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COMPARISON OF INTEGRO QUADRATIC AND QUARTIC SPLINE INTERPOLATION

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ABSTRACT. In this paper quadratic and quartic B-splines were used for reconstruction of an approximating function, where the integral values of successive subintervals were used instead of function values at the knots. After introducing integro quadratic and quartic interpolation a comparison was done between them through presenting numerical examples. The interpolation errors for quadratic and quartic integro interpolation are studied. Numerical results illustrate the efficiency and effectiveness of the new interpolation methods.

Integral values, quadratic and quartic B-spline, integro interpolation, super convergence, error analysis

AMS Subject Classification: 83-02, 99A00

1. Introduction

Interpolation is one of the most important issues in applied mathematics and numerical computations [1, 2, 3]. Interpolations with spline functions, particularly, act differently based on their degrees [4]. In some cases, we deal with some conditions or phenomena, which contain only integral values of $y(x)$. The question that arises here is: by using the integral values, how can we reconstruct an estimating function? Let the interval $[a, b]$ is partitioned to n successive subintervals $[x_i, x_{i+1}]$ where $i = 0, 1, ..., n-1$ and $a = x_0 \leq x_1 \leq ... \leq x_n$. Suppose that the function values $y_i = y(x_i)$ are not given, but the integral values I_i of $y(x)$ on n successive subinterval $[x_i, x_{i+1}]$ is identified. Our aim is to determine an integro-interpolating spline function $s(x)$ in a way that

$$\int_{x_i}^{x_{i+1}} s(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, \qquad i = 0, 1, ..., n-1. \tag{1}$$

Obviously, it would be a generalization for the classical interpolation problems. Various studies have been conducted in the field of integro spline interpolation. At first, Behforouz

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[5] introduced new method for construct integro cubic splines, where he used integral values instead of $y = y(x)$ values at the knots. Inferring the method was carried out by Hermit cubic polynomial interpolation formula along with three boundary conditions. After that, integro quinte spline method was developed by Behforouz [6], which was based on quintic Hermite–Birkhoff interpolation polynomial. In addition to integral values of function $y(x)$, these methods required several additional boundary conditions. Also, in this work, the derivatives approximation of $y(x)$ were studied. Zhanlav and Mijiddorj [7] discussed local integro cubic splines and their approximation properties in order to solve such problems and eliminate such barriers. In [8] a comparison between the convergence of integro quartic an sextic B-spline interpolation has been done. However, all integro interpolation methods use integral values given on successive subintervals, since uniform B-splines bases are used in all of them. Regarding quadratic B-spline, integro interpolation problem, the proposed method has some advantages compared to other methods. For example we can use this method to express integro interpolation problem, using $y(x)$ integral values in the desired subintervals. Moreover, computational cost is decreased due to lower degree of spline function and it basically needs to solve a system of $(n-1)$ linear equations with three-band coefficient matrix. Moreover, the proposed method possesses super convergence order in estimation of function values at the nodes. Meanwhile, regarding integro interpolation problem, using quartic B-spline, the new proposed method has also lots of advantages. As an example, in this method computational costs reaches its lowest volume and it basically needs to solve a system of $(n+4)$ linear equations with five-band coefficient matrix. Hence, it is easy to run. On the other hand, the proposed algorithm estimates numerical results with higher accuracy compared to other methods. This can estimate $y(x)$ with approximation error $O(h^6)$ at the knots. This shows that integro quartic spline has super convergence in estimation function values at the knots.

2. Integro quadratic spline interpolation

Assume that $[a, b]$ has been divided into n subintervals by equidistance points $x_i = a+ih$, where $\triangle_i = [x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ and $h = \frac{b-a}{n}$. The univariate quadratic spline space with respect to $\triangle = \{\triangle_i\}_{i=0}^{n-1}$ is defined as

$$S_2(I) = \left\{s(x) \in C^1(I)|s_i(x) \in P_2, i = 0, 1, ..., n-1\right\}, \tag{2}$$

where, $s_i(x)$ is a restriction for $s(x)$ in the interval $\triangle_i$ and P_2 is set of all univariate quadratic polynomials. It goes without saying that $S_2(I)$ is a linear space and its dimension is $n+2$. To use B-representation of quadratic spline $s(x)$ of class of $C^1(I)$, partition interval $[a, b]$ is expanded to the left and right sides by equidistance knots

$$x_{-2} < x_{-1} < x_0 = a, \;\; b = x_n < x_{n+1} < x_{n+2}. \tag{3}$$

Using basic results of B-splines [9, 10, 11], the explicit representation of uniform quadratic B-splines are defined as

$$B_i(x) = \frac{1}{2!h^2}\begin{cases} (x - x_{i-2})^2; & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-2})^2 - C_3^1(x - x_{i-1})^2; & x \in [x_{i-1}, x_i] \\ (x - x_{i+1})^2; & x \in [x_i, x_{i+1}] \\ 0; & else. \end{cases} \tag{4}$$

