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φ-BEST PROXIMITY POINT THEOREMS IN METRIC SPACES WITH APPLICATIONS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce the notions of (F, φ, θ)-proximal contraction and (F, φ, θ)-weak proximal contraction for non-self mappings and utilize the same to prove some existence and uniqueness of φ-best proximity point for such mappings. Some illustrative examples are also given to exhibit the utility of our results. As an application of the concept of φ-best proximity point, we deduce some best proximity point theorems in the context of partial metric spaces.

Keywords: (F, φ, θ)-proximal contraction, (F, φ, θ)-weak proximal contraction, φ-best proximity point, partial metric space.

AMS Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

Let (A, B) be a pair of non-empty subsets of a metric space (X, d) and $T : A \to B$ a mapping. Naturally, Banach contraction principle implies that every self-contraction T defined on a complete subsets A of a metric space X admits a unique fixed point. But a non self-mapping $T : A \to B$ need not to have a fixed point. In case T is free from fixed point, it is of interest to find an element x in A which is close to Tx (belonging to B) in the sense that $d(x, Tx)$ is minimum. In view of the fact that $d(A, B) \leq d(x, Tx)$, for all $x \in A$, a best proximity point is a point x that satisfies the condition $d(x, Tx) = d(A, B)$. A best proximity theorem enunciates sufficient conditions for the existence of a best proximity point of the mapping T. In fact, best proximity theorems are natural generalizations of fixed point theorems. For the convergence and existence theorems pertaining to best proximity points for several variants of contractions, one can be referred to [1, 2, 3, 4]. Results on best proximity point for cyclic mappings can be found in[5, 6] whereas Abkar and Gabeleh [7] studied best proximity point for noncyclic mappings in metric spaces.

In this article, we introduce the notions of (F, φ, θ)-proximal contraction and (F, φ, θ)-weak proximal contraction for non-self mappings wherein the control function F can be discontinuous(unlike [8]) and utilize the same to prove some existence and uniqueness results for φ-best proximity point for such mappings. We also adopt some examples to

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exhibit the utility of our results. These examples also demonstrate that our results are proper generalizations of the results due to Işlk et al. [8]. As an application of the concept of φ-best proximity point, we deduce some best proximity point theorems in the context of partial metric spaces.

In 2014, Jleli et al. [9] introduced the control function $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(F1) $\max\{a,b\} \le F(a,b,c)$, for all $a,b,c \in [0,\infty)$;
(F2) $F(0,0,0) = 0$;
(F3) F is continuous.

The class of all functions F satisfying conditions (F1)-(F3) is denoted by $\mathcal{F}$.

Example 1.1. [9] *The following functions $F : [0,\infty)^3 \to [0,\infty)$ belong to $\mathcal{F}$:*

(1) $F(a,b,c) = a + b + c$;
(2) $F(a,b,c) = max\{a,b\} + c$;
(3) $F(a,b,c) = a + a^2 + b + c$.

Recently, Asadi. [10] replaced condition (F3) by the following condition :

(F3′) $\limsup_{n\to\infty} F(x_n, y_n, 0) \le F(x,y,0)$, when $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

The class of all functions F satisfying the conditions (F1), (F2) and (F3′) is denoted by $\mathbb{F}$. Observe that $\mathcal{F} \subseteq \mathbb{F}$ but the converse is not true as substantiated by the following example:

Example 1.2. [10] *The following functions $F : [0,\infty)^3 \to [0,\infty)$ belong to $\mathbb{F}$ but not in $\mathcal{F}$ as they are not continuous:*

(1) $F(a,b,c) = a + b + [c]$;
(2) $F(a,b,c) = max\{a,b\} + [c]$.

