+2\pi r)|^2(1-\beta+2^{2(j+1)}\beta|(\eta+2\pi r)|^2)$$

converges almost everywhere, belongs to $\mathbb{L}_{loc}^1(\mathbb{R})$ and 2π-periodic. Moreover, for every $l\in\mathbb{Z}$, we have

$$\begin{aligned}
&\int_{\mathbb{T}} M(\eta)e^{-i\eta(k-l)}d\eta \\
&\quad = \sum_{r\in\mathbb{Z}}\int_{\mathbb{T}} |\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\eta+2\pi r)|^2(1-\beta+2^{2(j+1)}\beta|(\eta+2\pi r)|^2)e^{-i\eta(k-l)}d\eta \\
&\quad = \int_{\mathbb{R}} |\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\nu)|^2(1-\beta+2^{2(j+1)}\beta|(\nu)|^2)e^{-i\nu(k-l)}d\nu \\
&\quad = 2^{-(j+1)}\int_{\mathbb{R}} |\hat{\mathfrak{p}}_{[n/2^d]}^{(j+1)}(2^{-j-1}u)|^2(1-\beta+\beta|u|^2)e^{-i2^{-j-1}u(k-l)}du \\
&\quad = \int_{\mathbb{R}} (1-\beta+\beta|u|^2)e^{-i2^{-j-1}uk}2^{-(j+1)/2}\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(2^{-j-1}u) \\
&\qquad\quad \times \overline{e^{-i2^{-j-1}ul}2^{-(j+1)/2}\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}}(2^{-j-1}u)du
\end{aligned}$$

$$= \int_{\mathbb{R}} (1-\beta+\beta|u|^2)2^{(j+1)/2}\mathcal{F}\left[\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(2^{j+1}t-k)\right](u)\mathcal{F}\left[\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(2^{j+1}t-l)\right](u)du$$
$$= 2\pi\langle 2^{(j+1)/2}\mathfrak{p}^{(j+1)}_{[n/2]}(2^{j+1}t-k), 2^{(j+1)/2}\mathfrak{p}^{(j+1)}_{[n/2]}(2^{j+1}t-l)\rangle_W.$$

Since $\{1/2\pi e^{-i\eta.(k-l)}\} : k,l \in \mathbb{Z}$ is an orthonormal basis for $L^2(\mathbb{T})$, then

$$\frac{1}{2\pi}\int_{\mathbb{T}} M(\eta)e^{-i\eta(k-l)}d\eta = \langle 2^{(j+1)/2}\mathfrak{p}^{(j+1)}_{[n/2]}(2^{j+1}t-k), 2^{(j+1)/2}\mathfrak{p}^{(j+1)}_{[n/2]}(2^{j+1}t-l)\rangle_W = \delta_{k,l},$$

if $M(\eta) = 1$. □

Theorem 3.2. *Let j and n be the integers with $n \geq 0$ and $k,l \in \mathbb{Z}$. Then*

$$\langle \mathfrak{p}^{(j)}_{j,k,n}(t), \mathfrak{p}^{(j)}_{j,l,n}(t)\rangle_W = \delta_{k,l}.$$

Proof.

$$\begin{aligned}
&\langle \mathfrak{p}^{(j)}_{j,k,n}(t), \mathfrak{p}^{(j)}_{j,l,n}(t)\rangle_W \\
&= 2^j\langle \mathfrak{p}^{(j)}_{n}(2^jt-k), \mathfrak{p}^{(j)}_{n}(2^jt-l)\rangle_W \\
&= \frac{2^{-j}}{2\pi}\int_{\mathbb{R}} |\hat{\mathfrak{p}}^{(j)}_n(2^{-j}\eta)|^2 e^{-i\eta 2^{-j}(k-l)}(1-\beta+\beta|\eta|^2)d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\mathfrak{p}}^{(j)}_n(u)|^2 e^{-iu(k-l)}(1-\beta+\beta 2^{2j}|u|^2)du \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |m^{(j+1)}_{\epsilon_1}(u/2)|^2|\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(u/2)|^2 e^{-iu(k-l)}(1-\beta+\beta 2^{2j}|u|^2)du \\
&= \frac{2}{2\pi}\int_{\mathbb{R}} |m^{(j+1)}_{\epsilon_1}(\nu)|^2|\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(\nu)|^2 e^{-i2\nu(k-l)}(1-\beta+\beta 2^{2(j+1)}|\nu|^2)d\nu \\
&= \frac{1}{\pi}\int_{\mathbb{R}} |m^{(j+1)}_{\epsilon_1}(\nu)|^2|\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(\nu)|^2 e^{-i2\nu(k-l)}(1-\beta+\beta 2^{2(j+1)}|\nu|^2)d\nu \\
&= \frac{1}{\pi}\int_{\mathbb{T}} |m^{(j+1)}_{\epsilon_1}(\nu)|^2\sum_{r\in\mathbb{Z}}|\hat{\mathfrak{p}}^{(j+1)}_{[n/2]}(\nu+2\pi r)|^2(1-\beta+\beta 2^{2(j+1)}|\nu+2\pi r|^2)e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi}\int_{\mathbb{T}} |m^{(j+1)}_{\epsilon_1}(\nu)|^2 e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi}\int_{[0,\pi)} \left(|m^{(j+1)}_{\epsilon_1}(\nu)|^2 + |m^{(j+1)}_{\epsilon_1}(\nu+\pi)|^2\right) e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi}\int_{[0,\pi)} e^{-i2\nu(k-l)}d\nu = \delta_{k,l}.
\end{aligned}$$