The values of $B_i(x)$ and $B_i'(x)$ in the knots have been presented in the table (1). The

TABLE 1. The values of $B_i^{(k)}$, $(k = 0, 1)$at the given knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	$else$
$B_i(x)$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$B_i'(x)$	0	$\frac{1}{h}$	$-\frac{1}{h}$	0	0

integral values of $B_i(x)$ on each subinterval are given by

$$\int_{x_{i-2}}^{x_{i-1}} B_i(x)dx = \int_{x_i}^{x_{i+1}} B_i(x)dx = \frac{1}{6}h,$$
$$\int_{x_{i-1}}^{x_i} B_i(x)dx = \frac{4}{6}h.$$

For constructing integro quadratic spline, suppose that I_i is integral values of $y(x)$ on n subintervals of $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$. Also, assume that $y_0 = y(x_0)$ and $y_n = y(x_n)$ are two boundary conditions. Our aim is to construct a quadratic spline of $s(x) \in S_2(I)$ in a way that

$$\int_{x_i}^{x_{i+1}} s(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, i = 0, 1, ..., n-1, \tag{5}$$

and

$$s_0 = y(x_0), s_n = y(x_n). \tag{6}$$

Each $s(x) \in S_2(I)$ can exclusively be expressed as follows

$$s(x) = \sum_{j=-2}^{n-1} c_j B_j(x). \tag{7}$$

Using integral values we get

$$\int_{x_i}^{x_{i+1}} s(x)dx = \int_{x_i}^{x_{i+1}} \sum_{j=-2}^{n-1} c_j B_j(x)dx = \sum_{j=-2}^{n-1} c_j \int_{x_i}^{x_{i+1}} B_j(x)dx = I_i, \tag{8}$$

then it implies the linear equations

$$\frac{h}{120}(c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}) = I_i. \tag{9}$$

Moreover, two boundary condition give two linear equations

$$\begin{cases} s(x_0) = y_0 \\ s(x_n) = y_n \end{cases} \Rightarrow \begin{cases} c_{-2} - c_{-1} = 2y_0 \\ c_{n-2} - c_{n-1} = 2y_n \end{cases} \tag{10}$$

So, (9) and (10) present a linear system with a three-band matrix coefficients of order $(n+2) \times (n+2)$. It can be written in matrix form

$$AC = D, \tag{11}$$

where

$$A = \begin{bmatrix} 1 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 1 \end{bmatrix} \tag{12}$$

and

$$C = (c_{-2}, c_{-1}, c_0, c_1, ..., c_n, c_{n+1})^t, \tag{13}$$

$$D = (2y_0, \frac{6}{h}I_0, ..., \frac{6}{h}I_{n-1}, 2y_n)^t. \tag{14}$$

Theorem 2.1. *The integro quadratic spline interpolation* (5) *and* (6) *can be solved uniquely.*

Proof. The determinant of coefficient matrix is nonzero with Gauss elimination. □

After solving the system (11), the quadratic spline of $s(x) = \sum_{j=-2}^{n-1} c_j B_j(x)$ is obtained, which is called an integro quadratic spline. The values of $s(x)$ at the knots are obtained as follows

$$s_j = s(x_j) = \sum_{i=-2}^{n+1} c_i B_i(x_j) = \frac{1}{2}(c_{j-2} + 5c_{j-1}). \tag{15}$$