For any pair (A,B) of non-empty subsets of a metric space (X,d), we employ the following notions:

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x,y) = d(A,B), \quad \text{for some} \quad y \in B\},$$

$$B_0 = \{y \in B : d(x,y) = d(A,B), \quad \text{for some} \quad x \in A\}.$$

Throughout this work, $B_{est}(T)$ denotes the set of all best proximity points of non-self mapping T defined on a metric space (X,d). Particularly, for a mapping $T : A \to B$ we have

$$B_{est}(T) = \{x \in A : d(x,Tx) = d(A,B)\}.$$

Let $\varphi : A \to [0,\infty)$ be a function. Then, we denote the set of all zeros of the function φ by Z_φ. That is,

$$Z_\varphi = \{x \in A : \varphi(x) = 0\}.$$

In 2017, Işik et al.[8] introduced the concepts of φ-best proximity point of non-self mapping, (F,φ)-proximal contraction and (F,φ)-weak proximal contraction as follows:

Definition 1.1. [8] *Let (A,B) be a pair of non-empty subsets of a metric space (X,d) and $\varphi : A \to [0,\infty)$ a given function. An element $x^* \in A$ is called a φ-best proximity point of the mapping $T : A \to B$ if x^* is best proximity point of T and $\varphi(x^*)$=0, that is, $x^* \in B_{est}(T) \cap Z_\varphi$.*

Definition 1.2. [8] *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function and $F \in \mathcal{F}$. We say that the non-self mapping $T : A \to B$ is an (F, φ)-proximal contraction if there exists $k \in (0, 1)$ such that (for all $u, v, x, y \in A$)*

$$\left.\begin{array}{c} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow F(d(u, v), \varphi(u), \varphi(v)) \leq kF(d(x, y), \varphi(x), \varphi(y)). \quad (1)$$

Definition 1.3. [8] *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function and $F \in \mathcal{F}$. We say that the non-self mapping $T : A \to B$ is an (F, φ)-weak proximal contraction if there exist $k \in (0, 1)$ and $L \geq 0$ such that (for all $u, v, x, y \in A$)*

$$\left.\begin{array}{c} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow \begin{array}{l} F(d(u, v), \varphi(u), \varphi(v)) \leq kF(d(x, y), \varphi(x), \varphi(y)) \\ \qquad + L[F(d(y, u), \varphi(y), \varphi(u)) \\ \qquad - F(0, \varphi(y), \varphi(u))]. \end{array} \quad (2)$$

Let J be the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

$(j1)$ θ is a nondecreasing function;
$(j2)$ θ is continuous;
$(j3)$ $\sum_{k=1}^{\infty} \theta^n(t) < \infty$, for all $t > 0$.

The above control function is used in [10, 11]

Lemma 1.1. [11]

(i) If $\theta \in J$, then $\theta(t) < t$, for all $t > 0$;
(ii) If $\theta \in J$, then $\theta(0) = 0$.

Remark 1.1. *Observe that (j3) implies $\lim_{n\to\infty} \theta^n(t) = 0$, for all $t \in (0, \infty)$.*

2. Main results

We begin this section by introducing the notion of (F, φ, θ)-proximal contraction as follows:

Definition 2.1. *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function, $F \in \mathbb{F}$ and $\theta \in J$. A mapping $T : A \to B$ is said to be an (F, φ, θ)-proximal contraction if (for all $u, v, x, y \in A$)*

$$\left.\begin{array}{c} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow F(d(u, v), \varphi(u), \varphi(v)) \leq \theta(F(d(x, y), \varphi(x), \varphi(y))). \quad (3)$$

Now, we state and prove our first result as follows:

Theorem 2.1. *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function, $F \in \mathbb{F}$ and $\theta \in J$. Suppose that the following conditions are satisfied:*

$(H1)$ A_0 is non-empty and complete with respect to the topology induced by d;
$(H2)$ $T(A_0) \subseteq B_0$;
$(H3)$ $\varphi : A \to [0, \infty)$ is lower semi-continuous;
$(H4)$ $T : A \to B$ is an (F, φ, θ)-proximal contraction.

Then the following assertions hold:

(i) $B_{est}(T) \subseteq Z_\varphi$;
(ii) T has a unique φ-best proximity point $x^* \in A$. Moreover, $\lim_{n\to\infty} T^n x = x^*$, for all $x \in X$.