Thus,

$$\langle \mathfrak{p}^{(j)}_{j,k,n}(t), \mathfrak{p}^{(j)}_{j,l,n}(t)\rangle_W = \delta_{k,l}.$$

□

Theorem 3.3. *For any $n \in \mathbb{N}$ we have*

$$\langle \mathfrak{p}^{(j)}_{j,k,2n}(t), \mathfrak{p}^{(j)}_{j,l,2n+1}(t)\rangle_W = 0.$$

Proof. By using change of variable technique and (6), we have

$$\langle \mathfrak{p}^{(j)}_{j,k,2n}(t), \mathfrak{p}^{(j)}_{j,l,2n+1}(t)\rangle_W = 2^j\langle \mathfrak{p}^{(j)}_{2n}(2^jt-k), \mathfrak{p}^{(j)}_{2n+1}(2^jt-k)\rangle_W$$

$$
\begin{aligned}
&= \frac{2^{-j)}}{2\pi} \int_{\mathbb{R}} (1-\beta+\beta|\eta|^2)\hat{\mathfrak{p}}_{2n}^{(j)}(2^{-j}\eta)\overline{\hat{\mathfrak{p}}_{2n+1}^{(j)}(2^{-j}\eta)}e^{-i\eta 2^{-j}(k-l)}d\eta \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} (1-\beta+\beta 2^{2j}|u|^2)\hat{\mathfrak{p}}_{2n}^{(j)}(u)\overline{\hat{\mathfrak{p}}_{2^d n+1}^{(j)}(u)}e^{-iu(k-l)}du \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} m_{\epsilon_i}^{(j+1)}(u/2)\overline{m_{\epsilon_m}^{(j+1)}(u/2)}(1-\beta+\beta 2^{2j}|u|^2)^s\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(u/2)\overline{\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(u/2)}e^{-iu(k-l)}du \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} m_0^{(j+1)}(\nu)\overline{m_1^{(j+1)}(\nu)}(1-\beta+\beta 2^{2(j+1)}|\nu|^2)|\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\nu)|^2 e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi} \int_{\mathbb{T}} m_0^{(j+1)}(\nu)\overline{m_1^{(j+1)}(\nu)}\sum_{r\in\mathbb{Z}}|\hat{\mathfrak{p}}_{[n/2]}^{(j+1)}(\nu+2\pi r)|^2(1-\beta+\beta 2^{2(j+1)}|\nu+2\pi r|^2)e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi} \int_{\mathbb{T}} m_0^{(j+1)}(\nu)\overline{m_1^{(j+1)}(\nu)}e^{-i2\nu(k-l)}d\nu \\
&= \frac{1}{\pi} \int_{\mathbb{T}_1} \left(m_0^{(j+1)}(\nu)\overline{m_1^{(j+1)}(\nu)} + m_0^{(j+1)}(\nu+\pi)\overline{m_1^{(j+1)}(\nu+\pi)}\right) e^{-i2\nu(k-l)}d\nu \\
&= 0.
\end{aligned}
$$

□

With the help of orthogonal system in $L^2(\mathbb{R})$ of wavelet packets, we can achieve orthogonality of wavelet packets in $W_2^1(\mathbb{R})$ at j^{th} in an alternative form by the theory of convolution of Fourier transform.