3. Integro quartic spline interpolation

Suppose that $I = [a, b]$ is divided into n subintervals with equidistance and successive knots $x_i = a + ih$, where $\triangle_i = [x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ and $h = \frac{b-a}{n}$. Consider the univariate quartic spline space over the uniform partition which is defined as follows

$$S_4(I) = \{s(x) \in c^3(I) | s_i(x) \in P_4, i = 0, 1, ..., n-1\}, \tag{16}$$

where $s_i(x)$ denotes the restriction of $s(x)$ on $\triangle_i$ and P_4 denotes the set of univariate quartic polynomials. It is obvious that $S_4(I)$ is a linear space with dimension $n+4$ and its members are called quartic splines. Quartic spline $s(x)$ is basically, a piecewise quartic polynomial such that $s(x)$, $s'(x)$, $s''(x)$ and $s'''(x)$ are continuous on $[a, b]$. The interval $I = [a, b]$ is extended to $\tilde{I} = [a - 4h, b + 4h]$ through equidistance knots $x_i = a + ih, (i = -4, -3, ..., n+4)$. Quartic B-spline of $B_i(x), (i = -2, -1, ..., n+1)$ is defined as follows [12]

$$B_i(x) = \frac{1}{24h^4} \begin{cases} (x - x_{i-2})^4, & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-2})^4 - {C_4}^1 (x - x_{i-1})^4, & x \in [x_{i-1}, x_i] \\ (x - x_{i-2})^4 - {C_4}^1 (x - x_{i-1})^4 + {C_4}^2 (x - x_i)^4, & x \in [x_i, x_{i+1}] \\ (x - x_{i+3})^4 - {C_4}^1 (x - x_{i+2})^4, & x \in [x_{i+1}, x_{i+2}] \\ (x - x_{i+3})^4, & x \in [x_{i+2}.x_{i+3}] \\ 0. & else \end{cases} \tag{17}$$

Values of $B_i(x)$, $B_i'(x)$, $B_i''(x)$ and $B_i'''(x)$ at the knots have been shown in the table(1). Moreover, the integral values of $B_i(x)$ on each interval is as follows

Table 2. The values of $B_i^{(k)}$, $(k = 0, 1, 2, 3)$at the given knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$B_i(x)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$B_i'(x)$	0	$\frac{1}{6h}$	$\frac{3}{6h}$	$-\frac{3}{6h}$	$-\frac{1}{6h}$	0
$B_i''(x)$	0	$\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$\frac{1}{2h^2}$	0
$B_i'''(x)$	0	$\frac{1}{h^3}$	$-\frac{3}{h^3}$	$\frac{3}{h^3}$	$-\frac{1}{h^3}$	0

$$\int_{x_{i-2}}^{x_{i-1}} B_i(x)dx = \int_{x_{i+2}}^{x_{i+3}} B_i(x)dx = \frac{1}{120}h$$
$$\int_{x_{i-1}}^{x_i} B_i(x)dx = \int_{x_{i+1}}^{x_{i+2}} B_i(x)dx = \frac{26}{120}h,$$
$$\int_{x_i}^{x_{i+1}} B_i(x)dx = \frac{66}{120}h,$$
$$\int_{x_j}^{x_{j+1}} B_i(x)dx = \frac{66}{120}h,\ (j \geq i+3,\ or\, j \leq i-3).$$

The problem of integro quartic spline interpolation is expressed as follows:
Consider the integral values I_i of $y(x)$ on the n intervals $[x_i, x_{i+1}]$, $j = 0, 1, ..., n-1$ and four boundary conditions $y_0 = y(x_0)$, $y_1 = y(x_1)$, $y_{n-1} = y(x_{n-1})$ and $y_n = y(x_n)$. By using the conditions $s_0 = y_0$, $s_1 = y_1$, $s_{n-1} = y_{n-1}$ and $s_n = y_n$, we construct a quartic spline $s(x)$ in a way that

$$\int_{x_i}^{x_{i+1}} s(x)\, dx = I_i = \int_{x_i}^{x_{i+1}} y(x)\, dx,\ \ i = 0, 1, ..., n-1, \tag{18}$$

and

$$s(x_0) = y_0, s(x_1) = y_1, s(x_{n-1}) = y_{n-1}, s(x_n) = y_n. \tag{19}$$

Each spline $s(x) \in S_4(I)$ can be displayed as $s(x) = \sum\limits_{j=-2}^{n+1} c_j B_j(x)$. Using the integral values of $B_i(x)$, we have

$$\int_{x_i}^{x_{i+1}} s\,(x)\, dx = \int_{x_i}^{x_{i+1}} \sum_{j=-2}^{n+1} c_j B_j\,(x)\, dx = \sum_{j=i-2}^{i+2} c_j \int_{x_i}^{x_{i+1}} B_j\,(x)\, dx = I_i, \tag{20}$$

which lead to

$$\frac{h}{120}\,(c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}) = I_i. \tag{21}$$