Proof. (i) Suppose that $\xi \in A$ is a best proximity point of T so that $d(\xi, T\xi) = d(A, B)$. Applying (3) with $u = v = x = y = \xi$, we obtain

$$F(0, \varphi(\xi), \varphi(\xi)) \leq \theta(F(0, \varphi(\xi), \varphi(\xi))),$$

which in view of Lemma 1.1 gives rise

$$F(0, \varphi(\xi), \varphi(\xi)) = 0. \tag{4}$$

On the other hand, from (F1), we have

$$\varphi(\xi) \leq F(0, \varphi(\xi), \varphi(\xi)). \tag{5}$$

Using (4) and (5), we obtain $\varphi(\xi) = 0$ so that $\xi \in Z_\varphi$. This implies that

$$B_{est}(T) \subseteq Z_\varphi.$$

(ii) Next, choose an element $x_0 \in A_0$. As $Tx_0 \in T(A_0) \subseteq B_0$, we can find x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$. Again, $Tx_1 \in T(A_0) \subseteq B_0$ implies that there exists an element $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Repeating this process, we construct a sequence $\{x_n\} \subseteq A_0$ satisfying

$$d(x_{n+1}, Tx_n) = d(A, B), \text{ for all } n \in \mathbb{N}. \tag{6}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

Hence x_{n_0} is a best proximity point of T and we are done. Now, assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. On using (3) and (6), we have

$$F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \leq \theta(F(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))),$$

for all $n \in \mathbb{N}$. By induction on n, for each $n \in \mathbb{N}$, we have

$$F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \leq \theta^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))),$$

which implies that (in view of (F1))

$$max\{d(x_n, x_{n+1}), \varphi(x_n)\} \leq \theta^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))), \tag{7}$$

for all $n \in \mathbb{N}$. Hence

$$d(x_n, x_{n+1}) \leq \theta^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))), \tag{8}$$

for all $n \in \mathbb{N}$. We assert that $\{x_n\}$ is a Cauchy sequence. To prove our assertion, let $m, n \in \mathbb{N}$ such that $m > n$. On using (8) and triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \theta^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) + \theta^{n+1}(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) \\ &+ ... + \theta^{m-1}(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) \\ &= \sum_{i=1}^{m-1} \theta^i(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) - \sum_{j=1}^{n-1} \theta^j(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))). \end{aligned} \tag{9}$$

From (9), $(j3)$ and Remark 1.1, we obtain $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Thus, our claim is established.
Now, as A_0 is complete, there exists $x^* \in A_0$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{10}$$

We now prove that x^* is a φ-best proximity point of T. From (7), we can write

$$\varphi(x_n) \leq \theta^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))).$$

letting $n \to \infty$ in the above inequality and Making use of Remark 1.1, we have

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{11}$$

Since φ is lower semi-continuous, from (10) and (11) we get

$$\varphi(x^*) = 0. \tag{12}$$

Also, since $x^* \in A_0$ and $T(A_0) \subseteq B_0$, we can find $z \in A_0$ such that

$$d(z, Tx^*) = d(A, B). \tag{13}$$

Using $(H4)$, (6) and (13), we obtain

$$F(d(x_{n+1}, z), \varphi(x_{n+1}), \varphi(z)) \leq \theta(F(d(x_n, x^*), \varphi(x_n), \varphi(x^*))), \tag{14}$$

which on making use of (12), (14) and Lemma 1.1 (i), gives rise

$$\begin{aligned} d(x_{n+1}, z) &\leq max\{d(x_{n+1}, z), \varphi(x_{n+1})\} \\ &\leq F(d(x_{n+1}, z), \varphi(x_{n+1}), \varphi(z)) \\ &\leq \theta(F(d(x_n, x^*), \varphi(x_n), \varphi(x^*))) \\ &< F(d(x_n, x^*), \varphi(x_n), \varphi(x^*)) \\ &= F(d(x_n, x^*), \varphi(x_n), 0). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ of both the sides and making use of (F3′), we obtain

$$\limsup_{n\to\infty} d(x_{n+1}, z) \leq \limsup_{n\to\infty} F(d(x_n, x^*), \varphi(x_n), 0) \leq F(0, 0, 0) = 0,$$

which implies that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{15}$$

From (10) and (15), we get $z = x^*$. This together with (12) and (13) imply that x^* is a φ-best proximity point of T, that is

$$x^* \in B_{est}(T) \cap Z_\varphi.$$

Finally, we show that x^* is unique φ-best proximity point. Suppose that $w \in A$ is another φ-best proximity point of T. So, we have

$$d(x^*, Tx^*) = d(w, Tw) = d(A, B), \quad \varphi(x^*) = 0, \quad \text{and} \quad \varphi(w) = 0.$$