Theorem 3.4. *Let* $\sigma = \mathcal{F}^{-1}\left[(1-\beta+\beta|\eta|^2)^{-1/2}\right]$ *and* $\mathfrak{p}_{j,k,n}^{(j)}(t) = 2^{j/2}\mathfrak{p}_n^{(j)}(2^j t - k)$. *Then*

$$2\pi\langle \sigma * \mathfrak{p}_{j,k,n}^{(j)}, \sigma * \mathfrak{p}_{j,l,n}^{(j)}\rangle_W = \delta_{k,l},$$

if

$$\langle \mathfrak{p}_{j,k,n}^{(j)}, \mathfrak{p}_{j,l,n}^{(j)}\rangle_2 = \delta_{k,l},$$

where $\langle\cdot\rangle_2$ *is inner product in* $L^2(\mathbb{R})$.

Proof. Using the convolution theorem for Fourier transform, we have

$$
\begin{aligned}
&\langle \sigma * \mathfrak{p}_{j,k,n}^{(j)}, \sigma * \mathfrak{p}_{j,l,n}^{(j)}\rangle_W \\
&\qquad = \frac{1}{2\pi}\int_{\mathbb{R}}(1-\beta+\beta|\eta|^2)\mathcal{F}(\sigma * \mathfrak{p}_{j,k,n}^{(j)})\overline{\mathcal{F}(\sigma * \mathfrak{p}_{j,l,n}^{(j)})}d\eta.
\end{aligned}
$$

Let us choose $\mathcal{F}(\sigma)(\eta) = (1-\beta+\beta|\eta|^2)^{-1/2}$, then the above expression becomes

$$\langle \sigma * \mathfrak{p}_{j,k,n}^{(j)}, \sigma * \mathfrak{p}_{j,l,n}^{(j)}\rangle_W = \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}(\mathfrak{p}_{j,k,n}^{(j)})\overline{\mathcal{F}(\mathfrak{p}_{j,l,n}^{(j)})}d\eta.$$

By Plancherel's equation, we get

$$
\begin{aligned}
2\pi\langle \sigma * \mathfrak{p}_{j,k,n}^{(j)}, \sigma * \mathfrak{p}_{j,l,n}^{(j)}\rangle_W &= \int_{\mathbb{R}} \mathfrak{p}_{j,k,n}^{(j)}\overline{\mathfrak{p}_{j,l,n}^{(j)}}d\eta \\
&= \langle \mathfrak{p}_{j,k,n}^{(j)}, \mathfrak{p}_{j,l,n}^{(j)}\rangle_2 \\
&= \delta_{k,l}.
\end{aligned}
$$

□

Now we give an example of wavelet packet by considering the usual spline functions of order m

$$g = \chi_{[0,1]} * \cdots * \chi_{[0,1]},$$

then the Fourier of g is

$$\hat{g}(\eta) = e^{-i(m+1)\eta/2}\left(\frac{\sin(\eta/2)}{\eta/2}\right)^{m+1}.$$

For every $j \in \mathbb{Z}$, let us define

$$\hat{\mathfrak{p}}_0^{(j)}(\eta) = \hat{\varphi}^{(j)}(\eta) := \frac{\hat{g}(\eta)}{\sqrt{\omega_g^{(j)}(\eta)}},$$

where

$$\omega_g^{(j)}(\eta) = \sum_{k=-\infty}^{\infty}\left(1-\beta+2^{2j}\beta|\eta+2k\pi|^2\right)|\hat{g}(\eta+2k\pi)|^2.$$

It is easy to see that $\varphi^{(j)} \in W_2^1(\mathbb{R})$. By using scaling relation

$$\hat{\varphi}^{(j)}(2\eta) = m_0^{(j+1)}(\eta)\hat{\varphi}^{(j+1)}(\eta)$$

we get

$$m_0^{(j+1)}(\eta) = e^{-i(m+1)\eta/2}\cos^{m+1}\left(\frac{\eta}{2}\right)\sqrt{\frac{\omega_g^{(j+1)}(\eta)}{\omega_g^{(j)}(2\eta)}}. \tag{7}$$