Moreover, considering the four boundary conditions, we have

$$\begin{cases} s(x_0) = y_0 \\ s(x_1) = y_1 \\ s(x_{n-1}) = y_{n-1} \\ s(x_n) = y_n \end{cases} \Rightarrow \begin{cases} c_{-2} + c_{-1} + c_0 + c_1 = 24y_0 \\ c_{-1} + c_0 + c_1 + c_2 = 24y_1 \\ c_{n-1} + c_n + c_{n+1} + c_{n+2} = 24y_{n-1} \\ c_{n-2} + c_{n-1} + c_n + c_{n+1} = 24y_n \end{cases} \tag{22}$$

Thus, (21) and (22) give us a system of linear equations with a five-band coefficient matrix of order $n+5$ by $n+5$. It can be written in matrix form

$$AC = F, \tag{23}$$

where

$$A = \begin{bmatrix} 1 & 11 & 11 & 1 & 0 & & & & \\ 0 & 1 & 11 & 11 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 11 & 11 & 1 & 0 \\ & & & & 0 & 1 & 11 & 11 & 1 \end{bmatrix}, \tag{24}$$

and

$$C = (c_{-2}, c_{-1}, c_0, c_1, ..., c_n, c_{n+1})^t, \tag{25}$$

$$F = \left(24y_0, 24y_1, \frac{120}{h} I_0, ..., \frac{120}{h} I_{n-1}, 24y_{n-1}, 24y_n\right)^t. \tag{26}$$

Theorem 3.1. [12] *The integro quartic spline interpolation problem (18) and (19) is uniquely solvable.*

After solving $AC = F$, integro quartic spline $s(x) = \sum_{j=-2}^{n+1} c_j\, B_j(x)$ is obtained. By using data from table (1) in the knots x_i, $(i = 0, 1, ..., n)$ we have

$$s_i = s(x_i) = \sum_{j=-2}^{n+1} c_j\, B_j(x_i) = \frac{1}{24}(c_{i-2} + 11c_{i-1} + 11c_i + c_{i+1}). \tag{27}$$

4. Error analysis

In this section we will proceed on super convergence of integro quadratic and quartic interpolation. For a step size h and an infinitely differentiable $y(x)$, shift operator, differential operator and identity operator are defined as follows, respectively

$$\begin{aligned} &\mathbb{E}y\left(x\right) = y\left(x + h\right), \\ &\mathbb{D}y\left(x\right) = y'\left(x\right), \\ &\mathbb{I}y\left(x\right) = y\left(x\right). \end{aligned} \tag{28}$$

Moreover for a positive integer m we have

$$\begin{aligned} &\mathbb{E}^m y\left(x\right) = y\left(x + mh\right), \\ &\mathbb{E}^{-m} y\left(x\right) = y\left(x - mh\right), \\ &\mathbb{D}^m y\left(x\right) = y^{(m)}\left(x\right), \\ &\mathbb{I}^m y\left(x\right) = y\left(x\right). \end{aligned} \tag{29}$$

We also have

$$\mathbb{E}y\left(x\right) = y\left(x + h\right) = \sum_{i=0}^{+\infty} \frac{h^i y^{(i)}\left(x\right)}{i!} = \left(\sum_{i=0}^{+\infty} \frac{\left(h\mathbb{D}\right)^i}{i!}\right) y\left(x\right) = e^{h\mathbb{D}} y\left(x\right). \tag{30}$$

By using the above relation, we may write

$$\mathbb{E} = e^{h\mathbb{D}}, \tag{31}$$

where, similarly, we have

$$\mathbb{E}^{-1} = e^{-h\mathbb{D}},\ \mathbb{E}^m = e^{mh\mathbb{D}},\ \mathbb{E}^{-m} = e^{-mh\mathbb{D}}. \tag{32}$$