Now, on using $(H4)$, we have

$$F(d(x^*, w), \varphi(x^*), \varphi(w)) \leq \theta(F(d(x^*, w), \varphi(x^*), \varphi(w))),$$

which implies that

$$F(d(x^*, w), 0, 0) \leq \theta(F(d(x^*, w), 0, 0)).$$

In view of Lemma 1.1, we conclude that $F(d(x^*, w), 0, 0) = 0$ so that $d(x^*, w) = 0$, that is $x^* = w$. Hence, the φ-best proximity point of T is unique. This concludes the proof. □

Remark 2.1. *Notice that for any $F \in \mathcal{F}$ (as $\mathcal{F} \subseteq \mathbb{F}$) with $\theta(t) = kt$, for all $t \in [0, \infty)$, (where $k \in [0, 1)$) in Theorem 2.1, we obtain* [8, Theorem 7].

The following examples show that Theorem 2.1 remains genuine extensions of the corresponding theorems due to Işik et al.[8]

Example 2.1. *Let* $X = [0,1]$ *endowed with the usual metric. Suppose that* $A = [\frac{1}{2}, 1]$ *and* $B = [0, \frac{1}{2}]$. *Then* A *and* B *are non-empty subsets of* X, $A_0 = \{\frac{1}{2}\}$, $B_0 = \{\frac{1}{2}\}$ *and* $d(A,B) = 0$. *Let* $F : [0,\infty)^3 \to [0,\infty)$ *and* $\varphi : A \to [0,\infty)$ *be defined by*

$$F(a,b,c) = a + b + [c], \text{ for all } a,b,c \in [0,\infty) \text{ and } \varphi(x) = \ln(x + \frac{1}{2}), \text{ for all } x \in A,$$

where $[c]$ *is the integer of* c. *Then* $F \in \mathbb{F}$ *and* $Z_\varphi = \{\frac{1}{2}\}$. *Define* $T : A \to B$ *and* $\theta : [0,\infty) \to [0,\infty)$ *by*

$$T(x) = 1 - x, \text{ for all } x \in A \text{ and } \theta(t) = kt, \text{ for all } t \in [0,\infty),$$

where $k \in [0,1)$. *Then* $T(A_0) \subseteq B_0$ *and* $\theta \in J$. *Observe that* T *is a* (F,φ,θ)*-proximal contraction mapping, because if* $d(u,Tx) = d(A,B)$ *and* $d(v,Ty) = d(A,B)$, *wherein* $u = v = x = y = \frac{1}{2}$, *we have*

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d(u,v) + \varphi(u) + [\varphi(v)] \\ &= d(x,y) + \varphi(x) + [\varphi(y)] \\ &= F(d(x,y),\varphi(x),\varphi(y)) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

Hence, all conditions of Theorem 2.1 are satisfied so that T *has a unique* φ*-best proximity point (namely* $B_{est}(T) \cap Z_\varphi = \{\frac{1}{2}\}$*).*