Let $\mathfrak{p}_1^{(j)} = \psi^{(j)}$ be wavelet function corresponding to scaling function $\varphi^{(j)}$, then the distributions $2^{j/2}\psi^{(j)}(2^j x - k),\ j,k \in \mathbb{Z}$, where

$$\hat{\mathfrak{p}}_1^{(j)}(2\eta) = \hat{\psi}^{(j)}(2\eta) = e^{-i\eta}\overline{m_0^{(j+1)}(\eta+\pi)}\hat{\varphi}^{(j+1)}(\eta),$$

with

$$\begin{aligned} m_1^{(j)}(\eta) &= e^{-i\eta}\overline{m_0^{(j+1)}(\eta+\pi)} \\ &= -e^{-i\eta}e^{-i(m+1)(\eta+\pi)/2}\sin^{m+1}\left(\frac{\eta}{2}\right)\sqrt{\frac{\omega_g^{(j+1)}(\eta+\pi)}{\omega_g^{(j)}(2\eta)}} \end{aligned} \tag{8}$$

are orthonormal basis for $W_2^1(\mathbb{R})$. Define functions $\mathfrak{p}_n^{(j)},\ n \geq 0$, as follows:

$$\hat{\mathfrak{p}}_{2n}^{(j)}(2\eta) = m_0^{(j+1)}(\eta)\hat{\mathfrak{p}}_n^{(j+1)}(\eta),$$

and

$$\hat{\mathfrak{p}}_{2n+1}^{(j)}(2\eta) = m_1^{(j+1)}(\eta)\hat{\mathfrak{p}}_n^{(j+1)}(\eta),$$

where $m_0^{(j+1)},\ m_1^{(j+1)}$ as given in (7) and (8).

From theorem 3.4, we define wavelet packet in convolution form as

$$(\sigma * \mathfrak{p}_{j,l,n}^{(j)})(t) = \int_{\mathbb{R}}\sigma(x-t)2^{(j+1)/2}\sum_{k\in\mathbb{Z}}\alpha_{k,\epsilon}^{(j+1)}\mathfrak{p}_{[n/2]}^{(j+1)}(2^{(j+1)}t-k)dx, \tag{9}$$

where (j) denotes the jth level and if n is even then $\epsilon = 0$ or if n is odd then $\epsilon = 1$. In figure 1,2, we consider Haar scaling function $\varphi(x) = \chi_{[0,1]}$ in (9).

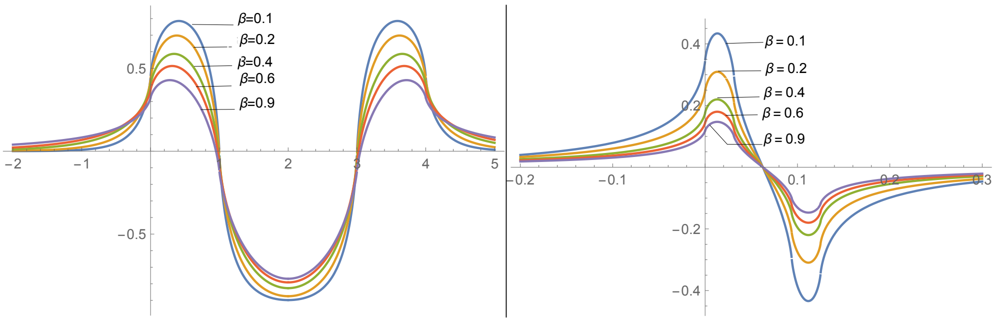

FIGURE 1. Wavelet packets in $W_2^1(\mathbb{R})$ for a) $n = 5,\ j = -3$ and b) $n = 4,\ j = 2$.

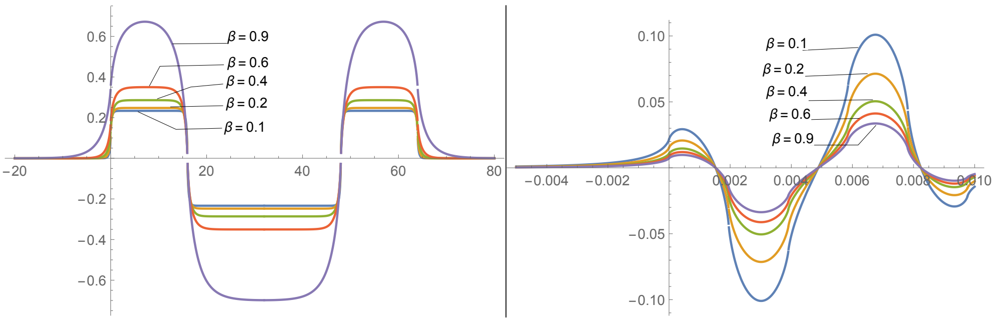

FIGURE 2. Wavelet packets in $W_2^1(\mathbb{R})$ for c) $n = 6,\ j = -7$ and b) $n = 4,\ j = 2$.