Lemma 4.1. [12] *Suppose $y_i = y(x_i)$ for $i = 0, 1, ..., n$, then we have*

$$I_i = \frac{\mathbb{E} - \mathbb{I}}{\mathbb{D}} y_i.$$

4.1. Super convergence of integro quadratic interpolation.

Theorem 4.1. *Assume that $y(x)$ is an infinitely differentiable function and $s(x)$ be the integro quadratic spline obtained by (5) and (6). For $i = 0, 1, ..., n$ we have*

$$s_i = y(x_i) - \frac{h^4}{30} y^{(4)}(x_i) + O\left(h^6\right). \tag{33}$$

Proof. By using (9) and (15) we have

$$s_i + 4s_{i+1} + s_{i+2} = \frac{3}{h}(I_i + I_{i+1}). \tag{34}$$

Using the above operators, Lemma 4.1 and Taylor formula we have

$$\begin{aligned} s_i &= \frac{3}{h}(I_i + I_{i+1}) = \frac{3}{h}\frac{\mathbb{I}+\mathbb{E}}{\mathbb{I}+\mathbb{E}4+\mathbb{E}^2} I_i \\ &= \frac{3}{h\mathbb{D}}\frac{(\mathbb{I}+\mathbb{E})(\mathbb{I}-\mathbb{E})}{\mathbb{I}+\mathbb{E}4+\mathbb{E}^2} y_i = \frac{3}{h\mathbb{D}}\frac{(\mathbb{I}+e^{h\mathbb{D}})(e^{h\mathbb{D}}-\mathbb{E})}{\mathbb{I}+4e^{h\mathbb{D}}+e^{2h\mathbb{D}}} y_i \\ &= \frac{-\mathbb{I}+\mathbb{I}+2h\mathbb{D}+\frac{(2h\mathbb{D})^2}{2!}+\frac{(2h\mathbb{D})^3}{3!}+\frac{(2h\mathbb{D})^4}{4!}+...}{\mathbb{I}+4(\mathbb{I}+h\mathbb{D}+\frac{(h\mathbb{D})^2}{2!}+\frac{(h\mathbb{D})^3}{3!}+...)+(\mathbb{I}+2h\mathbb{D}+\frac{(2h\mathbb{D})^2}{2!}+\frac{(2h\mathbb{D})^3}{3!}+...)} \\ &= \frac{-\mathbb{I}+\mathbb{I}+2h\mathbb{D}+\frac{(2h\mathbb{D})^2}{2!}+\frac{(2h\mathbb{D})^3}{3!}+\frac{(2h\mathbb{D})^4}{4!}+...}{\mathbb{I}+4(\mathbb{I}+h\mathbb{D}+\frac{(h\mathbb{D})^2}{2!}+\frac{(h\mathbb{D})^3}{3!}+...)+(\mathbb{I}+2h\mathbb{D}+\frac{(2h\mathbb{D})^2}{2!}+\frac{(2h\mathbb{D})^3}{3!}+...)} \\ &= (\mathbb{I} - \frac{(h\mathbb{D})^4}{30} + ...)y_i = y(x_i) - \frac{h^4}{30}y^{(4)}(x_i) + O\left(h^6\right). \end{aligned}$$

□

4.2. **Super convergence of integro quartic interpolation.**

Lemma 4.2. [12] *Suppose that $s(x)$ be an integro quartic spline interpolation obtained from (18) and (19) for $y(x)$. For $i = 0, 1, .., n$ we have*

$$s_j = \frac{5}{h}\left(\frac{\mathbb{E}^{-2} + 11\mathbb{E}^{-1} + 11\mathbb{I} + \mathbb{E}}{\mathbb{E}^{-2} + 26\mathbb{E}^{-1} + 66\mathbb{I} + 26\mathbb{E} + \mathbb{E}^2}\right) I_j. \tag{35}$$

Theorem 4.2. [12] *Let $y\left(x\right) \in C^{\infty}\left[a,b\right]$ and $s(x)$ be integro quartic spline interpolation obtained from (18) and (19). For $i = 0, 1, ..., n$ we have*

$$s_i = y\left(x_i\right) + \frac{1}{5040}h^6 y^{(6)}\left(x_j\right) + O\left(h^8\right). \tag{36}$$

Theorem 4.3. [12] *Suppose that $y\left(x\right) \in C^{\infty}\left[a,b\right]$ and $s(x)$ are integro quartic spline interpolation obtained form (18) and (19). Then we have*

$$||y\left(x\right) - s\left(x\right)||_{\infty} = O\left(h^5\right) \tag{37}$$

where $\|f\|_{\infty} = \max\limits_{a \le x \le b} |f(x)|$.