Example 2.2. *Let* $X = \mathbb{R}^2$ *endowed with the usual metric,* $A = \{(x,0), x \geq 1\}$ *and* $B = \{(0,y), y \geq 1\}$. *Then, we have* $A_0 = \{(1,0)\}$, $B_0 = \{(0,1)\}$ *and* $d(A,B) = \sqrt{2}$. *Let* $F : [0,\infty)^3 \to [0,\infty)$ *and* $\varphi : A \to [0,\infty)$ *be defined by*

$$F(a,b,c) = max\{a,b\} + [c], \quad \text{for all } a,b,c \in [0,\infty) \quad \text{and} \quad \varphi(x,y) = \ln(x) \quad \text{for all } x,y \in A,$$

where $[c]$ *is the integer of* c. *Then* $F \in \mathbb{F}$ *and* $Z_\varphi = \{(0,1)\}$. *Define* $T : A \to B$ *and* $\theta : [0,\infty) \to [0,\infty)$ *by*

$$T(x,y) = \Big(0, \frac{x+1}{2}\Big), \quad \text{for all } x,y \in A \text{ and } \theta(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ k\ln(t), & t \geq 1. \end{cases}$$

where $k \in [0,1)$. *So,* $T(A_0) \subseteq B_0$ *and* $\theta \in J$. *Now, we claim that* T *is a* (F,φ,θ)*-proximal contraction mapping. Consider*

$$d(u,Tx) = d(A,B) \quad \text{and} \quad d(v,Ty) = d(A,B),$$

then we have $u = v = x = y = (1,0)$. *Hence*

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= max\,\{d(u,v) + \varphi(u)\} + [\varphi(v)] \\ &= max\,\{d(x,y) + \varphi(x)\} + [\varphi(y)] \\ &= F(d(x,y),\varphi(x),\varphi(y)) \\ &= \theta(F(d(x,y),\varphi(x),\varphi(y))). \end{aligned}$$

Therefore, all conditions of Theorem 2.1 are satisfied so that T *has a unique* φ*-best proximity point (namely* $B_{est}(T) \cap Z_\varphi = \{(1,0)\}$*).*

Next, we define (F,φ,θ)-weak proximal contraction as follows:

Definition 2.2. *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function, $F \in \mathbb{F}$ and $\theta \in J$. A mapping $T : A \to B$ is said to be an (F, φ, θ)-weak proximal contraction if there exists $L \geq 0$ such that (for all $u, v, x, y \in A$)*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Rightarrow F(d(u,v),\varphi(u),\varphi(v)) \leq \theta(F(d(x,y),\varphi(x),\varphi(y))) + L[F(d(y,u),\varphi(y),\varphi(u)) - F(0,\varphi(y),\varphi(u))]. \tag{16}$$

Now, we prove the following result under (F, φ, θ)-weak proximal contraction.

Theorem 2.2. *Let (A, B) be a pair of non-empty subsets of a metric space (X, d), $\varphi : A \to [0, \infty)$ a given function, $F \in \mathbb{F}$ and $\theta \in J$. Suppose that the following conditions are satisfied:*
(H1) A_0 is non-empty and complete with respect to the topology induced by d;
(H2) $T(A_0) \subseteq B_0$;
(H3) $\varphi : A \to [0, \infty)$ is lower semi-continuous;
(H4) $T : A \to B$ is an (F, φ, θ)-weak proximal contraction.
Then the following assertions hold:
(i) $B_{est}(T) \subseteq Z_\varphi$,
(ii) T has at least one φ-best proximity point $x^* \in A$. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to a φ-best proximity point of T.

Proof. (i) Suppose that $\xi \in A$ is a best proximity point of T, then $d(\xi, T\xi) = d(A, B)$. Applying (16) with $u = v = x = y = \xi$, we get

$$\begin{aligned} F(0,\varphi(\xi),\varphi(\xi)) &\leq \theta(F(0,\varphi(\xi),\varphi(\xi))) + L(F(0,\varphi(\xi),\varphi(\xi)) - F(0,\varphi(\xi),\varphi(\xi))) \\ &= \theta(F(0,\varphi(\xi),\varphi(\xi))). \end{aligned}$$

This together with Lemma 1.1 imply that

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \tag{17}$$

Now, from (F1), we have

$$\varphi(\xi) \leq F(0,\varphi(\xi),\varphi(\xi)). \tag{18}$$

Using (17) and (18), we obtain that $\varphi(\xi) = 0$ so that

$$B_{est}(T) \subseteq Z_\varphi.$$

(ii) We choose an element x_0 in A_0. As $T(x_0) \in B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. By repeating this process, we construct a sequence $\{x_n\} \subseteq A_0$ satisfying