Acknowledgements

The authors would like to thank the anonymous referees for their insightful comments and suggestions.

Funding

The work of corresponding author is supported by the Senior Research Fellowship of University Grants Commission (UGC), India(Grant no. SRF/AA/139/F-236/2013-14).

References

[1] Bastin, F. and Boigelot, C., (1998), Biorthogonal wavelets in $H^m(\mathbb{R})$, J. Fourier Anal. Appl., 4, pp. 749–768.

[2] Bastin, F. and Laubin, B., (1997), Compactly supported wavelets in Sobolev spaces of integer order, Appl. Comput. Harmon. Anal., 4, pp. 51–57.

[3] Bastin, F. and Laubin, B., (1997), Regular compactly supported wavelets in Sobolev spaces, Duke Math. J., 87, pp. 481–508.

[4] Buzykanov, S., (2012), Enhancement of poor resolution text images in the weighted Sobolev space, in Proceedings of the 19th International Conference on Systems, Signals and Image Processing (IWSSIP 12), Vienna, Austria, April, pp. 536–539.

[5] Cohen, A. and Daubechies, I., (1993), On the instability of arbitrary biorthogonal wavelet packets, SIAM J. Math. Anal., 24, pp. 1340–1354.

[6] Coifman, R. and Meyer, Y., Orthogonal wave packet bases, preprint.

[7] Coifman, R., Meyer, Y., Quake, S. and Wickerhauser, M. V., (1989), Signal Processing and compression with wave packets, In: proceedings of the conference on wavelets, Marseilles: Spring.

[8] Chui, C. R. and Li, C., (1993), Non-orthogonal wavelet packets, SIAM J. Math. Anal., 24, pp. 712–738.

[9] Dayong, L. and Dengfeng, L., (2011), A characterization of orthonormal wavelet families in Sobolev spaces, Acta Math. Sci. Ser. B Engl. Ed., 31, pp. 1475–1488.
[10] Ehler, M., (2010), The multiresolution structure of pairs of dual wavelet frames for a pair of Sobolev spaces, Jaen J. Approx, 2(2), pp. 193–214.
[11] Han, B. and Shen, Z., (2009), Daul wavelet frames and rise bases in Sobolev spaces, Constr. Approx., 29, pp. 369–406.
[12] Hernández, E. and Weiss, G., (1996), A First Course on Wavelets, Boca Raton (FL): CRC Press.
[13] Chen, D. R., (2000), On the splitting trick and wavelet frame packets, SIAM J. Math. Anal., 31, pp. 729–739.
[14] Long, R. and Chen, W., (1997), Wavelet basis packets and wavelet frame packets, J. Fourier Anal. Appl., 3, pp. 239–256.
[15] Pathak, R. S., (2005), Wavelets in a generalized Sobolev space, Comput. Math. Appl., 49, 823–839.
[16] Pathak, R. S. and Kumar, M., (2015), Wavelet packet in Sobolev space, Applicable Anal., 94, pp. 1068–1084.
[17] Shen, Z., (1995), Nontensor product wavelet packets in $L^2(\mathbb{R}^s)$, SIAM J. Math. Anal., 26, pp. 1061–1074.
[18] Walter, G. G., (1992), Discrete wavelets, SIAM J. Math. Anal., 23, pp. 1004–1014.
[19] Walter, G. G., (1994), Wavelets and other orthogonal systems with applications, Boca Raton (FL): CRC Press.
[20] Wickerhauser, M. V., (1992), Acoustic signal compression with wavelet packets, in Wavelets: A Tutorial in Theory and Applications , Chui, C. K. ed., Academic Press, Boston, MA., pp. 679–700.

Raj Kumar got his M.Sc. degree in Mathematics from IIT Delhi, India. He received his Ph.D. degree in mathematics from University of Delhi, India. Currently, he is a senior assistant professor in the Department of Mathematics at the Kirori Mal College, University of Delhi, India. He has given talks at national/international workshops/conferences. His area of research is Theory of Frames for Hilbert and Banach spaces and Wavelets.

Manish Chauhan graduated from JNVU, Jodhpur, Rajasthan, in 2007. He got his M.Sc. degree in mathematics from JNVU, Jodhpur, Rajasthan, in 2009. Currently, he is pursuing Ph.D. from Department of Mathematics, University of Delhi, India. His area of research are Theory of Frames, wavelets, signal processing.