5. Numerical results

In this section we are going to present some numerical examples to show super convergence properties, accuracy test and efficiency of integro quadratic and quartic spline interpolation. Numerical test is carried out by MATLAB software. Suppose that $y \in C^{\infty}\left[0,1\right]$ and consider following test functions

$$y_1(x) = \sin(\pi x), \quad y_2(x) = \cos(\pi x), \quad y_3(x) = e^x, \quad y_4(x) = \frac{1}{x+2}. \tag{38}$$

Suppose that $s(x)$ is integro quadratic spline interpolation and $p(x)$ is integro quartic spline interpolation for $y(x)$. Maximum absolute error between the approximating function of $y(x)$ and its corresponding integro quadratic and quartic interpolation at the knots is defined as follows

$$E_s\left(n\right) = \max_{0 \le j \le n} |s_j - y\left(x_j\right)|, \tag{39}$$

and

$$E_p(n) = \max_{0 \le j \le n} |p_j - y(x_j)|, \quad (40)$$

where $E_s(n)$ and $E_p(n)$ are maximum absolute error for $s(x)$ and $p(x)$, respectively. The tables 3-6 shows maximum absolute error between approximating function $y(x)$ and integro quadratic and quartic spline interpolation.

TABLE 3. The maximum errors for $y_1(x) = sin\pi x$

n	$E_s(n)$	$E_p(n)$
10	5.4755×10^{-5}	1.9197×10^{-7}
20	3.3922×10^{-6}	2.9982×10^{-9}
30	6.6897×10^{-7}	2.6233×10^{-10}
40	2.1154×10^{-7}	4.6638×10^{-11}
50	8.6626×10^{-8}	1.2217×10^{-11}

TABLE 4. The maximum errors for $y_2(x) = cos\pi x$

n	$E_s(n)$	$E_p(n)$
10	6.6747×10^{-5}	2.4899×10^{-7}
20	4.2593×10^{-6}	4.3090×10^{-9}
30	8.4455×10^{-7}	3.8504×10^{-10}
40	2.6757×10^{-7}	6.8950×10^{-11}
50	1.0966×10^{-7}	11.8128×10^{-11}

TABLE 5. The maximum errors for $y_3(x) = e^x$

n	$E_s(n)$	$E_p(n)$
10	1.7689×10^{-6}	6.8170×10^{-10}
20	1.1503×10^{-7}	1.1570×10^{-11}
30	2.3025×10^{-8}	1.0427×10^{-12}
40	7.3335×10^{-9}	1.9984×10^{-13}
50	3.0156×10^{-9}	4.8405×10^{-14}

TABLE 6. The maximum errors for $y_4(x) = \frac{1}{x+2}$

n	$E_s(n)$	$E_p(n)$
10	4.3450×10^{-7}	9.4265×10^{-10}
20	2.9930×10^{-8}	1.9518×10^{-11}
30	6.1084×10^{-9}	1.8892×10^{-12}
40	1.9646×10^{-9}	3.5388×10^{-13}
50	8.1265×10^{-10}	9.8310×10^{-14}

According to these tables it can be concluded that integro quadratic and quartic spline interpolation have super convergence order in approximating function values at the knots.

Integro quadratic and quartic spline interpolation errors for the test functions $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ have been shown in the figures (1)-(4).

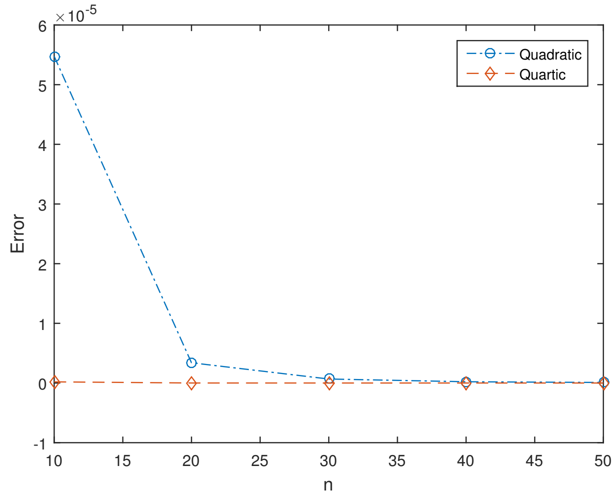

FIGURE 1. Comparison between the integro quadratic and quartic interpolation errors for $y_1(x) = sin(\pi x)$.