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \text{for all } n \in \mathbb{N}.$$

By using (16), we have (for all $n \in \mathbb{N}$)

$$\begin{aligned} F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) &\leq \theta(F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))) \\ &\quad + L(F(d(x_n,x_n),\varphi(x_n),\varphi(x_n)) \\ &\quad - F(d(0,\varphi(x_n),\varphi(x_n))) \\ &= \theta(F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))). \end{aligned}$$

Inductively, for each $n \in \mathbb{N}$, we get

$$F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \leq \theta^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))).$$

The rest of the proof follows by using similar arguments as in the proof of Theorem 2.1. □

Remark 2.2. *Notice that for any $F \in \mathcal{F}$ (as $\mathcal{F} \subseteq \mathbb{F}$) with $\theta(t) = kt$ for all $t \in [0, \infty)$, (where $k \in [0, 1)$) in Theorem 2.2, we obtain* [8, Theorem 10].

3. Application to partial metric spaces

We begin this section by recalling some definitions and basic results which are needed in the sequel.

Definition 3.1. [12] *Let X be a non-empty set. A mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X, if for all $x, y, z \in X$, we have*
(P1) $p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
The pair (X, d) is called a partial metric space.

Remark 3.1. [12] *If $p(x, y) = 0$, then (p1) and (p2) imply that $x = y$ but the converse is not true in general.*

Remark 3.2. [12] *Let X be a non-empty set.*

(a) Every partial metric p on X generates a T_0 topology τ_p on X with base the family of the open balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$.
(b) If p is a partial metric on X, then the function $d_p : X \times X \to [0, \infty)$ defined by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

Definition 3.2. [12] *Let (X, p) be a partial metric space.*

(i) A sequence $\{x_n\} \subseteq X$ is said to be convergent and converges to a point $x \in X$ with respect to τ_p if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.
(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if $\lim_{n\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A partial metric space (X, p) is called a complete partial metric space if every Cauchy sequence in X converges (with respect to τ_p) to a point in X.

Lemma 3.1. [12] *Let (X,p) be a partial metric space. Then the following assertions hold:*
(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p);
(b) the partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete.

Lemma 3.2. [13] *Let (X, p) be a partial metric space and $\varphi : X \to [0, \infty)$ defined by $\varphi(x) = p(x, x)$, $\forall x \in X$. Then the function φ is lower semi-continuous in the metric space (X, d_p).*

Before giving our best proximity point results in the partial metric space, we adapt the following notations
Let (A, B) be a pair of non-empty subsets of a partial metric space (X, p) and $T : A \to B$. Then

$$p(A, B) = \inf\{p(x, y) : x \in A, y \in B\};$$
$$A_0 = \{x \in A : p(x, y) = p(A, B), \text{ for some } y \in B\};$$
$$B_0 = \{y \in B : p(x, y) = p(A, B), \text{ for some } x \in A\}.$$

As an application of our main results, we derive the following results in the setting of partial metric space.

Theorem 3.1. *Let (A, B) be a pair of non-empty subsets of a partial metric space (X, p), $T : A \to B$ and $\theta \in J$. Suppose that the following conditions are satisfied:*
($\overline{H}1$) *A_0 is non-empty and complete with respect to the topology induced by p;*
($\overline{H}2$) *$T(A_0) \subseteq B_0$;*
($\overline{H}3$) *$\theta(2t) = 2\theta(t)$, for all $t \in [0, \infty)$;*
($\overline{H}4$) *the mapping T satisfies (for all $x, y, u, v \in X$)*

$$\left.\begin{array}{l} p(u, Tx) = p(A, B) \\ p(v, Ty) = p(A, B) \end{array}\right\} \Rightarrow p(u, v) \leq \theta(p(x, y)).$$

Then the following assertions hold:
(i) T has a unique best proximity point $x^* \in A$,
(ii) $p(x^*, x^*) = 0$.

Proof. Let $x, y \in A$. Obviously, ($\overline{H}3$) and ($\overline{H}4$) imply that

$$2p(u, v) \leq \theta(2p(x, y)). \tag{19}$$

Consider the metric d_p defined by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$ and the function $\varphi : A \to [0, \infty)$ defined by $\varphi = p(x, x)$, for all $x \in A$. Now, we can write

$$p(x, y) = \frac{d_p(x, y) + p(x, x) + p(y, y)}{2}, \quad \text{for all } x, y \in X. \tag{20}$$

Using (19) and (20), we get

$$d_p(u, v) + \varphi(u) + \varphi(v) \leq \theta(d_p(x, y) + \varphi(x) + \varphi(y)).$$

Define $F : [0, \infty)^3 \to [0, \infty)$ by $F(a, b, c) = a + b + c$, we obtain

$$F(d_p(u, v), \varphi(u), \varphi(v)) \leq \theta(F(d_p(x, y), \varphi(x), \varphi(y))).$$

By using Lemmas 3.1 and 3.2, the hypotheses of Theorem 2.1 are satisfied. Hence, T has a unique best proximity point $x^* \in A$ such that $p(x^*, x^*) = 0$. □

Similarly, from Theorem 2.2, we derive the following result:

Theorem 3.2. *Let (A, B) be a pair of non-empty subsets of a partial metric space (X, d), $T : A \to B$ and $\theta \in J$. Suppose that the following conditions are satisfied:*
($\overline{H}1$) *A_0 is non-empty and complete with respect to the topology induced by p;*
($\overline{H}2$) *$T(A_0) \subseteq B_0$;*
($\overline{H}3$) *$\theta(2t) = 2\theta(t)$, for all $t \in [0, \infty)$;*
($\overline{H}4$) *the mapping T satisfies (for all $x, y, u, v \in X$ and $L \geq 0$)*

$$\left.\begin{array}{l} p(u, Tx) = p(A, B) \\ p(v, Ty) = p(A, B) \end{array}\right\} \Rightarrow p(u, v) \leq \theta\left(p(x, y)\right) + L\left(p(y, u) - \frac{p(y, y) + p(u, u)}{2}\right).$$

Then the following assertions hold:
(i) T has at least one best proximity point $x^* \in A$,
(ii) $p(x^*, x^*) = 0$.

On setting $\theta(t) = kt$ for all $t \in [0, \infty)$, (where $k \in (0, 1)$) in Theorems 3.1 and 3.2, we deduce the following corollaries:

Corollary 3.1. [8] *Let (A, B) be a pair of non-empty subsets of a partial metric space (X, p) and $T : A \to B$. Assume that the following conditions hold:*
($\overline{H}1$) *A_0 is non-empty and complete with respect to the topology induced by p;*
($\overline{H}2$) *$T(A_0) \subseteq B_0$;*

$(\overline{H}3)$ *there exists* $k \in (0,1)$ *such that (for all* $u, v, x, y \in A$*)*

$$\left.\begin{array}{l} p(u,Tx) = p(A,B) \\ p(v,Ty) = p(A,B) \end{array}\right\} \Rightarrow p(u,v) \leq kp(x,y).$$

Then T *has a unique best proximity point* $x^* \in A$.

Corollary 3.2. [8] *Let* (A,B) *be a pair of non-empty subsets of a partial metric space* (X,p) *and* $T : A \to B$. *Assume that the following conditions hold:*

$(\overline{H}1)$ A_0 *is non-empty and complete with respect to the topology induced by p;*

$(\overline{H}2)$ $T(A_0) \subseteq B_0$*;*

$(\overline{H}3)$ *there exists* $k \in (0,1)$ *and* $L \geq 0$ *such that (for all* $u, v, x, y \in A$*)*

$$\left.\begin{array}{l} p(u,Tx) = p(A,B) \\ p(v,Ty) = p(A,B) \end{array}\right\} \Rightarrow p(u,v) \leq k\left(p(x,y)\right) + L\left(p(y,u) - \frac{p(y,y)+p(u,u)}{2}\right).$$

Then T *has at least one best proximity point* $x^* \in A$.

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