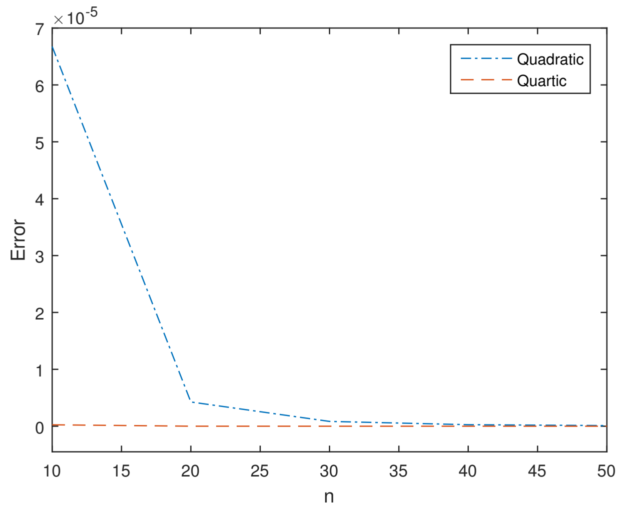

FIGURE 2. Comparison between the integro quadratic and quartic interpolation errors for $y_2(x) = cos(\pi x)$.

According to these figures it can be understood that quadratic integro interpolation error is more than quartic integro interpolation error. When n is increased, integro quadratic interpolation error is decreased, tending toward integro quartic interpolation error. For both integro interpolations, as n is increased, the integro interpolation errors tend toward zero. Moreover, computational cost of integro quadratic interpolation is less than integro quartic interpolation.

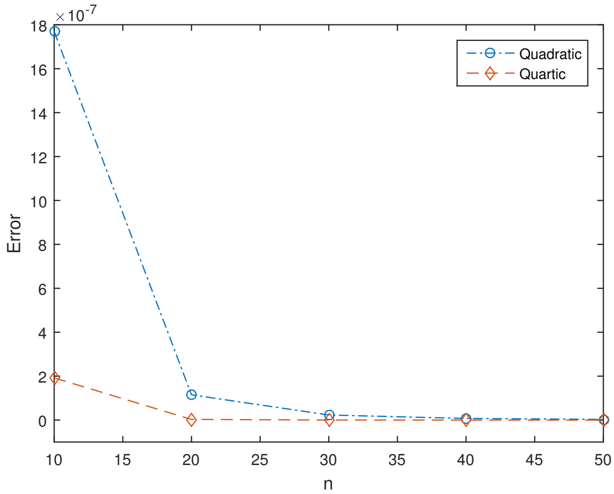

FIGURE 3. Comparison between the integro quadratic and quartic interpolation errors for $y_3(x) = e^x$.

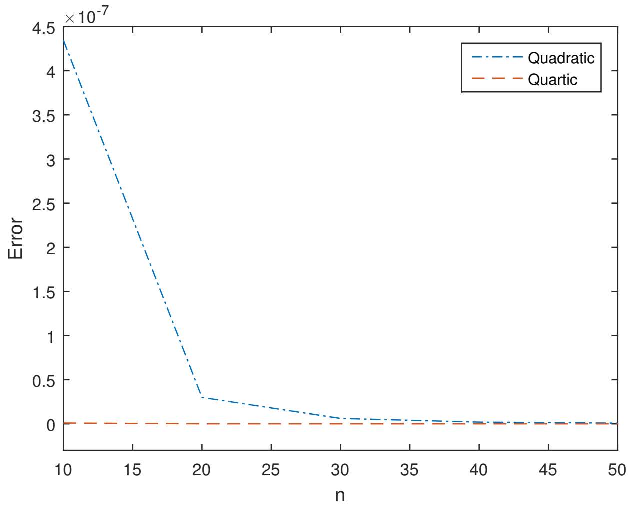

FIGURE 4. Comparison between the integro quadratic and quartic interpolation errors for $y_4(x) = \frac{1}{x+2}$.

6. CONCLUSION

In this paper we studied integro interpolation by using quadratic and quartic splines. Such new type of interpolation method can be used in different fields. Integro interpolation can construct approximating function by using integral values on successive intervals instead of using function values at knots. This interpolation has super convergence property in approximation of function values at the knots. Integro quartic interpolation error is lower than integro quadratic interpolation error. In return, approximation cost of integro quadratic interpolation is lower than quartic interpolation. By increasing the number n, as interpolation interval is partitioned into more parts, interpolation error is decreased and tended toward zero.

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